Lecture 4

- Feynman’s rules for gauge fields
- Divergences in QFT
- Dimensional regularization
- Regularization and renormalization of the scalar theory

Feynman's rules for non abelian gauge theories

Remember the generating functional for non-abelian gauge theories

$$Z[\eta,\eta^*,\eta_\mu]=$$
$$=\int D(A_\mu)D(c,c^*)e^{iS+iS_f+iS_{FPG}}e^{i\int d^4x\left[\eta_\mu^A A_A^\mu\right]}e^{i\int d^4x\left[c^{A*}\eta_A+\eta^{A*}c_A\right]}$$

In order to set up the perturbative expansion we need to separate the quadratic (free) part of the action from the interactions. For the gauge fields we have

$$L_A = -\frac{1}{4}\sum_C F_{\mu\nu C}F_C^{\mu\nu} = \underbrace{-\frac{1}{4}\left(\partial_\mu A_{\nu C} - \partial_\nu A_{\mu C}\right)\left(\partial^\mu A_C^\nu - \partial^\nu A_C^\mu\right)}_{L_A^{(2)}} +$$

$$+\underbrace{g f_C^{AB} A_{\mu A} A_{\nu B} \partial^\mu A_C^\nu - \frac{1}{4} g^2 f_C^{AB} f_C^{DE} A_A^\mu A_B^\nu A_{\mu D} A_{\nu E}}_{L_A^I} \equiv L_A^{(2)} + L_A^I$$

Analogously for the FP ghosts

$$L_{FPG} = \underbrace{c_A^*(x)\Box c_A(x)}_{L_{FPG}^{(2)}} + \underbrace{(-g) f_B^{AC} c_A^*(x) \partial_\mu A_C^\mu c_B(x)}_{L_{FPG}^I} \equiv L_{FPG}^{(2)} + L_{FPG}^I$$

The final term is the one from the gauge fixing. In the Lorentz gauge we have

$$f_A[A_\mu] = \partial_\mu A_A^\mu \Rightarrow L_f = -\frac{1}{2\beta}\sum_A (\partial_\mu A_A^\mu)^2$$

Introducing sources also for the ghost fields we get

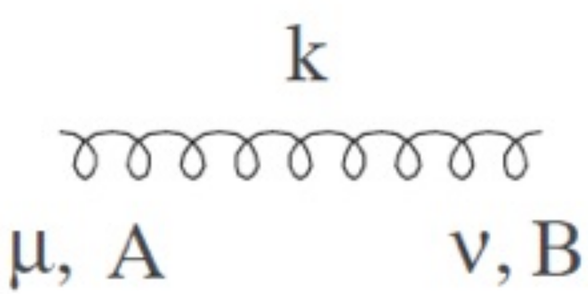
$$Z[\eta,\eta^*,\eta_\mu] = e^{iS_A^{(I)}\left[\frac{1}{i}\frac{\delta}{\delta\eta_\mu}\right]} e^{iS_{FPG}^{(I)}\left[-\frac{1}{i}\frac{\delta}{\delta\eta},\frac{1}{i}\frac{\delta}{\delta\eta^*}\right]} Z_0[\eta,\eta^*,\eta_\mu]$$

with

$$Z_0[\eta,\eta^*,\eta_\mu] = \int D(A_\mu) D(c,c^*) e^{i\int d^4x\left[L_A^{(2)} - \frac{1}{2\beta}\sum_A (\partial_\mu A_A^\mu)^2 + \eta_\mu^A A_A^\mu\right]} \times$$

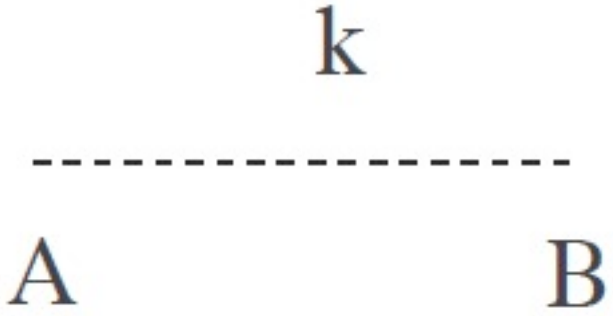
$$\times e^{i\int d^4x\left[c_A^* \Box c_A + c_A^* \eta^A + \eta^{*A} c_A\right]}$$

