Advanced Quantum Field Theory

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Summary

★ Lecture 1) - Path integral in QM

★ Lecture 2) - LSZ formalism and path integral for scalar fields

★ Lecture 3) - Path integral for fermions and gauge fields

★ Lecture 4) - Renormalization for scalar fields

★ Renormalization group and anomalies in QFT

Bibliography

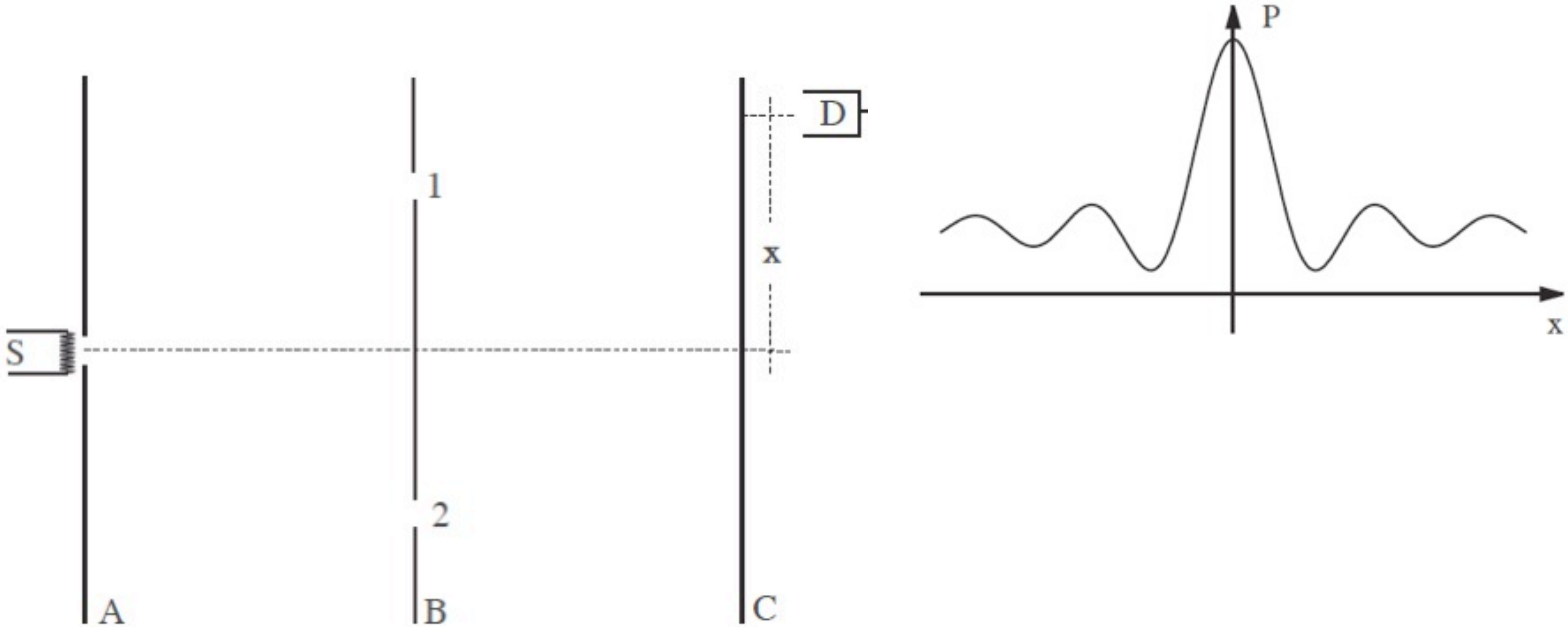
- ★ P. Ramond, Field Theory a modern primer
- ★ M.E. Peskin and D.V. Schroder, An introduction to Quantum Field Theory
- ★ L.H. Ryder, Quantum Field Theory
- ★ R. Casalbuoni, Lecture notes in Advanced Quantum Field Theory, http://theory.fi.infn.it/casalbuoni/lezioni99.pdf

Lecture 1

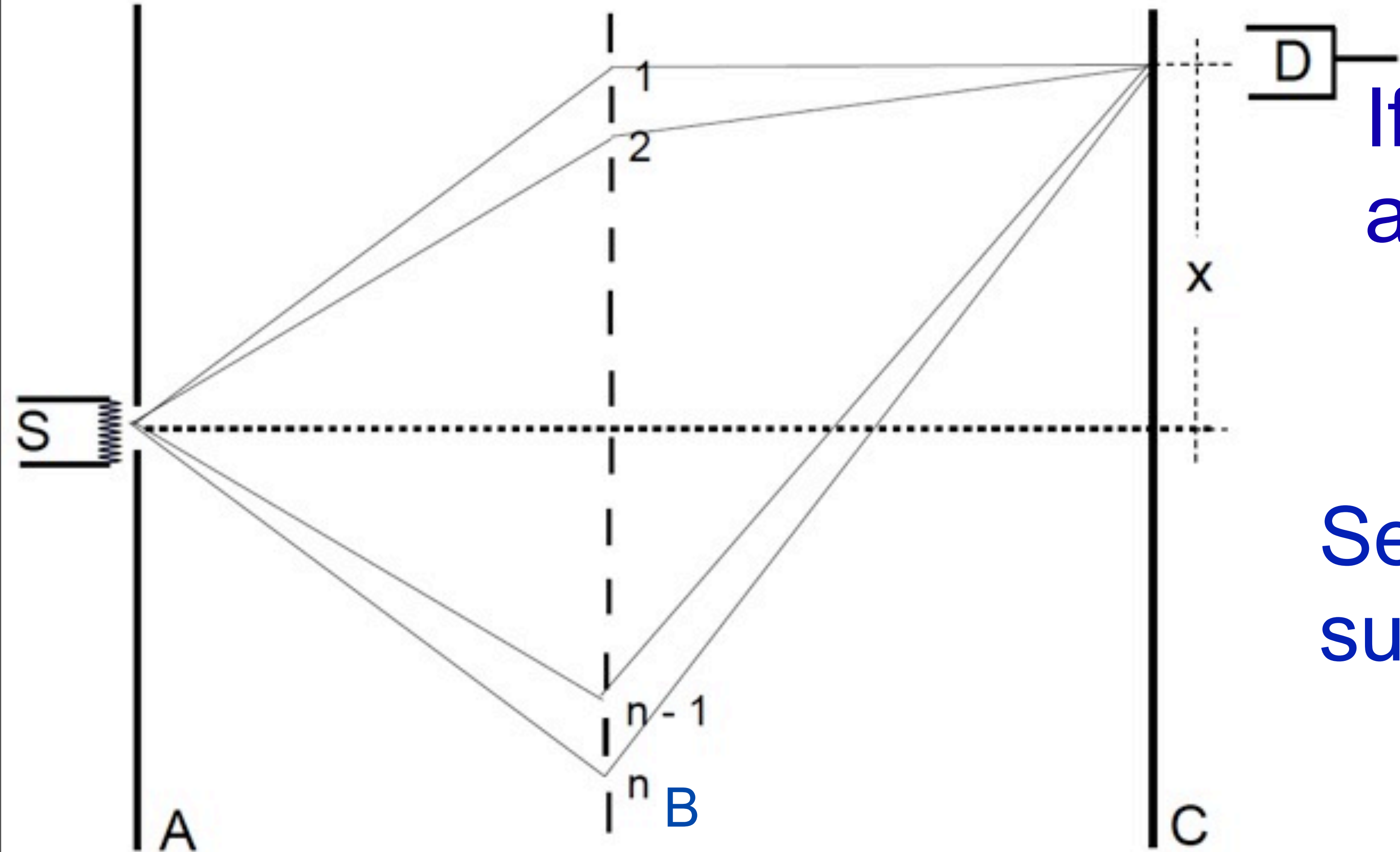
- ★ Path integral in QM
- ★ Functional formalism
- ★ Generating functionals

