Lecture 2

★ LSZ formalism and S-matrix

★ Path integral for scalar fields

★ Perturbative expansion and Feynman's rules

LSZ formalism

We will need a formulation of field theory showing that any element of the scattering matrix (S-matrix) can be reduced to the evaluation of VEV (vacuum expectation value) of a T-product of field operators. Such a formulation is called LSZ (Lehmann, Symanzik and Zimmermann).
We start using the Heisenberg representation and we will assume:

❖ The eigenvalues of the four-momentum lie within the forward light cone

$$p^2 = p^\mu p_\mu \geq 0, \qquad p^0 \geq 0$$

❖ There exists a non degenerate Lorentz invariant state of minimum energy (vacuum state)

$$\Phi_0 \equiv |0\rangle, \quad P_\mu |0\rangle = 0$$

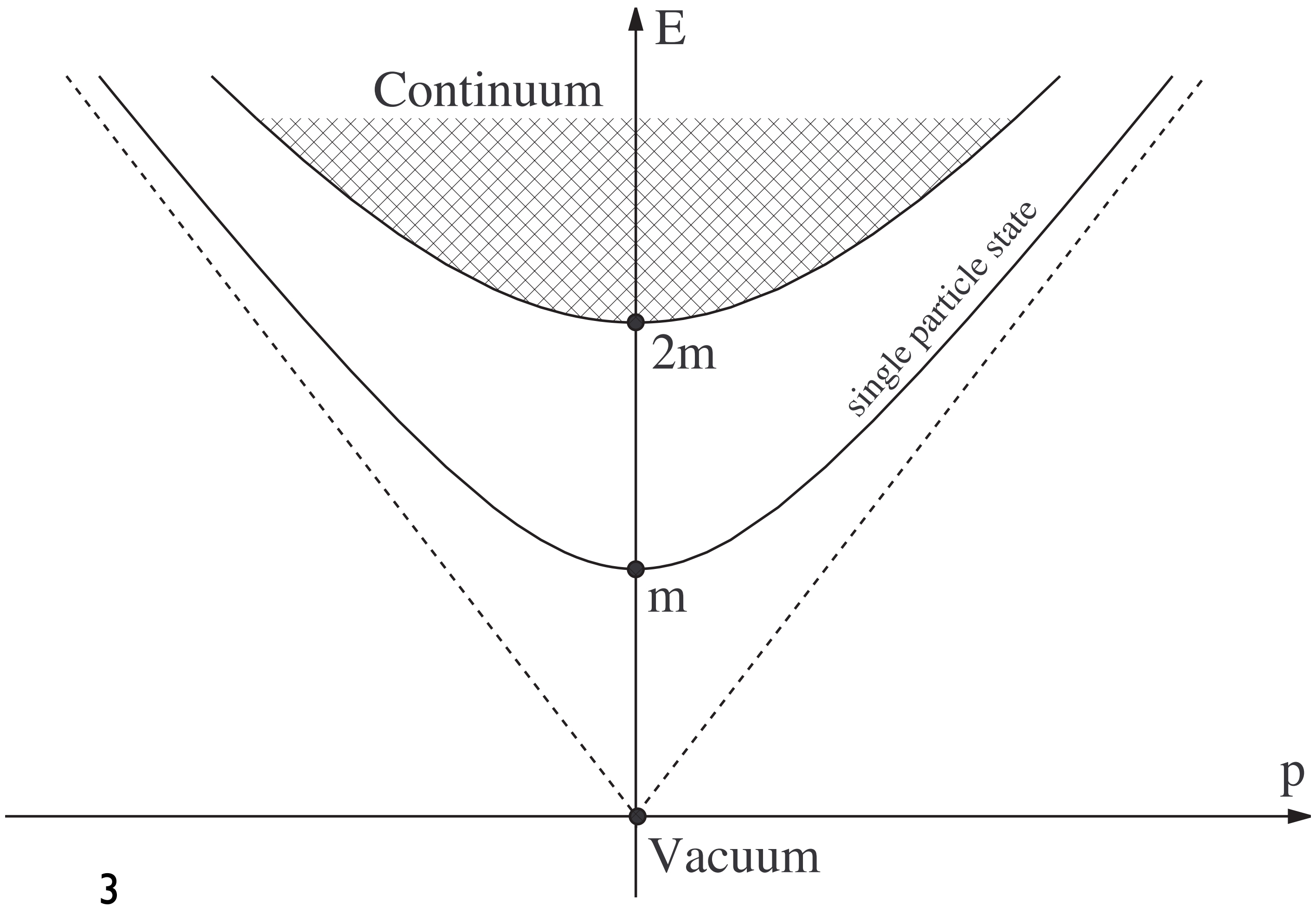
❖ For each stable particle of mass m, there exists a stable single particle state

$$\Phi_1 \equiv | p \rangle$$

eigenstate of momentum with

$$p^2 = m^2$$

❖ Except for $p^2 = 0$, the mass spectrum is discrete (vacuum and one particle state) + a continuum



The main tool the scattering experiments.