From these expressions one can easily derive the Feynman rules



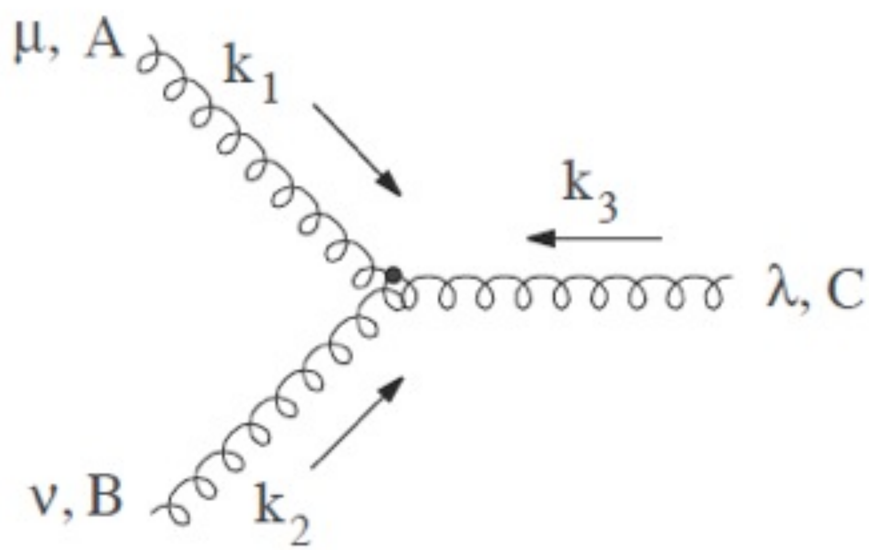
$$-i\delta_{AB}\left(g_{\mu\nu} - (1-\beta)\frac{k_\mu k_\nu}{k^2+i\epsilon}\right)\frac{1}{k^2+i\epsilon}$$