Path integral in Quantum Mechanics

Let us consider the classical double slit experiment



$$P = |A|^2 = |A_1 + A_2|^2 = |A_1|^2 + |A_2|^2 + A_1^\star A_2 + A_1 A_2^\star \neq |A_1|^2 + |A_2|^2 = P_1 + P_2$$

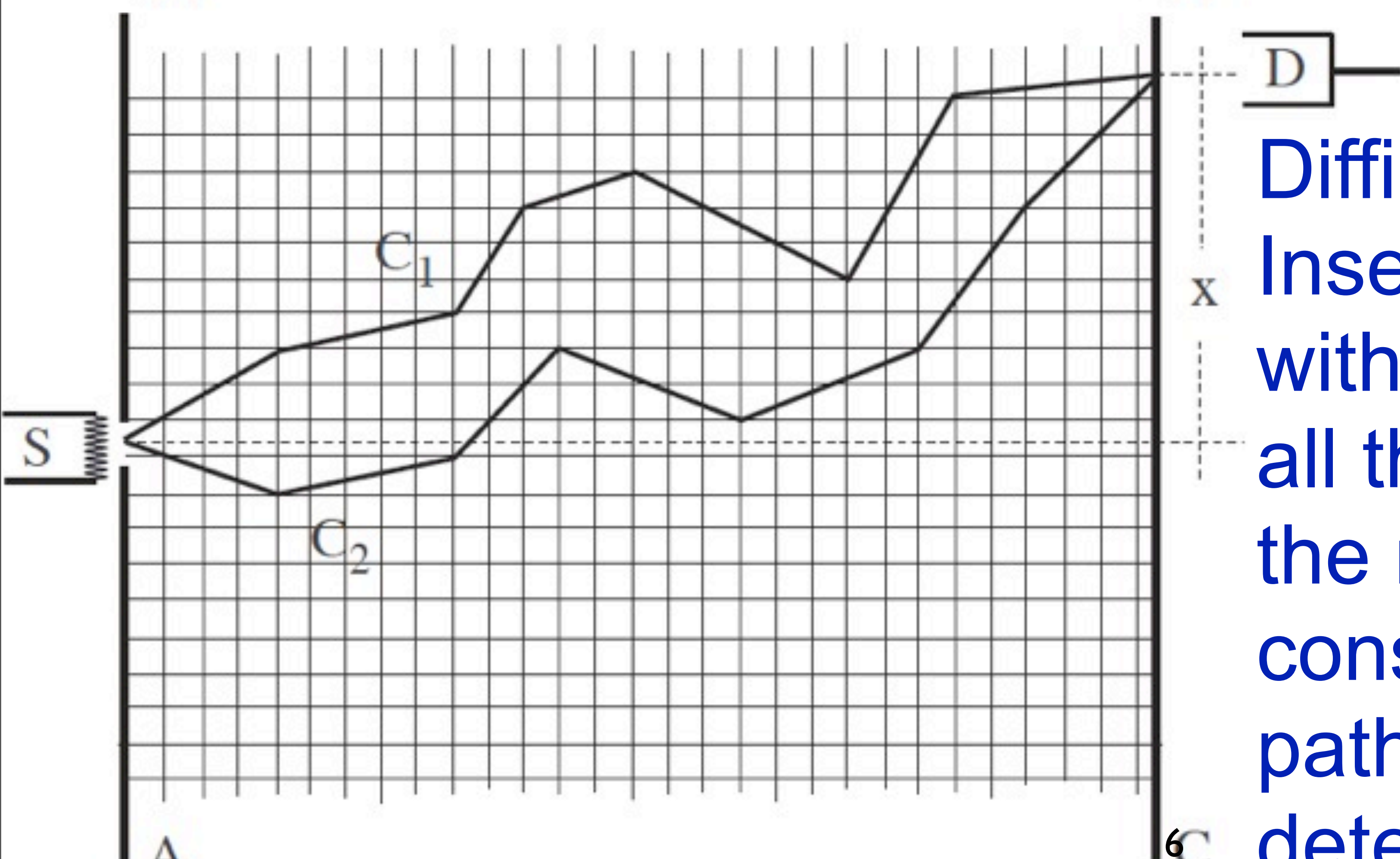


If we consider n-holes the amplitude is given by

$$A = \sum_{i=1}^{n} A_i$$

Sending n to infinity or suppressing the scren B

$$A = \int_B A(x)\,dx$$



Difficult to evaluate A directly. Insert m screens as B, each with n-holes and considering all the possible paths along the mxn holes. Equivalently consider all the possible paths between S and the detector at x.