In a very sketchy way we consider a system of free particles at $t = -\infty$. These particles interact with a target at some finite time. After the scattering has taken place the emerging particles behave again as free particles at $t=+\infty$.

In real life we prepare the beam at some finite time - T, the scattering process occurs around some finite time (say $t = 0$) and we detect and measure some property of the outgoing particles at a time T. This ideal description considered before can be considered as correct if the interaction time is much smaller than T.

Imagine an electron prepared in an eigenstate of momentum p, impinging over an atom. Notice that to prepare an eigenstate of momentum in a finite volume is an ideal process.

For this reasons in order to analyze a scattering process it is convenient to quantize in a box of the dimension of the experiment itself. The same consideration holds for the energy. One has always to remember that the experiment is taking place in a finite interval of time (-T, +T).

Therefore we describe the particles at $|t| = +\infty$ in terms of free fields, except that they will be subject to self-interaction (that cannot be neglected). These fields will be denoted by

$$\phi_{\text{in}}(x), \text{for } t \to -\infty, \quad \phi_{\text{out}}(x), \text{for } t \to +\infty$$

The interacting field $\phi(x)$ can be thought of as constructed in terms of the free fields operators and we will assume that the interacting field reduces to the in-out fields for $t \to \mp\infty$
We will require the following properties for the in-out fields.

1) The in (out) fields should transform as the interpolating field w.r. t. the symmetries of the theory. For instance:

$$[\mathrm{P}_\mu, \phi_{\mathrm{in}}(x)] = -i \frac{\partial \phi_{\mathrm{in}}(x)}{\partial x^\mu}$$

2) The in (out) fields should satisfy the KG equation (or the appropriate wave equation for other fields)

$$(\Box + m^2)\phi_{\mathrm{in}}(x) = 0$$

The in (out) field creates the physical one-particle state from the vacuum:

$$-i\frac{\partial}{\partial x^\mu}\langle p|\phi_{\text{in}}(x)|0\rangle = \langle p|[\mathrm{P}_\mu,\phi_{\text{in}}(x)]|0\rangle = p_\mu\langle p|\phi_{\text{in}}(x)|0\rangle$$

Iterating

$$-\Box\langle p|\phi_{\text{in}}(x)|0\rangle = p^2\langle p|\phi_{\text{in}}(x)|0\rangle$$

and using the KG equation

$$(p^2-m^2)\langle p|\phi_{\text{in}}(x)|0\rangle = 0$$

The in (out) fields are free ones, therefore we can apply the standard formalism, e.g.

$$\phi_{\text{in}}(x) = \int d^3\vec{k}\left[a_{\text{in}}(\vec{k})f_{\vec{k}}(x) + a^\dagger_{\text{in}}(\vec{k})f^*_{\vec{k}}(x)\right] \quad \begin{cases} f_{\vec{k}}(x) = \dfrac{1}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}}e^{-ikx}, \\ \omega_{\vec{k}} = \sqrt{\vec{k}^2+m^2} \end{cases}$$

We can derive the creation (annihilation) operators inverting the previous relation

$$a_{\text{in}}(\vec{k}) = i\int d^3\vec{x}\left[f_{\vec{k}}^*(x)\partial_0\phi_{\text{in}}(x) - \partial_0 f_{\vec{k}}^*(x)\phi_{\text{in}}(x)\right]$$

Notice that for any two solutions of the KG equation, f_1 and f_2

$$Q = \int (f_1\partial_0 f_2 - f_2\partial_0 f_1)$$

is a conserved charge. One can prove easily that

$$[\mathrm{P}_\mu, a_{\text{in}}(\vec{k})] = -k_\mu a_{\text{in}}(\vec{k}), \quad [\mathrm{P}_\mu, a_{\text{in}}^\dagger)] = k_\mu a_{\text{in}}^\dagger(\vec{k})$$

and from this that $a_{in}(k)$ and its h.c. destroy and create respectively states with fourmomentum k ($k^2 = m^2$) from the vacuum.

The interpolating field satisfies a KG equation describing the interaction with a source j(x) (including also the self-interactions)

$$(\Box + m^2)\phi(x) = j(x)$$

★ We need to solve this equation with the boundary conditions at $|t| = \infty$ relating the interpolating field with the in-out fields.