Gauge field propagator



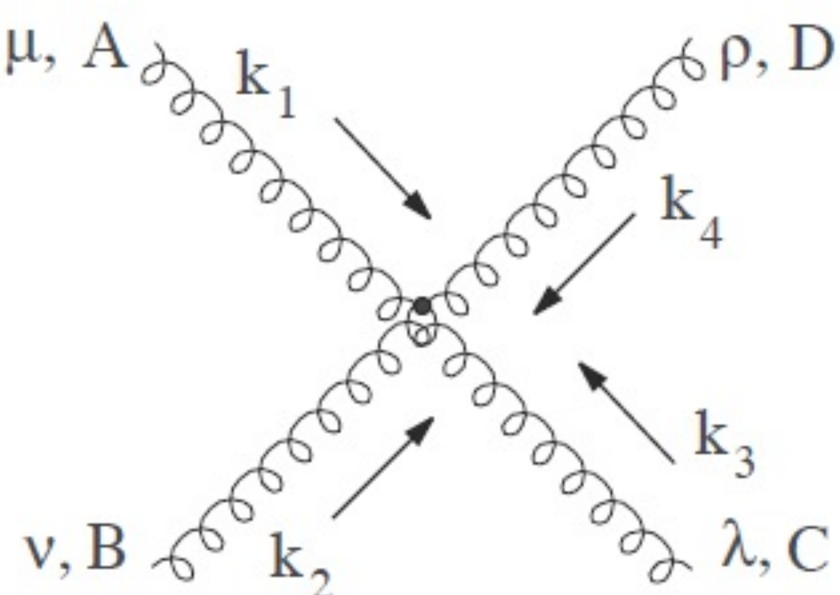
$$i\delta_{AB}\frac{1}{k^2+i\epsilon}$$

Ghost field propagator



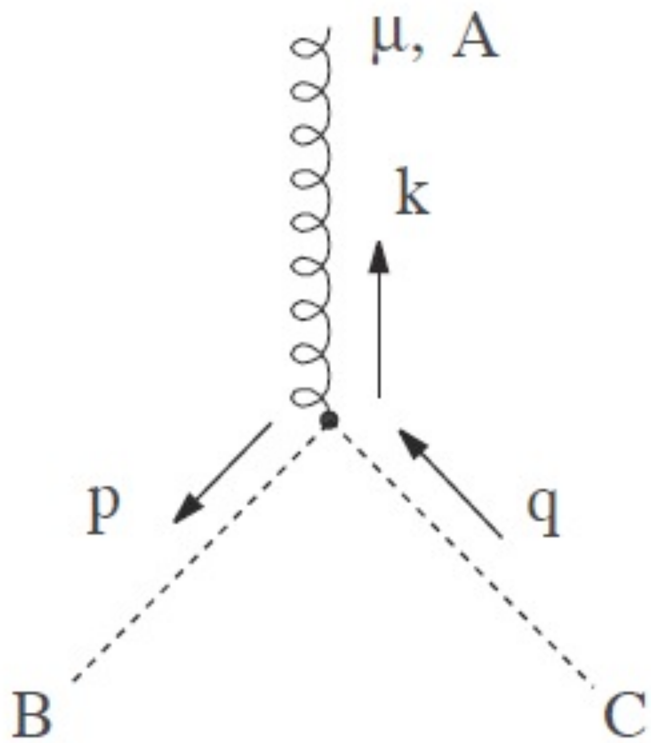
Trilinear gauge field vertex

$$gf_{ABC}(2\pi)^4\delta^4(k_1+k_2+k_3)\left[g_{\mu\nu}(k_1-k_2)_\lambda+g_{\nu\lambda}(k_2-k_3)_\mu+g_{\lambda\mu}(k_3-k_1)_\nu\right]$$



Four-linear gauge field vertex

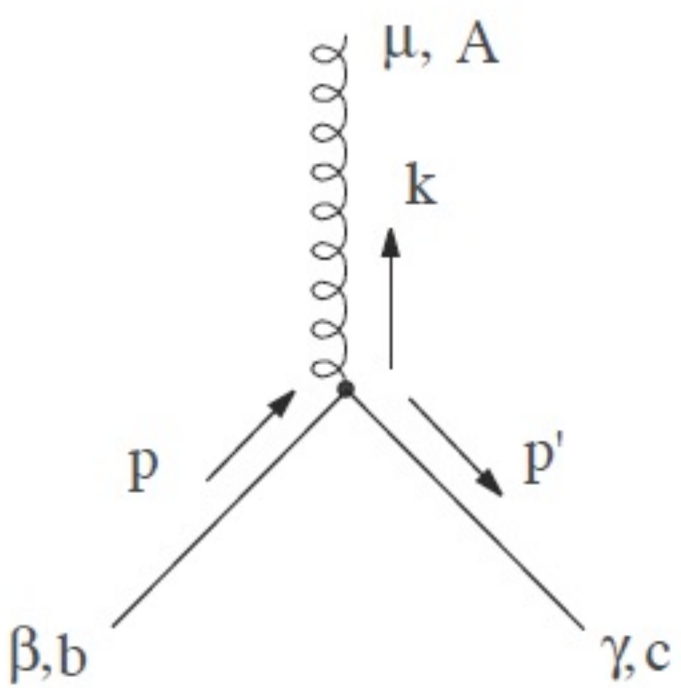
$$-ig^2(2\pi)^4\delta^4\Big(\sum_{i=1}^{4}k_i\Big)\Big[f_{ABE}f_{CDE}(g_{\mu\lambda}g_{\nu\rho}-g_{\nu\lambda}g_{\mu\rho})+$$
$$+f_{ACE}f_{BDE}(g_{\mu\nu}g_{\lambda\rho}-g_{\lambda\nu}g_{\mu\rho})+f_{ADE}f_{CBE}(g_{\mu\lambda}g_{\rho\nu}-g_{\rho\lambda}g_{\mu\nu})\Big]$$



Ghost gauge field vertex

$$-gf_{ABC}(2\pi)^4\delta^4(k+p-q)p_\mu$$

We then need the fermion gauge field vertex which is as in QCD except for the group factor. The propagator of the Dirac field is the usual one.