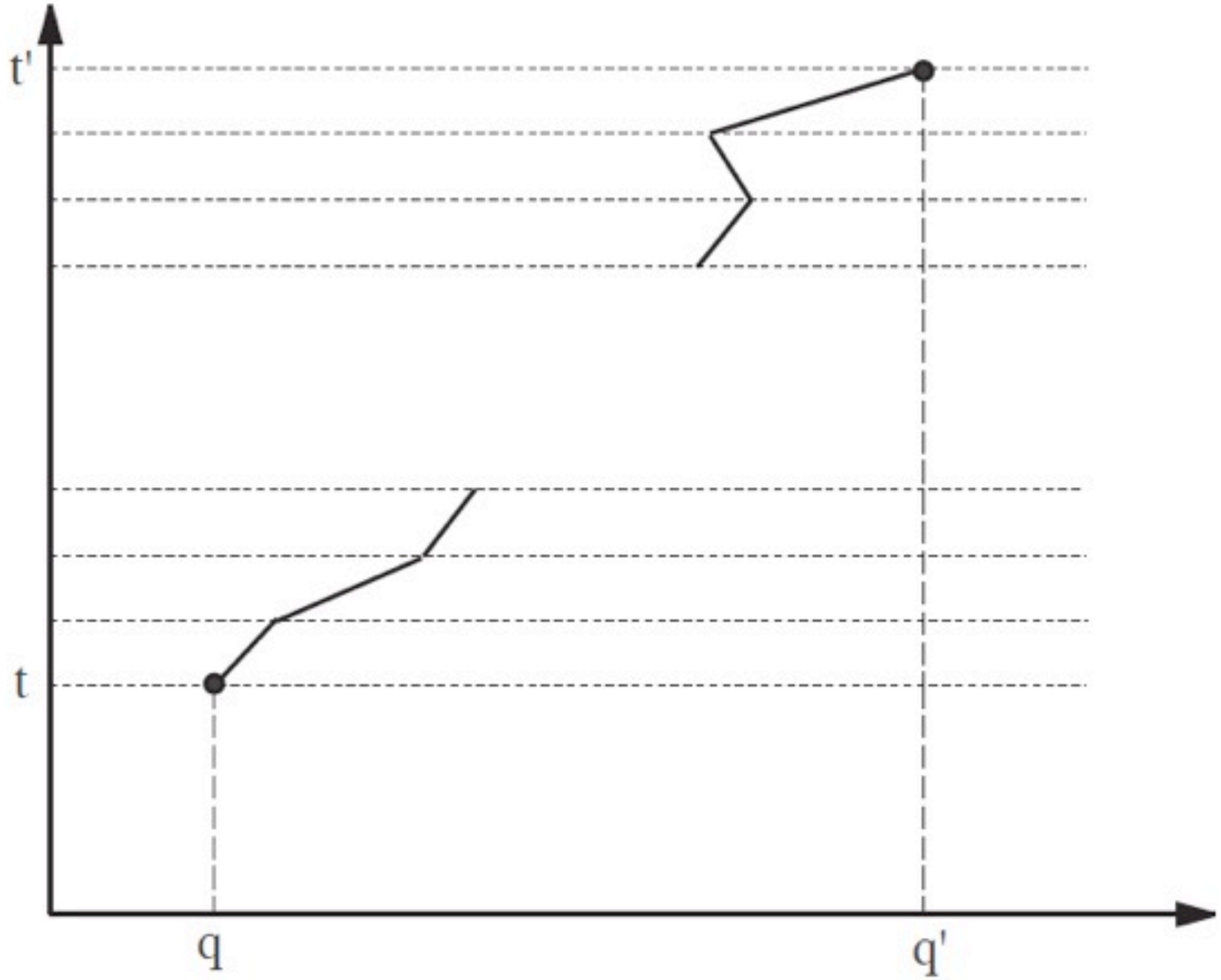
That is to say $$A = \sum_{C} A_c$$

with C's all the possible paths joining S and the detector at x. But now they are decomposed in terms of infinitesimal segments. So we need to calculate the amplitude for a linear path joining any two infinitesimal close points.

Considering the one-dimensional case Dirac proved in his book on quantum mechanics that such an amplitude is given by

$$\langle q', t + \Delta t \mid q, t \rangle \approx e^{iS/\hbar}$$

with S the classical action evaluated along the segment joining thr positions q at time t and q'at time t'.



In the 1- dim case, to evaluate the matrix element

$$\langle q', t' | q, t \rangle, \quad t' \geq t,$$

$$\mathrm{q}(t)|q,t\rangle = q|q,t\rangle,$$

$$|q,t\rangle = e^{iHt}|q,0\rangle$$

we divide the interval (t,t') into infinitesimal pieces and use the completeness relation in q-space

$$\langle q', t' | q, t \rangle =$$

$$= \lim_{n\to\infty} \int dq_1 \cdots dq_{n-1} \langle q', t' | q_{n-1}, t_{n-1} \rangle \langle q_{n-1}, t_{n-1} | q_{n-2}, t_{n-2} \rangle \cdots \langle q_1, t_1 | q, t \rangle$$

$$t_k = t + k\epsilon, \quad t_0 \equiv t, \quad t_n \equiv t'$$

Inserting the completeness in p-space

$$\langle q_{k+1}, t_k + \epsilon | q_k, t_k \rangle = \int dp \langle q_{k+1}, t_k + \epsilon \, | p, \tilde{t} \rangle \langle p, \tilde{t} | q_k, t_k \rangle$$

$$t_k \leq \tilde{t} \leq t_{k+1}$$

$$\langle q_{k+1}, t_k + \epsilon | p, \tilde{t} \rangle \simeq \langle q_{k+1}, 0 | e^{-iH(\mathrm{p},\mathrm{q})(t_k + \epsilon - \tilde{t})} | p, 0 \rangle$$

Ordering p and q inside H to the right and to the left respectively and denoting the result by H_+

$$\langle q_{k+1}, t_k + \epsilon | p, \tilde{t} \rangle \simeq \langle q_{k+1}, 0 | p, 0 \rangle e^{-iH_+(p,q)(t_k + \epsilon - \tilde{t})}$$

we get, using $t_{k+1} = t_k + \epsilon$

$$\langle q_{k+1}, t_k + \epsilon | p, \tilde{t} \rangle \simeq \frac{1}{\sqrt{2\pi}} e^{i(p q_{k+1} - H_+(q_{k+1}, p)(t_{k+1} - \tilde{t}))}$$