★ However asymptotically the interpolating field cannot be the same as the in-out fields but it differs by a factor $Z^{1/2}$, where Z is the so called wave function renormalization. To understand this point consider the following matrix elements

$$\langle k|\phi_{in}(x)|0\rangle \quad \text{and} \quad \langle k|\phi(x)|0\rangle$$

Both matrix elements are proportional to exp(ikx) due to translational invariance and therefore they may differ by a constant. This cannot be one, because the in field creates only the single particle state, whereas the interpolating field, in general, creates also multiparticle states.

Therefore we will require

$$\lim_{t\to-\infty}\phi(x)=\lim_{t\to-\infty}\sqrt{Z}\phi_{\text{in}}(x),\quad \lim_{t\to+\infty}\phi(x)=\lim_{t\to+\infty}\sqrt{Z}\phi_{\text{out}}(x)$$

To be rigorous these relations are valid only in a weak sense, that is not among operators but rather among their matrix elements between any two normalizable states

The S matrix

Let us consider a scattering process. The initial state will be described by

$$|\, p_1, \cdots, p_n; \mathrm{in}\rangle = a_{in}^{\dagger}(p_1)\cdots a_{in}^{\dagger}(p_n)\,|\,0;\mathrm{in}\rangle \equiv |\,\alpha;\mathrm{in}\rangle$$

and the final state by

$$|\, p'_1, \cdots, p'_n; \mathrm{out}\rangle = a_{out}^{\dagger}(p'_1)\ldots a_{out}^{\dagger}(p'_n)\,|\,0;\mathrm{in}\rangle \equiv |\,\alpha;\mathrm{out}\rangle$$

The corresponding probability amplitude is called the matrix element of the S-matrix

$$\boxed{S_{\beta\alpha} = \langle \beta;\mathrm{out}|\alpha;\mathrm{in}\rangle}$$

or, defining $\boxed{\langle \beta;\mathrm{in}\,|\,S = \langle \beta;\mathrm{out}\,|}$

equivalent to

$$\boxed{S_{\beta\alpha} = \langle \beta; \mathrm{in} | S | \alpha; \mathrm{in} \rangle}$$

Properties of the S-matrix

1) Since the vacuum is stable and non degenerate, follows S_{00} = 1, implying $\langle 0, \mathrm{in} | S = \langle 0, \mathrm{out} | = e^{i\theta} \langle 0, \mathrm{in} |$

2) For the same reason $| p; \mathrm{in} \rangle = | p; \mathrm{out} \rangle$

3) The S-matrix maps in and out fields $\phi_{\mathrm{in}}(x) = S \phi_{\mathrm{out}}(x) S^{-1}$

4) The S-matrix is unitary (probability conservation) $SS^{\dagger} = 1$

5) The S-matrix is invariant under the symmetry properties of the theory (out and in states possess the same symmetries)

The reduction formalism

This formalism allows to evaluate the S-matrix elements in terms of VEV's of T-products of fields. Consider an in state composed by a set of particles + a single particle state of momentum p

$$S_{\beta,\alpha p} = \langle \beta;\mathrm{out} | \alpha, p; \mathrm{in} \rangle$$

The single particle can be "extracted" from the in state

$$\langle \beta;\mathrm{out} | \alpha, p; \mathrm{in} \rangle = \langle \beta;\mathrm{out} | a^{\dagger}_{in}(p) | \alpha; \mathrm{in} \rangle =$$

$$= \langle \beta;\mathrm{out} | a^{\dagger}_{out}(p) | \alpha; \mathrm{in} \rangle + \langle \beta;\mathrm{out} | \mathrm{a}^{\dagger}_{in}(p) - \mathrm{a}^{\dagger}_{out}(p) | \alpha; \mathrm{in} \rangle =$$

$$= \langle \beta - p;\mathrm{out} | \alpha; \mathrm{in} \rangle -$$

conserved charge

$$-i \int d^3\vec{x} \Big[f_{\vec{p}}(x) \Big(\partial_0 \langle \beta;\mathrm{out} | \phi_{in}(x) - \phi_{out}(x) | \alpha; \mathrm{in} \rangle \Big) -$$

$$- \Big(\partial_0 f_{\vec{p}}(x) \Big) \langle \beta;\mathrm{out} | \phi_{in}(x) - \phi_{out}(x) | \alpha; \mathrm{in} \rangle \Big]$$