Fermion gauge field vertex

$$-igT^A_{bc}(\gamma_\mu)_{\beta\gamma}(2\pi)^4\delta^4(p-p'-k)$$

Divergences in QFT

Let us recall that the euclidean time is defined as $t_E = it$, and defining an euclidean energy as $E_E = iE$.

$$i\Delta_F(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\Delta(t,\omega_k^2)e^{i\vec{p}\cdot\vec{x}} = \int \frac{d^3\vec{p}}{(2\pi)^3}D_E(t_E,\omega_k^2)e^{i\vec{p}\cdot\vec{x}} = \int \frac{d^4 p_E}{(2\pi)^4}\frac{e^{-ip_E x_E}}{p_E^2+m^2}$$

Then, the Feynman' rule for the propagator is $\dfrac{1}{p^2+m^2}$

and for the vertex $-\dfrac{\lambda}{4!}$

This is because in the euclidean the weight in the path integral goes from exp[iS] to exp[-S_E]

These expressions will become useful when we will deal with the renormalization of the theory

We will use the previous results to analyze the renormalization problem in QFT.

➡ The need of renormalization does not depend on the presence of infinities. Consider the experimental definition of the mass in terms of scattering amplitudes. This can be defined as the pole position in the propagator. Consider the first order correction

$$G_c^{(2)}(p_1,p_2) = (2\pi)^4 \delta^4(p_1+p_2) \frac{i}{p_1^2 - m^2 + i\epsilon} \left(1 + \Sigma(p_1)\right) \approx$$

$$\approx (2\pi)^4 \delta^4(p_1+p_2) \frac{i}{p_1^2 - (m^2 + \Sigma(p_1)) + i\epsilon}$$

$$\Sigma(p_1) = -i\frac{\lambda}{2}\left(\frac{i}{p_1^2 - m^2 + i\epsilon}\right) \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon}$$

Therefore after having taken into account the first order correction the pole is not anymore at $p^2 = m^2$, but rather at

$$p^2 = m^2 + \Sigma(p) \rightarrow p^2 = m^2_{\text{ren}} \neq m^2$$

We see that, due to the corrections the mass is not given by the parameter m appearing in the lagrangian, but it gets modified by higher order diagrams. This implies, at least in principle, that we should re-express all the physical quantities, in terms of the physical (measured) parameters. This is the **renormalization process**.

The complication in QFT arises from the fact that, in general, the loop corrections are divergent. Then, before proceed to the renormalization we need to define properly the theory by making the corrections finite. This other process (coceptually uncorrelated to renormalization) is called **regularization**. **The regularization must precede the renormalization.**

To define the regularization procedure we need to know and classify the divergences occurring in the theory.

For simplicity we will consider the previous case, a real scalar particle with a quartic interaction. Since the divergences arise from the integration, we begin to count the number of independent integration L (equal to the number of loops). The number of integration remaining after using the delta-functions at each vertex is I - V (I= # internal lines, V = # vertices). However, by translational invariance the conservation of the external momenta should hold, meaning that one of the delta functions does not depend on the internal momenta. Therefore

$$L = I - (V - 1) = I - V + 1$$

- The L independent integrations give a factor (d are the space-time dimensions)

$$\prod_{k=1}^{L} d^d p_k$$

- The internal lines give a factor (staying euclidean)

$$\prod_{i=1}^{I} \frac{1}{p_i^2 + m^2}$$

The superficial degree of divergence of the corresponding diagram is defined as

$$\boxed{D = dL - 2I}$$

If
D < 0 we say that the integral is superficially convergent
D = 0 logarithmically divergent
D > 0 divergent

Notice that for D < 0 some subdiagram could diverge.

If we denote by V_N the # of vertices with N lines, since an external line ends in a vertex and an internal one connects two vertices, the following relation holds (E = # external lines)

$$NV_N = E + 2I \rightarrow I = \frac{NV_N - E}{2}$$

and substituting in D

$$D = d - \frac{1}{2}(d-2)E + \left(\frac{N-2}{2}d - N\right)V_N$$

In d = 4 dimensions

$$D = 4 - E + (N - 4)V_N$$

and, in our previous example, N - 4

$$D = 4 - E$$

In this case there are only 2 diagrams showing a superficial divergence, E = 2 (2-point function) and E = 4 (4-point function).