In analogous way

$$\langle p,\tilde{t}|q_k,t_k\rangle \simeq \frac{1}{\sqrt{2\pi}} e^{i(-pq_k - H_-(q_k,p)(\tilde{t}-t_k))}$$

where H_- is defined bringing p and q to the left and to the right respectively. Putting everything together and defining

$$\tilde{t} = \frac{t_{k+1}+t_k}{2}$$

$$\langle q_{k+1},t_k+\epsilon|q_k,t\rangle \simeq \int \frac{dp_k}{2\pi} e^{i\epsilon(p(q_{k+1}-q_k)/\epsilon - H_c(q_{k+1},q_k,p))}$$

$$H_c(q_{k+1},q_k,p) = \frac{1}{2}(H_+(q_{k+1},p)+H_-(q_k,p))$$

Using $$\langle q_{k+1},t_k+\epsilon|q_k,t\rangle \xrightarrow[\epsilon\to 0]{} \delta(q_{k+1}-q_k)$$

We define the "velocity"

$$\dot{q}_k = \frac{q_{k+1} - q_k}{\epsilon}$$

and in the same limit we get

$$H_c(q_{k+1}, q_k, p) \equiv H(q_k, p)$$

which is nothing but the Hamiltonian in phase space, and

$$\langle q_{k+1}, t_k + \epsilon | q_k, t \rangle \simeq \int \frac{dp}{2\pi} e^{i\epsilon L(q_k, p)}, \quad L(q_k, p) = p\dot{q}_k - H(q_k, p)$$

the Dirac's result.

For a finite time interval

$$\langle q',t'|q,t\rangle = \lim_{n\to\infty}\int\left(\prod_{k=1}^{n-1} dq_k\right)\left(\prod_{k=0}^{n-1}\frac{dp_k}{2\pi}\right)e^{i\sum_{k=0}^{n-1}\epsilon L(q_k,p_k)}$$

or, in a more symbolic way

$$\langle q',t'|q,t\rangle = \int_{q,t}^{q',t'} d\mu(q(t))d\mu(p(t))e^{i\int_t^{t'} dt L(q,p)},$$

$$d\mu(q(t)) = \prod_{t<t''<t'} dq(t''),\quad d\mu(p(t)) = \prod_{t\le t''\le t'}\frac{dp(t'')}{2\pi}$$

assuming

$$H(q,p) = \frac{p^2}{2m} + V(q)$$

one can perform trivially the integration over the momenta with the result

$$\langle q',t'|q,t\rangle = \lim_{n\to\infty}\int\left(\sqrt{\frac{m}{2i\pi\epsilon}}\right)^{n}\left(\prod_{k=1}^{n-1}dq_k\right)\times$$

$$\times e^{i\epsilon\sum_{k=0}^{n-1}[m((q_{k+1}-q_k)/\epsilon)^2/2-V(q_k)]}$$

Or, in a more symbolic way

$$\langle q',t'|q,t\rangle = \int_{q,t}^{q',t'} D(q(t))e^{i\int_t^{t'} dtL(q,\dot{q})}, \quad D(q(t)) = \lim_{n\to\infty}\left(\sqrt{\frac{m}{2i\pi\epsilon}}\right)^{n}\prod_{k=1}^{n-1}dq_k$$

Functional Formalism

- The mathematical setting for path integration is the theory of functionals.

- Function = mapping from a manifold M to R^1

- Real functional = mapping from a space of functions (infinite dimensional space) to R^1

- In other words, a real functional associate a real number to a real function. If we denote the space of functions by L, a real functional is the mapping

$$F: \quad L \to R^1$$

To denote a functional we will make use of the following notations

$$F[\eta], \quad F, \quad F[\cdot], \qquad\qquad \eta \in L$$

Examples of functionals:

1) L^1 the space of the integrable functions in R^1 w. r. t. the measure w(x) dx

$$F_1[\eta] = \int_{-\infty}^{+\infty} dx\, w(x)\, \eta(x)$$

2) L^2 the space of the square integrable functions in R^1

$$F_2[\eta] = e^{-\frac{1}{2}\int dx \eta^2(x)}$$

The variation of F is then given by

$$F(\eta_i + \delta\eta_i) - F(\eta_i) = \sum_{j=1}^{n} \frac{\partial F(\eta_i)}{\partial \eta_j} \delta\eta_j$$

Defining

$$K_i = \frac{1}{\epsilon} \frac{\partial F}{\partial \eta_i}$$

we get

$$F(\eta_i + \delta\eta_i) - F(\eta_i) = \sum_{j=1}^{n} K_j \delta\eta_j \epsilon \xrightarrow[n\to\infty]{} \int K(x) \delta\eta(x) dx,$$

$$\delta F[\eta] = \int K(x) \delta\eta(x) dx$$

$$\delta F[\eta] = \int \frac{\delta F}{\delta \eta(x)} \delta\eta(x) dx, \quad \text{or,} \quad \frac{\delta F[\eta]}{\delta \eta(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \frac{\partial F(\eta_i)}{\partial \eta_i}$$

Generalization of the Taylor expansion (Volterra's series

$$F[\eta_0 + \eta_1] = \sum_{k=0}^{\infty} \frac{1}{k!} \int dx_1 \cdots dx_k \left. \frac{\delta^k F[\eta]}{\delta\eta(x_1) \cdots \delta\eta(x_k)} \right|_{\eta=\eta_0} \eta_1(x_1) \cdots \eta_1(x_k)$$

General properties of path integral

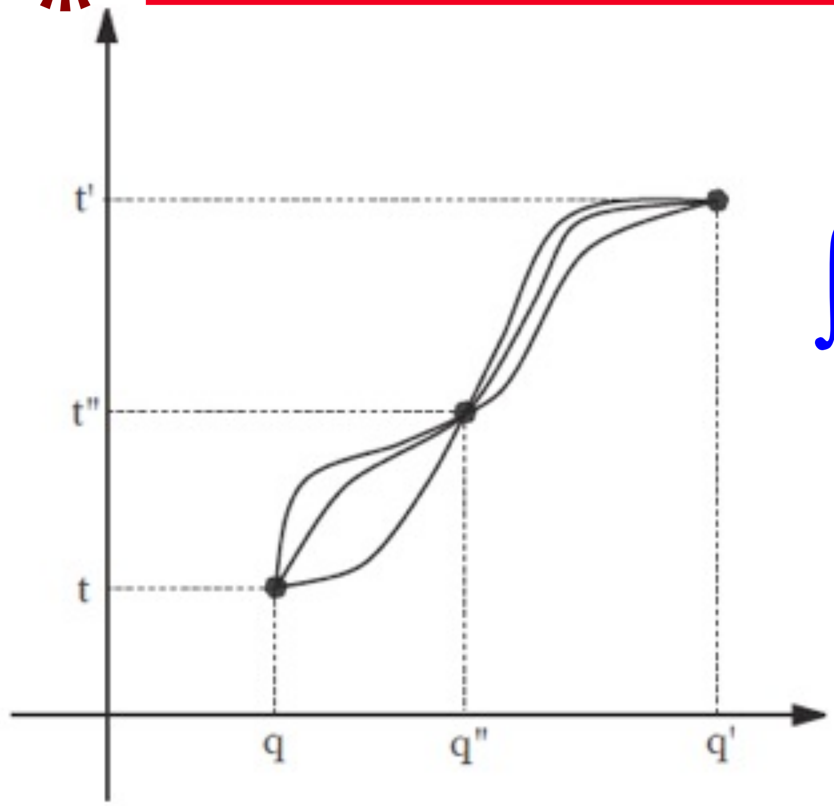
Invariance of the functional measure under the following translations