The red expression in the previous slide can be evaluated at any time and we can use the asymptotic conditions

$$\lim_{x^0\to-\infty}\phi_{in}(x)=\lim_{x^0\to-\infty}\frac{1}{\sqrt{Z}}\phi(x),\quad \lim_{x^0\to+\infty}\phi_{out}(x)=\lim_{x^0\to+\infty}\frac{1}{\sqrt{Z}}\phi(x)$$

obtaining

$$\langle\beta;\mathrm{out}|\alpha,p;\mathrm{in}\rangle=\langle\beta-p;\mathrm{out}|\alpha;\mathrm{in}\rangle+$$

$$+\frac{i}{\sqrt{Z}}\left(\lim_{x^0\to+\infty}-\lim_{x^0\to-\infty}\right)\int d^3\vec{x}\Big[f_{\vec{p}}(x)\big(\partial_0\langle\beta;\mathrm{out}|\phi(x)|\alpha;\mathrm{in}\rangle\big)-$$

$$-\big(\partial_0 f_{\vec{p}}(x)\big)\langle\beta;\mathrm{out}|\phi(x)|\alpha;\mathrm{in}\rangle\Big]$$

The 3-dim integral can be written in terms of a 4-dim one, using the identity

$$I=\left(\lim_{x^0\to+\infty}-\lim_{x^0\to-\infty}\right)\int d^3\vec{x}\left[g_1\partial_0 g_2-g_2\partial_0 g_1\right]=$$

$$=\int_{-\infty}^{+\infty}d^4x\partial_0\left[g_1\partial_0 g_2-g_2\partial_0 g_1\right]=\int_{-\infty}^{+\infty}d^4x\left[g_1(x)\ddot{g}_2(x)-\ddot{g}_1(x)g_2(x)\right]$$

obtaining

$$\langle\beta;\mathrm{out}|\alpha,p;\mathrm{in}\rangle=\langle\beta-p;\mathrm{out}|\alpha;\mathrm{in}\rangle+$$

$$\frac{i}{\sqrt{Z}}\int d^4x\langle\beta;\mathrm{out}|\left[f_{\vec{p}}(x)\ddot{\phi}(x)-\ddot{f}_{\vec{p}}(x)\phi(x)\right]|\alpha;\mathrm{in}\rangle$$

using KG for f_p and integrating by part

$$\boxed{\begin{aligned}&\langle\beta;\mathrm{out}|\alpha,p;\mathrm{in}\rangle=\langle\beta-p;\mathrm{out}|\alpha;\mathrm{in}\rangle+\\&+\frac{i}{\sqrt{Z}}\int d^4x f_{\vec{p}}(x)(\Box+m^2)\langle\beta;\mathrm{out}|\phi(x)|\alpha;\mathrm{in}\rangle\end{aligned}}$$

This procedure can be iterated. For instance, if the set $\beta = \gamma, p'$

$$\langle \beta;\text{out}|\alpha, p;\text{in}\rangle = \langle \beta - p;\text{out}|\alpha;\text{in}\rangle +$$

$$+\frac{i}{\sqrt{Z}}\int d^4x f_{\vec{p}}(x)(\Box+m^2)_x \langle \gamma;\text{out}|\phi(x)|\alpha - p';\text{in}\rangle +$$

$$+\left(\frac{i}{\sqrt{Z}}\right)^2 \int d^4x\, d^4y\, f_{\vec{p}}(x) f^*_{\vec{p}'}(y)(\Box+m^2)_x(\Box+m^2)_y \langle \gamma;\text{out}|T(\phi(y)\phi(x))|\alpha;\text{in}\rangle$$

$$\boxed{T(\phi(x)\phi)y)) = \theta(x_0 - y_0)\phi(x)\phi(y) + \theta(y_0 - x_0)\phi(y)\phi(x)}$$

More generally if all the final p's are different by the initial q's

$$\langle p_1,\cdots,p_m;\text{out}|q_1,\cdots,q_n;\text{in}\rangle =$$
$$= \left(\frac{i}{\sqrt{Z}}\right)^{m+n} \int \prod_{i,j=1}^{m,n} d^4x_i d^4y_j f_{\vec{q}_i}(x_i) f^*_{\vec{p}_j}(y_j) \cdot$$
$$\cdot(\Box+m^2)_{x_i}(\Box+m^2)_{y_j} \langle 0|T(\phi(y_1)\cdots\phi(y_n)\phi(x_1)\cdots\phi(x_m))|0\rangle$$

All the S-matrix elements can be evaluated in terms of VEV's of T-products

Analogous reduction formulas hold for all the other fields, charged bosons, vector fields. In the case of Fermi fields the only difference is that

$$\frac{i}{\sqrt{Z}} \to -\frac{i}{\sqrt{Z_2}}, \quad \text{fermions},$$

$$\frac{i}{\sqrt{Z}} \to +\frac{i}{\sqrt{Z_2}}, \quad \text{antifermions}$$

where Z_2 is the wave function renormalization for the spinor fields.