A deeper analysis (Weinberg's theorem) shows that a diagram is convergent if its superficial degree of divergence of a graph and that of all its subgraphs are negative.

In the case of our previous theory the Weinberg theorem tells us that all the divergences rely in the two and four point functions.

If it happens that all the divergent diagrams correspond to terms which are present in the original lagrangian, we say that the theory is renormalizable.

In the actual case the divergences correspond to the quadratic and to the quartic term in the lagrangian.

Recalling that in 4 dimensions

$$D = 4 - E + (N - 4)V_N$$

we see that for N > 4, D increases. There are infinite divergent diagrams and we say that the theory is not renormalizable. We can have a renormalizable theory only for

$$N \leq 4$$

An interesting case is in d = 6

$$D = 6 - 2E + (2N - 6)V_N$$

In a theory with cubic interaction, N = 3, again, D depends only on the external legs

$$D = 6 - 2E$$

The theory is renormalizable with divergent diagrams for E = 1,2,3. All these terms are present in the lagrangian.

The previous analysis can be simply expressed in terms of the canonical dimensions of the terms appearing in the lagrangian. The canonical dimension of fields (in mass) is defined in terms of the canonical kinetic terms:

$$\text{Scalar field}: \quad S = \int d^d x \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi \rightarrow \dim[\varphi] = \frac{d-2}{2}$$

$$\text{Fermion field}: S = \int d^d x \bar{\psi} \gamma^\mu \partial_\mu \psi \rightarrow \dim[\psi] = \frac{d-1}{2}$$

The dimension of a term made of elementary fields is obtained by trivial inspection. **Then the renormalizability of a QFT holds if the dimensions of the monomial appearing in the lagrangian are less or equal to the space-time dimension**. Therefore a scalar theory in 4 dimensions is renormalizable only if it contains terms of dimensions less or equal 4, whereas in 6 dim, since the dimension of a scalar field is 2, only monomials with 1, 2 or 3 fields are allowed.

The requirement of renormalizability can be formulated also in terms of the couplings appearing in the action:

$$S = \int d^d x \sum_j g_i O_i(\varphi)$$

Since

$$\dim[g_i O_i] = d$$

It follows from the ren. condition

$$\dim[O_i] = d - \dim[g_i] \leq d$$

that

$$\boxed{\dim[g_i] \geq 0}$$

The dimensions in mass of the coupling must be positive for the renormalizibility of the theory

Dimensional Regularization

To regularize the divergent integrals we will follw the method of dimensional regularization. A typical (euclidean) integral to be regularized is

$$I_4(k) = \int d^4p \; F(p,k)$$

with F behaving at infinity as $1/p^2$ or $1/p^4$. Suppose that we can define

$$I(\omega,k) = \int d^{2\omega}p \; F(p,k)$$

for complex ω. Then suppose that there exists a complex quantuty I'(ω,k) which, coincide with I(ω,k) in some region of the complex plane, then I' is the analytic continuation of I.

Begin defining

$$I_N = \int d^N p F(p^2)$$

and

$$d^N p = d\Omega_N p^{N-1} dp$$

$$I_N = S_N \int_0^\infty p^{N-1} F(p^2) dp$$

with S_N the surface of the N-dimensional sphere. This can be evaluated by using

$$\pi^{N/2} = \int dx_1 \cdots dx_N e^{-(x_1^2 + \cdots + x_N^2)}$$

and calculating the r.h.s. in polar coordinates

$$\pi^{N/2} = S_N \int_0^\infty \rho^{N-1} e^{-\rho^2} d\rho \rightarrow \pi^{N/2} = \frac{1}{2} S_N \int_0^\infty x^{N/2-1} e^{-x} dx = \frac{1}{2} S_N \Gamma\left(\frac{N}{2}\right)$$

follows

$$S_N = \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \quad \Rightarrow \quad I_N = \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \int_0^\infty x^{N/2-1} F(x) dx$$

$$I_N = \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \int_0^\infty x^{N/2-1} F(x) dx$$