$$q(\tau)\to q(\tau)+\eta(\tau), \qquad \eta(t)=\eta(t')=0$$

$$p(\tau)\to p(\tau)+\pi(\tau), \quad \text{for any } \pi(t), \pi(t')$$

$$\begin{cases} d\mu(q(t)) = \prod_{t<\tau<t'} dq(\tau), \\ d\mu(p(t)) = \prod_{t\le\tau\le t''} dp(\tau) \end{cases}$$

Factorization of the path integral (completeness)



$$\langle q',t'|q,t\rangle = \int_{q,t}^{q't'} d\mu(q)d\mu(p)e^{i\int_t^{t''}Ld\tau + i\int_{t''}^{t'}Ld\tau} =$$

$$\int dq''\int_{q,t}^{q'',t''} d\mu(q)d\mu(p)e^{i\int_t^{t''}Ld\tau}\int_{q'',t''}^{q',t'} d\mu(q)d\mu(p)e^{i\int_{t''}^{t'}Ld\tau} =$$

$$= \int dq''\langle q',t'|q'',t''\rangle\langle q'',t''|q,t\rangle$$

$$d\mu(q) = \prod_{t<\tau<t'} dq(\tau) = dq''\prod_{t<\tau<t''} dq(\tau)\prod_{t''<\tau<t'} dq(\tau),$$

$$d\mu(p) = \prod_{t\le\tau\le t'} dp(\tau) = \prod_{t\le\tau\le t''} dp(\tau)\prod_{t''\le\tau\le t'} dp(\tau)$$

Using these two properties it is possible to derive all the quantum mechanical formalism. Now let us show how to evaluate average values of the observables. Let us begin with

$$\int_{q,t}^{q',t'} d\mu(q)\, d\mu(p)\, q(t'')e^{iS}, \quad t \le t'' \le t'$$

Using

$$\int_{q,t}^{q',t'} d\mu(q)\, d\mu(p)\, q(t')e^{iS} = q' \int_{q,t}^{q',t'} d\mu(q)\, d\mu(p)\, e^{iS} = q' \langle q',t'|q,t\rangle$$

we get

$$\int_{q,t}^{q',t'} d\mu(q)\, d\mu(p)\, q(t'')e^{iS} =$$

$$= \int dq'' \int_{q,t}^{q'',t''} d\mu(q)\, d\mu(p)\, q(t'')e^{iS} \int_{q'',t''}^{q',t'} d\mu(q)\, d\mu(p)\, e^{iS} =$$

$$= \int dq'' \langle q',t'|q'',t''\rangle q'' \langle q'',t''|q,t\rangle = \langle q',t'|\mathrm{q}(t'')|q,t\rangle$$

Now let us study the average value of $q(t_1)q(t_2)$ with

$$t \le t_1 \le t_2 \le t'$$

By proceeding as before we will get the expectation value of

$$\mathrm{q}(t_1)\mathrm{q}(t_2)$$

However, if $t \le t_2 \le t_1 \le t'$ we get the expectation value of

$$\mathrm{q}(t_2)\mathrm{q}(t_1)$$

In conclusion we obtain

$$\int_{q,t}^{q',t'} d\mu(q)\, d\mu(p)\, e^{iS} q(t_1) q(t_2) = \langle q',t' | T(\mathrm{q}(t_1)\mathrm{q}(t_2) | q,t \rangle$$

More generally

$$\int_{q,t}^{q',t'} d\mu(q)\, d\mu(p)\, e^{iS} q(t_1)\cdots q(t_n) = \langle q',t' | T(\mathrm{q}(t_1)\cdots \mathrm{q}(t_n)) | q,t \rangle$$

Considering a functional admitting a Volterra' series expansion

$$\int_{q,t}^{q',t'} d\mu(q)\, d\mu(p)\, e^{iS} F[q] = \langle q',t' | T(F[\mathrm{q}]) | q,t \rangle$$

where we have defined

$$\langle q',t' | T(F[\mathrm{q}]) | q,t \rangle \equiv$$

$$\equiv \sum_{k=0}^{\infty} \frac{1}{k!} \int dt_1 \cdots dt_k \frac{\delta^k F[q]}{\delta q(t_1)\cdots \delta q(t_k)} \Big|_{q=0} \langle q',t' | T(\mathrm{q}(t_1)\cdots \mathrm{q}(t_k)) | q,t \rangle$$

If the functional depends also on time derivatives a certain caution in defining the T-product is necessary. Consider, for instance, the product $q(t_1)\dot{q}(t_2)$

$$\int_{q,t}^{q',t'} d\mu(q)\,d\mu(p)\,e^{iS} q(t_1)\dot{q}(t_2) =$$

$$= \lim_{\epsilon\to 0} \int_{q,t}^{q',t'} d\mu(q)\,d\mu(p)\,e^{iS} q(t_1)\frac{q(t_2+\epsilon)-q(t_2)}{\epsilon} = \frac{d}{dt_2}\langle q',t'|T(\mathrm{q}(t_1)\mathrm{q}(t_2))|q,t\rangle$$