Path integral in Field Theory

The path-integral quantization can be, in principle, extended to field theory. Consider a real scalar field in d+1 dimensions. This is a mapping

$$R^d \times R^1 \to R^1$$

The quantum mechanical case corresponds to d = 0 (zero spatial dimensions). Quite clearly there is no difference in principle with the QM case, we have only to choose a lattice in d+1 dimensions and then proceed by the same principles. However we will be interested also in massless vector theories

$$R^d \times R^1 \to V^{d+1}$$

where V^{d+1} is a real vector space in d + 1 dimensions. Here we will have to be careful since, as we shall see, the Feynman integral is not well defined,
The last case we will be interested about is the one of spinor fields, typically

$$R^d \times R^1 \to C^4$$

This case also requires some modification due to the anticommuting nature of the quantum spinor fields. We shall see that this point will be solved by introducing “classical” spinor fields belonging to an infinite dimensional Grassmann algebra.

Path integral for scalar fields

Consider a real scalar field described by the following action

$$S=\int_V d^4x\frac{1}{2}\left[\partial_\mu\varphi\partial^\mu\varphi-m^2\varphi^2\right]\equiv\int_t^{t'}dt\int d^3\vec{x}\,\frac{1}{2}\left[\partial_\mu\varphi\partial^\mu\varphi-m^2\varphi^2\right]$$

As it is well know, by introducing normal modes, this field can be seen as a collection of non-interacting harmonic oscillators

$$\varphi(\vec{x},t)=\frac{1}{(2\pi)^3}\int d^3\vec{k}\,e^{i\vec{k}\cdot\vec{x}}q(\vec{k},t)$$

We get

$$S=\int_t^{t'}dt\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2}\left[|\dot{q}(\vec{k},t)|^2-\omega_{\vec{k}}^2|q(\vec{k},t)|^2\right],$$

$$\omega_{\vec{k}}^2=|\vec{k}|^2+m^2,\ \ q^*(\vec{k},t)=q(-\vec{k},t)$$

We get a continuous infinity of complex harmonic oscillators (with mass =1) satisfying the classical eqs. of motion

$$\ddot{q}(\vec{k},t)+\omega_{\vec{k}}^2 q(\vec{k},t)=0$$

In presence of an external source J

$$S_J=\int_V d^4x\frac{1}{2}\left[\partial_\mu\varphi\partial^\mu\varphi-m^2\varphi^2+J\varphi\right]$$

The normal modes decomposition gives

$$S_J=\int_t^{t'}dt\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2}\left[|\dot{q}(\vec{k},t)|^2-\omega_{\vec{k}}^2|q(\vec{k},t)|^2+J(-\vec{k},t)\varphi(-\vec{k},t)\right]$$

By decomposing q and J in their real and immaginary components, one can use the QM result by summing over all the oscillators, that is

$$-\frac{1}{2}\int_t^{t'}ds\,ds'J(s)\Delta(s-s')J(s')\to-\frac{1}{2}\int_t^{t'}ds\,ds'\int\frac{d^3\vec{k}}{(2\pi)^3}J(-\vec{k},s)\Delta(s-s';\omega_{\vec{k}})J(\vec{k},s')$$

In configuration space we get

$$-\frac{1}{2}\int d^4x d^4y J(x)\int\frac{d^3\vec{k}}{(2\pi)^3}\Delta(s-s';\omega_{\vec{k}})e^{i\vec{k}\cdot(\vec{x}-\vec{y})}J(y)\equiv$$

$$\equiv\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y;m^2)J(y)$$

where

$$i\Delta_F(x,m^2)=-\int\frac{d^3\vec{k}}{(2\pi)^3}\Delta(s;\omega_{\vec{k}})e^{i\vec{k}\cdot\vec{x}}=-\lim_{\epsilon\to 0^+}i\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2-m^2+i\epsilon}$$

we have made use of (remember that the mass of the oscillators is 1)

$$\Delta(s,\omega_{\vec{k}})=\lim_{\epsilon\to 0^+}\frac{i}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-i\nu s}}{\nu^2-\omega_{\vec{k}}^2+i\epsilon}d\nu,\quad \omega_{\vec{k}}^2=\vec{k}^2+m^2$$

and we have defined

$$x^\mu=(\vec{x},s),\, y^\mu=(\vec{y},s'),\quad k^\mu=(\nu,\vec{k})$$

The generating functional is

$$Z[J] = e^{\frac{i}{2}\int d^4x d^4y J(x)\Delta_F(x-y;m^2)J(y)} Z[0],$$

and the two-point function

$$\frac{1}{\langle 0|0\rangle}\langle 0|T(\varphi(x_1)\varphi(x_2))|0\rangle = -\frac{1}{Z[0]}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\Big|_{J=0} =$$