Typically $$F(x) = (x + a^2)^{-A}$$

$$I_N = \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \int_0^\infty \frac{x^{N/2-1}}{(x+a^2)^A} dx = (a^2)^{N/2-A} \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \int_0^\infty y^{N/2-1}(1+y)^{-A} dx$$

$$I_N = \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \int_0^\infty \frac{x^{N/2-1}}{(x+a^2)^A} dx = (a^2)^{N/2-A} \frac{\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \int_0^\infty y^{N/2-1}(1+y)^{-A} dx$$

$$x = a^2 y$$

and using

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty t^{x-1}(1+t)^{-(x+y)} dt$$

we get the final result

$$I_N = \int \frac{d^N p}{(p^2 + a^2)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{1}{(a^2)^{A-N/2}}$$

This expression is analytic in N and can be extended to all complex plane using the properties of the Gamma function. Another very useful formula that can be obtained manipulating the previous one is

$$\int \frac{d^N p}{(p^2 + 2p{\cdot}k + b^2)^A} = \pi^{\omega} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{1}{(b^2 - k^2)^{A-N/2}}$$

Regularization in $\lambda\varphi^4$

We start with the action in d = 2ω dimensions

$$S_{2\omega} = \int d^{2\omega}x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 \right]$$

and we have

$$\dim[\varphi] = \frac{d}{2} - 1 = \omega - 1,$$

$$\dim[m] = 1, \quad \dim[\lambda] = 4 - 2\omega$$

It will be useful to define a dimensionless coupling

$$\lambda_{\text{new}} = \lambda_{\text{old}}(\mu^2)^{\omega-2}$$

Therefore

$$S_{2\omega} = \int d^{2\omega}x \left[\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}(\mu^2)^{2-\omega}\varphi^4 \right]$$

with $\lambda = \lambda_{\text{new}}$

Feynman rules are modified

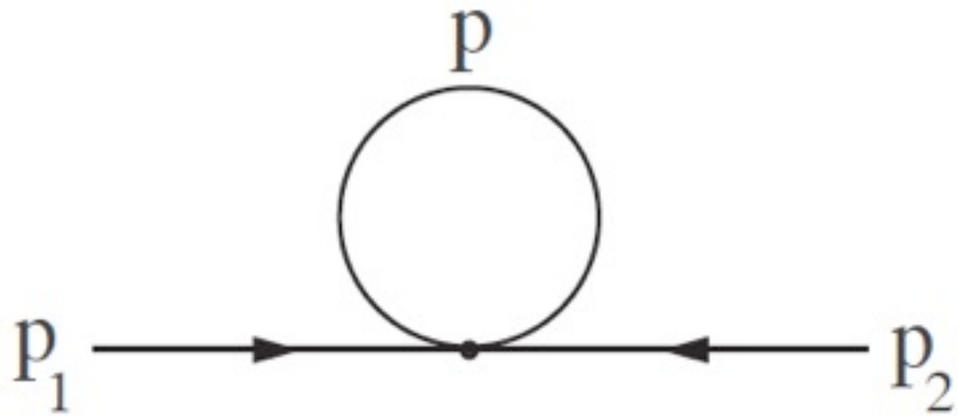
- The scalar product among two four-vectors becomes a sum over 2ω components.
- In any loop integration we have a factor $\int \frac{d^{2\omega}p}{(2\pi)^{2\omega}}$ with the delta function defined as

$$\int d^{2\omega}p\,\delta^{(2\omega)}(\sum_i p_i) = 1$$

- And for the coupling

$$\lambda \to \lambda(\mu^2)^{2-\omega}$$

We begin evaluating the one-loop contribution to the 2-point function



$$(2\pi)^{2\omega}\delta^{(2\omega)}(p_1+p_2)\frac{1}{p_1^2+m^2}T_2\frac{1}{p_1^2+m^2}$$