Defining the T*- product as

$$T^{\star}\big(\mathrm{q}(t_1)\dot{\mathrm{q}}(t_2)\big) = \frac{d}{dt_2} T\big(\mathrm{q}(t_1)\mathrm{q}(t_2)\big)$$

the general result reads

$$\int_{q,t}^{q',t'} d\mu(q)\,d\mu(p)\,F[q,\dot{q}]\,e^{iS} = \langle q',t'|T^{\star}(F[\mathrm{q},\dot{\mathrm{q}}])|q,t\rangle$$

Functional generator of Green's functions

As we have seen, in order to evaluate the S-matrix elements we need the T-products of fields (or Green functions). We start evaluating T-products in 1-dim QM, equivalent to a field theory in 0 space dim. The result is easily generalized to many or infinite degrees of freedom (field theory). The way will follow is to introduce a functional generating all the Green functions:

$$\langle q',t'|q,t\rangle_J = \int d\mu(q)\, d\mu(p)\, e^{iS_J}$$

$$S_J = \int_t^{t'} d\tau \left[p\dot{q} - H + Jq \right]$$

Differentiating the previous expression

$$\langle q',t'|q,t\rangle_J = \int d\mu(q)\,d\mu(p)\,e^{iS_J}$$

$$S_J = \int_t^{t'} d\tau\left[p\dot{q} - H + Jq\right]$$

we get

$$\langle q',t'|T(\mathrm{q}(t_1)\cdots\mathrm{q}(t_n))|q,t\rangle_{J=0} = \left(\frac{1}{i}\right)^n \frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}\langle q',t'|q,t\rangle_J\Big|_{J=0}$$

But what we really need is the matrix elements of the T-product in the fundamental state of the system (the vacuum in QFT)

To this end we consider the wave function of the fundamental state

$$\Phi_0(q,t)=\langle q,t|0\rangle=\langle q|e^{-iHt}|0\rangle=e^{-iE_0t}\langle q|0\rangle$$

then

$$\langle 0|T(\mathrm{q}(t_1)\cdots\mathrm{q}(t_n))|0\rangle=\int dq'dq\Phi_0^\star(q',t')\langle q',t'|T(\mathrm{q}(t_1)\cdots\mathrm{q}(t_n))|q,t\rangle\Phi_0(q,t)$$

Let us now define the generating functional of the vacuum amplitudes

$$\boxed{Z[J]=\int dq'dq\Phi_0^\star(q',t')\langle q',t'|q,t\rangle_J\Phi_0(q,t)=\langle 0|0\rangle_J}$$

Clearly

$$\langle 0|T(\mathrm{q}(t_1)\cdots\mathrm{q}(t_n))|0\rangle=\left(\frac{1}{i}\right)^n\frac{\delta^n}{\delta J(t_1)\cdots\delta J(t_n)}Z[J]\bigg|_{J=0}$$

Now we will show that the following relation holds for any value of q and q’ and with the ground state wave functions evaluated at t = 0.

$$Z[J] = \lim_{t\to+i\infty,\ t'\to-i\infty} \frac{e^{iE_0(t'-t)}}{\Phi_0^\star(q)\Phi_0(q')} \langle q',t'|q,t\rangle_J$$

To this end consider an external source J(t) vanishing outside the interval (t'',t''') with t'>t'''>t''>t. Then, we can write

$$\langle q',t'|q,t\rangle_J = \int dq'' dq''' \langle q',t'|q''',t'''\rangle \langle q''',t'''|q'',t''\rangle_J \langle q'',t''|q,t\rangle$$

and using

$$\langle q',t'|q''',t'''\rangle = \langle q'|e^{-iH(t'-t''')}|q'''\rangle = \sum_n \Phi_n(q')\Phi_n^\star(q''')e^{-iE_n(t'-t''')}$$

we get

$$\lim_{t'\to -i\infty} e^{iE_0 t'}\langle q',t'|q''',t'''\rangle = \lim_{t'\to -i\infty}\sum_n \Phi_n(q')\Phi_n^\star(q''')e^{iE_n t'''}e^{-i(E_n-E_0)t'}$$

$$= \Phi_0(q')\Phi_0^\star(q''')e^{iE_0 t'''} = \Phi_0^\star(q''',t''')\Phi_0(q')$$

and, in analogous way

$$\lim_{t\to +i\infty} e^{-iE_0 t}\langle q'',t''|q,t\rangle = \Phi_0(q'',t'')\Phi_0^\star(q)$$

proving that

$$\lim_{t\to +i\infty, t'\to -\infty} e^{iE_0(t'-t)}\langle q',t'|q,t\rangle_J =$$

$$= \int dq''dq'''\Phi_0^\star(q)\Phi_0(q')\Phi_0^\star(q''',t''')\langle q''',t'''|q'',t''\rangle_J \Phi_0(q'',t'') =$$

$$= \Phi_0^\star(q)\Phi_0(q')Z[J]$$

Notice again that the values q and q' are arbitrary, and that the coefficient in front of the matrix element is J-independent. Therefore it is physically irrelevant and we will consider the ratio

$$\frac{\langle 0|T(q(t_1)\cdots q(t_n))|0\rangle}{\langle 0|0\rangle} = \left(\frac{1}{i}\right)^n \frac{1}{Z[0]} \frac{\delta^n}{\delta J(t_1)\cdots \delta J(t_n)} Z[J]\Big|_{J=0}$$

which does not depend on the normalization and, assuming to be able to integrate on the momenta

$$Z[J] = \lim_{t\to +i\infty, t'\to -i\infty} N \int_{q,t}^{q',t'} d\mu(q) e^{i\int_t^{t'}(L+Jq)dt}$$

where the normalization factor does not depend on J. Furthermore the typical choice is to take q = q' = 0.