$$= -i\Delta_F(x_1 - x_2; m^2)$$

In the following we will use the following notation for the n-point function

$$G^{(N)}(x_1,\cdots,x_N)=\frac{\langle 0|T(\varphi(x_1)\cdots\varphi(x_N))|0\rangle}{\langle 0|0\rangle}$$

and we will use an index 0 for denoting the free case. In particular

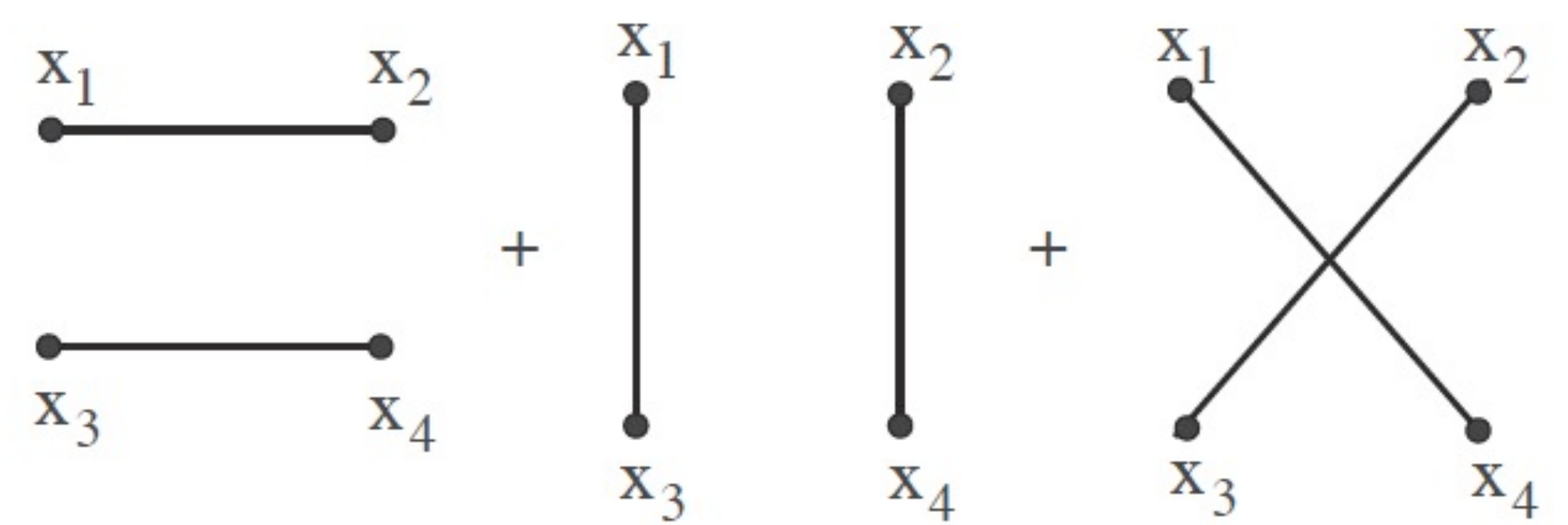
$$G_0^{(2)}(x_1,x)2)=-i\Delta_F(x_1-x_2;m^2)$$

It is a simple execise to show that

$$G_0^{(4)}(x_1,\cdots,x_4)=\Big[G_0^{(2)}(x_1-x_2)G_0^{(2)}(x_3-x_4)+$$
$$+G_0^{(2)}(x_1-x_3)G_0^{(2)}(x_2-x_4)+G_0^{(2)}(x_1-x_4)G_0^{(2)}(x_2-x_3)\Big]$$

If we associate to a two point function a line

The four point function we have just evaluated can be represented by



It is easily shown that all the n-point functions of the free case with n odd vanish, whereas the ones sith n even are obtained by combining all the points by a single line in all the possible ways. In other word, the only non trivial element is the 2-point function. All the higher order functions correspond to disconnected graphs, The previous considerations suggest to define the functional

$$Z[J] = e^{iW[J]}$$

and, in the free case

$$W_0[J] = \frac{1}{2}\int d^4x d^4y J(x)\Delta_F(x-y;m^2)J(y) + W[0]$$

in particular

$$\frac{1}{Z[0]}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\Big|_{J=0} = i\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)}\Big|_{J=0} = i\Delta_F(x_1-x_2)$$

One can check that W[J] generates the connected Green's functions through the Volterra expansion

$$iW[J] = \sum_{n=0}^{\infty}\frac{(i)^n}{n!}\int d^4x_1\cdots d^4x_n J(x_1)\cdots J(x_n)G_c^{(n)}(x_1,\cdots,x_n)$$

where

$$G_c^{(n)}(x_1,\cdots,x_n)=\frac{1}{\langle 0|0\rangle}\langle 0|T(\varphi(x_1)\cdots\varphi(x_n))|0\rangle_{\mathrm{conn}}$$

is the connected part of the n-point function. In the free case, only the two-point function is connected, all the other connected functions vanish as it is can be seen immediately.