with

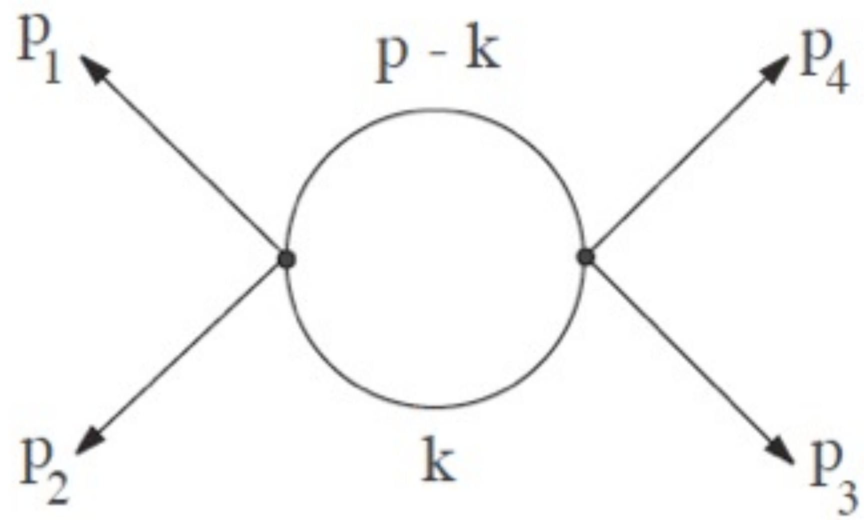
$$T_2=\frac{1}{2}(-\lambda)(\mu^2)^{2-\omega}\int\frac{d^{2\omega}p}{(2\pi)^{2\omega}}\frac{1}{p^2+m^2}=-\frac{\lambda}{2}\frac{m^2}{(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{2-\omega}\Gamma(1-\omega)\rightarrow$$

$$\xrightarrow[\omega\to 2]{}\frac{\lambda}{32\pi^2}m^2\left[\frac{1}{2-\omega}+\psi(2)+\log\frac{4\pi\mu^2}{m^2}+\cdots\right]$$

where we have made use of (γ = 0.5772... is the Euler-Mascheroni constant)

$$\Gamma(-n+\epsilon)=\frac{(-1)^n}{n!}\left[\frac{1}{\epsilon}+\psi(n+1)+O(\epsilon)\right],\quad \psi(n+1)=1+\frac{1}{2}+\cdots+\frac{1}{n}-\gamma$$

In analogous way we can evaluate the 1-loop contribution to the 4-point function



$$(2\pi)^{2\omega}\delta^{(2\omega)}(p_1+p_2+p_3+p_4)\prod_{k=1}^{4}\frac{1}{p_k^2+m^2}T_4$$

with

$$T_4=\frac{1}{2}(-\lambda)^2(\mu^2)^{4-2\omega}\int\frac{d^{2\omega}k}{(2\pi)^{2\omega}}\frac{1}{k^2+m^2}\frac{1}{(k-p)^2+m^2},\qquad p=p_1+p_2$$

Using the following identity to combine the denominators

$$\frac{1}{ab}=\frac{1}{b-a}\left[\frac{1}{a}-\frac{1}{b}\right]=\frac{1}{b-a}\int_a^b\frac{dx}{x^2}=\int_0^1\frac{dz}{[az+b(1-z)]^2}$$

we get for ω close to 2

$$T_4 \approx (\mu^2)^{2-\omega} \frac{\lambda^2}{32\pi^2} \Big[\frac{1}{2-\omega} + \psi(1) + 2 + \log \frac{4\pi\mu^2}{m^2} -$$

$$- \sqrt{1 + \frac{4m^2}{p^2}} \log \frac{\sqrt{1 + \frac{4m^2}{p^2}} + 1}{\sqrt{1 + \frac{4m^2}{p^2}} - 1} + \cdots \Big]$$