This expression suggests the definition of an euclidean generating functional $Z_E[J]$. This is obtained by introducing an euclidean time (Wick's rotation) $\tau = it$

$$Z_E[J] = \lim_{\tau' \to +\infty, \tau \to -\infty} N \int_{q,\tau}^{q',\tau'} d\mu(q) e^{-\int_\tau^{\tau'} (L_E(q, \frac{dq}{d\tau}) + Jq) d\tau}$$

where

$$L_E = \frac{1}{2} m \left(\frac{dq}{d\tau} \right)^2 + V(q)$$

more generally

$$Z_E[J] = \lim_{\tau' \to +\infty, \tau \to -\infty} N \int_{q,\tau}^{q',\tau'} d\mu(q) e^{-S_E + \int_\tau^{\tau'} Jq d\tau}$$

If exp{-S_E} is a positive definite functional, the path integral is well defined and convergent the vacuum expectation values of the operators q(t) can be evaluated by using Z_E and continuing the result for real times.

$$\frac{1}{Z[0]} \frac{\delta^n Z[J]}{\delta J(t_1) \cdots \delta J(t_n)} \Bigg|_{J=0} = (i)^n \frac{1}{Z_E[0]} \frac{\delta^n Z_E[J]}{\delta J(\tau_1) \cdots \delta J(\tau_n)} \Bigg|_{J=0, \tau_i = it_i}$$

Z[J] for the harmonic oscillator

We will evaluate Z[J] for the harmonic oscillator starting from the expression

$$\langle 0|0\rangle_J = Z[J] = \int \Phi_0^\star(q',t') \underbrace{\int d\mu(q) e^{iS_J}}_{\langle q',t'|q,t\rangle_J} \Phi_0(q,t)\, dq\, dq'$$

where

$$\Phi_0(q,t) = \left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}m\omega q^2} e^{-\frac{i}{2}\omega t}, \quad \Phi_0^\star(q',t') = \left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}m\omega q'^2} e^{\frac{i}{2}\omega t'},$$

$$S_J = \int_t^{t'} \left[\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 + Jq\right] d\tau$$

neglecting, for the moment being the time dependence, the argument of the exponential is

$$\arg = -\frac{1}{2}m\omega(q^2 + q'^2) + i\int_t^{t'} \left[\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 + Jq\right] d\tau$$

By changing variable $q(\tau) = x(\tau) + x_0(\tau)$

$$\arg = -\frac{1}{2} m\omega [x(t)^2 + x^2(t')] -$$

$$-\frac{1}{2} m\omega [x_0(t)^2 + x_0^2(t')] - \underbrace{m\omega [x(t)x_0(t) + x(t')x_0(t')]}_{\text{mixed term}} +$$

$$+i\int_t^{t'} \left[\frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2 \right] d\tau + i\int_t^{t'} \left[\frac{1}{2} m\dot{x}_0^2 - \frac{1}{2} m\omega^2 x_0^2 + Jx_0 \right] d\tau +$$

$$+i\int_t^{t'} \left[m\dot{x}\dot{x}_0 - m\omega^2 x x_0 + Jx \right] d\tau$$

Integrate by part last term and choose x_0 solution of

$$m\ddot{x}_0 + m\omega^2 x_0 = J \quad \Rightarrow \quad im[x\dot{x}_0]_t^{t'}$$

and with the boundar conditions

$$i\dot{x}_0(t') = \omega x_0(t'), \quad i\dot{x}_0(t) = -\omega x_0(t)$$

it cancels the mixed term, and we are left with

$$\arg = -\frac{1}{2}m\omega[x^2(t) + x^2(t')] - \frac{1}{2}m\omega[x_0^2(t) + x_0^2(t')] +$$
$$+i\int_t^{t'}[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2]d\tau + i\int_t^{t'}[\frac{1}{2}m\dot{x}_0^2 - \frac{1}{2}m\omega^2 x_0^2 + Jx_0]d\tau$$

Integrating by parts the last term and using again the b.c.'s

$$\arg = -\frac{1}{2}m\omega[x^2(t) + x^2(t')] + i\int_t^{t'}[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2]d\tau + \frac{i}{2}\int_t^{t'} Jx_0 d\tau$$

The first two terms do not depend on J and contribute to Z[0]

Therefore we can write

$$Z[J] = e^{\frac{i}{2}\int_t^{t'} J(\tau)x_0(\tau)d\tau} Z[0]$$

Notice that x_0 depends on J since it satisfies the classical eqs. of motion in the presence of J

$$m\ddot{x}_0 + m\omega^2 x_0 = J$$

We can solve this equation with the b.c.'s

$$i\dot{x}_0(t') = \omega x_0(t'), \quad i\dot{x}_0(t) = -\omega x_0(t)$$

by defining

$$x_0(\tau) = i\int_t^{t'} \Delta(\tau - s)J(s)ds$$

$$\left[\frac{d^2}{d\tau^2} + \omega^2\right]\Delta(\tau - s) = -\frac{i}{m}\delta(\tau - s),$$

$$i\dot{\Delta}(t' - s) = \omega\Delta(t' - s), \quad i\dot{\Delta}(t - s) = -\omega\Delta(t - s)$$

The equation can be easily solved by considering t' > s and t < s, We get

$$\Delta(\tau) = Ae^{-i\omega\tau}, \quad \tau > 0, \quad \Delta(\tau) = Be^{+i\omega\tau}, \quad \tau < 0$$

or

$$\Delta(\tau) = A\theta(\tau)e^{-i\omega\tau} + B\theta(-\tau)e^{+i\omega\tau}$$

A and B are fixed by the b.c.'s

$$\boxed{\Delta(\tau) = \frac{1}{2\omega m}\left[\theta(\tau)e^{-i\omega\tau} + \theta(-\tau)e^{+i\omega\tau}\right]}$$

and the final result is

$$\boxed{Z[J] = e^{-\frac{1}{2}\int_t^{t'} ds ds' J(s)\Delta(s-s')J(s')} Z[0]}$$

The VEV of two q operators is then given by

$$\Delta(s-s') = -\frac{1}{Z[0]}\frac{\delta^2 Z[J]}{\delta J(s)\delta J(s')}\bigg|_{J=0} = \frac{\langle 0|T(\mathrm{q}(s)\mathrm{q}(s'))|0\rangle}{\langle 0|0\rangle}$$

We could have done the same calculation in the euclidean space. The result would have been

$$Z_E[J] \approx e^{-S_E^{(cl)} + \int_{-\infty}^{+\infty} Jq\,d\tau} Z_E[0],$$

$$S_E^{(cl)} = \frac{1}{2}\left(m\dot{q}^2 + m\omega^2 q^2\right)$$

with q satisfying the classical euclidean equations of motion in presence of the source J

$$\ddot{q} - \omega^2 q = -\frac{1}{m}J$$

and witn b.c.'s $q \to 0, \text{for } \tau \to \infty$

We define the euclidean Green function

$$\left[\frac{d^2}{d\tau^2} - \omega^2\right] D_E(\tau) = -\frac{1}{m}\delta(\tau), \quad \lim_{\tau\to\pm\infty} D_E(\tau) = 0$$

then

$$q(\tau) = \int_{-\infty}^{+\infty} D_E(\tau - s) J(s) ds$$

Going to the F.T.