Perturbative expansion

Let us consider a real scalar field with a self-interaction

$$S=\int_V d^4x\left(\frac{1}{2}\left[\partial_\mu\varphi\partial^\mu\varphi-m^2\varphi^2\right]-V(\varphi)\right),\quad V(\varphi)=\frac{\lambda}{4!}\varphi^4$$

We want to evaluate the effects of the potential using a perturbative expansion. We start from the following identity

$$\int D(\varphi)F[\varphi]e^{iS+i\int d^4xJ(x)\varphi(x)}=F\left[\frac{1}{i}\frac{\delta}{\delta J}\right]\int D(\varphi)e^{iS+i\int d^4xJ(x)\varphi(x)}$$

from which

$$Z[J]=N\int D[\varphi]\,e^{iS+i\int d^4xJ(x)\varphi(x)}=$$

$$=N\int D[\varphi]e^{-i\int d^4xV(\varphi)}e^{iS_0+i\int d^4xJ(x)\varphi(x)}=e^{-i\int d^4xV\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)}Z_0[J]$$

Introducing the notation

$$\langle F(1,\cdots,n)\rangle \equiv \int d^4x_1\cdots d^4x_n F(x_1,\cdots,x_n)$$

one finds

$$W[J]=-i\log Z[J]=-i\log\left[e^{iW_0[J]}+\left(e^{-i\langle V\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\rangle}-1\right)e^{iW_0[J]}\right]=$$

$$=W_0[J]-i\log\left[1+\delta[J]\right]$$

$$\delta[J]=e^{-iW_0[J]}\left(e^{-i\langle V\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\rangle}-1\right)e^{iW_0[J]}$$

For the coupling going to zero this expression also vanishes, therefore we can perform a series expansion in $\delta[J]$. At second order

$$W[J] = W_0[J] - i\left(\delta - \frac{1}{2}\delta^2\right) + \cdots$$

Then, we expand δ in a series of λ

$$\delta = \lambda\delta_1 + \lambda^2\delta_2 + \cdots$$

getting

$$W[J] = W_0[J] - i\lambda\delta_1 - i\lambda^2\left(\delta_2 - \frac{1}{2}\delta_1^2\right) + \cdots$$

considering only the first order term

$$\delta_1 = -\frac{i}{4!}e^{-iW_0[J]}\left\langle\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4\right\rangle e^{iW_0[J]}$$

and differentiating

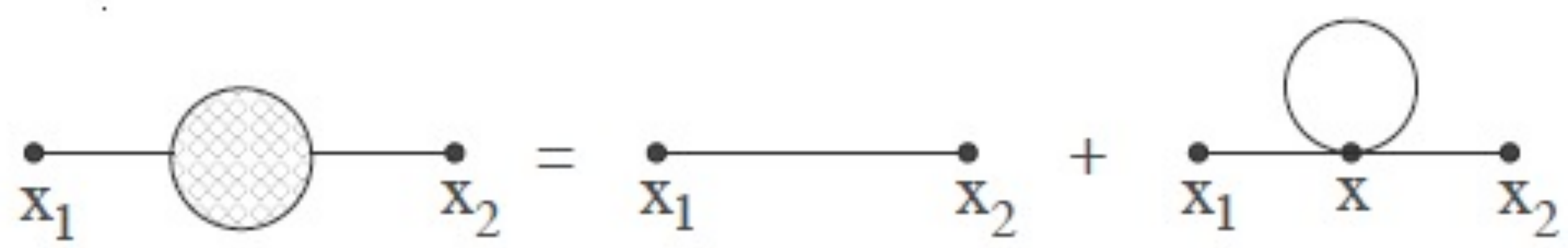
$$\delta_1 = -\frac{i}{4!}\Big[\langle \Delta(y,1)\Delta(y,2)\Delta(y,3)\Delta(y,4)J(1)J(2)J(3)J(4)\rangle +$$

$$+6i\langle \Delta(y,y)\Delta(y,1)\Delta(y,2)J(1)J(2)\rangle - 3\langle \Delta^2(y,y)\rangle\Big]$$

This contributes to the two-point connected function

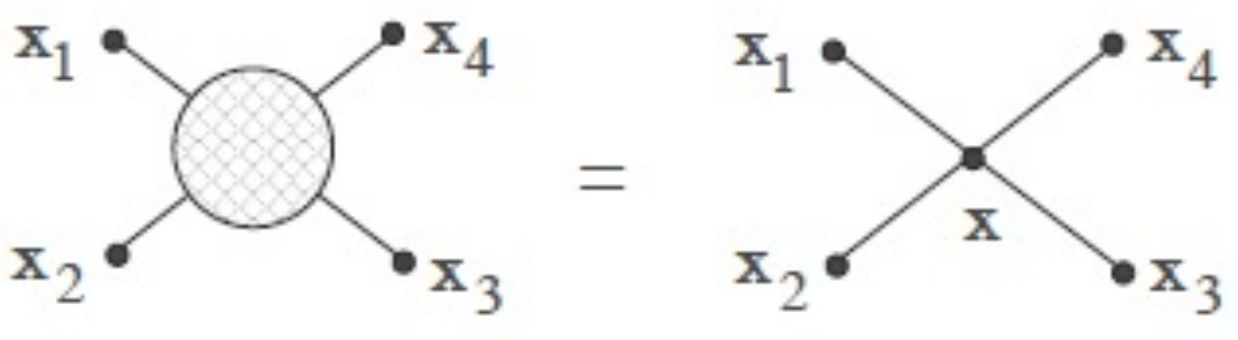
$$G_c^{(2)}(x_1,x_2) = -i\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)}\Bigg|_{J=0} =$$

$$= i\Delta_F(x_1-x_2) - \frac{\lambda}{2}\int d^4x \Delta_F(x_1-x)\Delta_F(x-x)\Delta_F(x-x_2)$$



and to the 4-point connected one

$$G_c^{(4)}(x_1,x_2,x_3,x_4) = \frac{\delta^4 W[J]}{\delta J(x)^4} = -i\lambda \int d^4x \big[\Delta(x,1)\Delta(x,2)\Delta(x,3)\Delta(x,4)\big]$$



Feynman's rules in momentum space

Since the scattering experiments in particle physics are the main tool of investigation, it is convenient to work in momentum space. Consider, for example, the two-point function, and define

$$G_c^{(2)}(p_1,p_2) = \int d^4x_1 d^4x_2 G_c^{(2)}(x_1,x_2) e^{ip_1x_1+ip_2x_2}$$

The result up to first order in the coupling is

$$G_c^{(2)}(p_1,p_2) = (2\pi)^4 \delta^4(p_1+p_2) \times$$

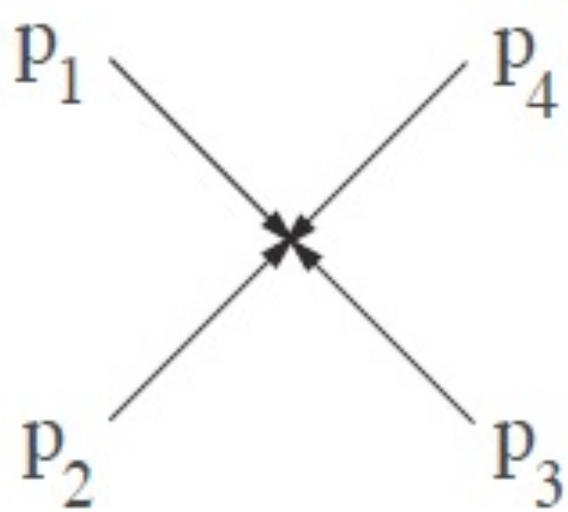
$$\times \left[\frac{i}{p_1^2 - m^2 + i\epsilon} - i\frac{\lambda}{2} \left(\frac{i}{p_1^2 - m^2 + i\epsilon} \right)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \right]$$

From these examples, it is easy to derive the following rules

- For each propagator draw a line with associated momentum p

$$: \quad \frac{i}{p^2 - m^2 + i\varepsilon}$$

- For each factor $-i\lambda/4!$ draw a vertex with the convention that the momentum flux is zero



$$: \quad -i\frac{\lambda}{4!}, \quad p_1 + p_2 + p_3 + p_4 = 0$$

▸ To get $G_c^{(n)}$ draw all the topological inequivalent diagrams after having fixed the external legs. The number of ways of drawing a given diagram is its topological weight. The contribution of any diagram has to be multiplied by its topologicalweight.

▸ After imposing conservation of the four-momentum at each vertex. integrate over all the independent internal four-momenta

$$\int \frac{d^4 q}{(2\pi)^4}$$

or, in a more systematic way, associate to each vertex

$$-\frac{i\lambda}{4!}(2\pi)^4 \delta^4\left(\sum_{i=1}^{4} p_i\right)$$

and integrate over all the internal momenta.