$$D_E(\tau) = \int d\nu \; e^{-i\nu\tau} D_E(\nu)$$

we find

$$D_E(\nu) = \frac{1}{2\pi m}\frac{1}{\nu^2 + \omega^2}, \quad D_E(\tau) = \frac{1}{2\pi m}\int_{-\infty}^{+\infty} d\nu \frac{1}{\nu^2 + \omega^2} e^{-i\nu\tau}$$

To evaluate this integral we close it with the circle at infinity in the lower plane for positive euclidean time

$$D_E(\tau) = \frac{1}{2\pi m}\frac{e^{-\omega\tau}}{(-2i\omega)}(-2\pi i) = \frac{1}{2m\omega}e^{-\omega\tau}$$

By operating in the upper plane for negative euclidean time we get the final result

$$D_E(\tau) = \frac{1}{2m\omega}e^{-\omega|\tau|}$$

and

$$Z_E[J] = e^{\frac{1}{2}\int_{-\infty}^{+\infty} J(s)D_E(s-s')J(s')ds ds'} Z_E[0]$$

we see that

$$\frac{1}{Z_E[0]}\frac{\delta^2 Z_E[J]}{\delta J(s)\delta J(s')}\Big|_{J=0} = D_E(s-s')$$

Using the relation between the euclidean functional and Z[J] one finds

$$\Delta(t) = D_E(it)$$

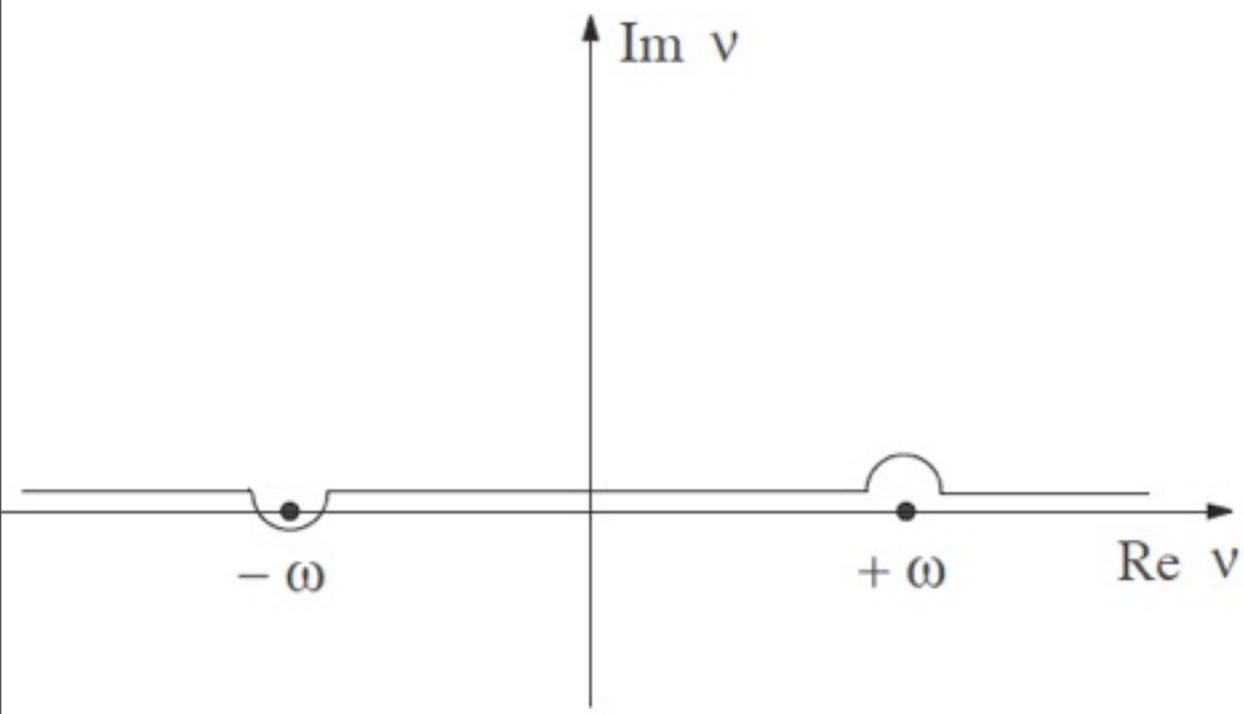
To compare the two expressions take the FT

$$\Delta(t) = \int d\nu e^{-i\nu t} \Delta(\nu),$$

$$(-\nu^2 + \omega^2)\Delta(\nu) = -\frac{i}{2\pi m}, \quad \Delta(\nu) = \frac{i}{2\pi m} \frac{1}{\nu^2 - \omega^2}$$

whereas there is no problem in the euclidean case, we see that this expression is singular and the integral should be defined properly

In fact, to reproduce the euclidean result we need to choose the path as follows



Or, in equivalent way we can do the integration along the real axis with the following prescription

$$\Delta(t) = \lim_{\epsilon\to 0^+} \frac{i}{2\pi m}\int_{-\infty}^{+\infty} \frac{e^{-i\nu t}}{\nu^2-\omega^2+i\epsilon}d\nu$$

With this prescription we can safely rotate the contour of integration anticlockwise getting

$$\Delta(t) = \frac{i}{2\pi m}\int_{-i\infty}^{+i\infty} \frac{e^{-i\nu t}}{\nu^2-\omega^2}d\nu$$

Changing variable

$$\nu = i\nu'$$

$$\Delta(t) = -\frac{1}{2\pi m}\int_{-\infty}^{+\infty} \frac{e^{+\nu' t}}{-\nu'^2-\omega^2}d\nu' = D_E(it)$$

Notice that the representation

$$\Delta(t) = \lim_{\epsilon \to 0^+} \frac{i}{2\pi m} \int_{-\infty}^{+\infty} \frac{e^{-i\nu t}}{\nu^2 - \omega^2 + i\epsilon} d\nu$$

can be obtained by the substitution

$$\omega^2 \to \omega^2 - i\epsilon$$

in the path integral. That is

$$e^{-\frac{i}{2} m\omega^2 \int q^2(t)dt} \to e^{-\frac{i}{2} m(\omega^2 - i\epsilon)\int q^2(t)dt}$$

We see that this adds to the path integral the convergence factor

$$\boxed{e^{-\frac{1}{2} m\epsilon \int q^2(t)dt}}$$