Renormalization in $\lambda\varphi^4$

We have regularized the divergent integrals in the perturbative expansion using dimensional regularization, As long as we stay in the analiticity domain of the integrals we can proceed in a series expansion in the coupling. This means that we are taking a double limit, in a precise order, first we develop for small coupling an then we take the limit of dimension 4. However all the higher corrections diverge in this limit. The way to cure this problem is to be careful about physical quantities. Consider the scattering amplitude of 2 into 2 scalar particles at some kinematical point. Assuming that this is linear in the physical coupling, we get a perturbative series for the physical coupling given by

$$\lambda = \lambda_B + c_2\lambda_B^2 + c_3\lambda_B^3 + \cdots$$

We have now called λ_B the coupling appearing in the original lagrangian. As we have seen the coefficients c_i ‘s are divergent. In any cae, divergent or not, we would like to replace the physical quantities expressed via λ_B in terms of the physical coupling. This means that the previous series should be inverted. This is possible as long as the coupling is small and the coefficients finite. So we perform this operation for the regularized quantities and after that we will go to the limit d = 4. However this can be done only for a particular set of theories where it is possible to arrange the perturbative series in such a way to get finite results for the physical quantities. We will show how to proceed perturbatively in the context of the $\lambda\varphi^4$ theory.

We will write the original lagrangian in the form

$$L = \frac{1}{2}\partial_\mu \varphi_B \partial^\mu \varphi_B + \frac{1}{2} m_B^2 \varphi_B^2 + \frac{\lambda_B}{4!}\varphi_B^4$$

where the index B stays for bare, implying that both fields and constants should be re-defined or renormalized in terms of physical quantities. Then, we separate L in two parts, a physical one (expressed in terms of physical couplings and fields)

$$L_p = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!}\mu^{4-2\omega}\varphi^4$$

and a counterterms part

$$L_{ct} = \frac{1}{2}\delta Z \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2}\delta m^2 \varphi^2 + \frac{\delta\lambda}{4!}\mu^{4-2\omega}\varphi^4$$

The sum of the two

$$L = \frac{1}{2}(1+\delta Z)\partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2}(m^2 + \delta m^2)\varphi^2 + \frac{(\lambda + \delta\lambda)}{4!}\mu^{4-2\omega}\varphi^4$$

gives back the original lagrangian if we identify

$$\varphi_B = (1+\delta Z)^{1/2}\varphi \equiv Z_\varphi^{1/2}\varphi,$$

$$m_B^2 = \frac{m^2 + \delta m^2}{1+\delta Z} = (m^2 + \delta m^2) Z_\varphi^{-1},$$

$$\lambda_B = \mu^{4-2\omega}\frac{\lambda + \delta\lambda}{(1+\delta Z)^2} = \mu^{4-2\omega}(\lambda + \delta\lambda) Z_\varphi^{-2}$$

The counter-terms are then evaluated by the requirement of removing the divergences arising in the limit d = 4, except for a finite part which is fixed by the renormalization conditions.

Consider, for instance, the contribution to the two point function (evaluated using L_p)

$$G_c^{(2)}(p_1,p_2)=(2\pi)^{2\omega}\delta^{2\omega}(p_1+p_2)\left[\frac{1}{p_1^2+m^2}+\frac{1}{p_1^2+m^2}T_2\frac{1}{p_1^2+m^2}+\cdots\right]\approx$$

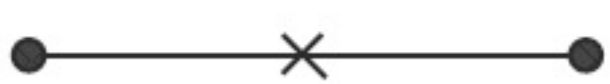
$$\approx(2\pi)^{2\omega}\delta^{2\omega}(p_1+p_2)\frac{1}{p_1^2+m^2-T_2}$$

where, remember ($\varepsilon = 4 - 2\omega$)

$$T_2\xrightarrow[\omega\sim 2]{}\frac{\lambda}{32\pi^2}m^2\left[\frac{2}{\epsilon}+\psi(2)+\log\frac{4\pi\mu^2}{m^2}+\cdots\right]$$

We then choose a counterterm such to cancel the divergent part

$$\frac{1}{2}\delta m^2\varphi^2=\frac{\lambda}{64\pi^2}m^2\left[\frac{2}{\epsilon}+F_1\left(\omega,\frac{m^2}{\mu^2}\right)\right]\varphi^2$$



$$= -\delta m^2$$

Summing together the two contributions we get

$$G_c^{(2)}(p_1,p_2)=(2\pi)^{2\omega}\delta^{2\omega}(p_1+p_2)\left[\frac{1}{p_1^2+m^2-T_2}+\frac{1}{p_1^2+m^2}(-\delta m^2)\frac{1}{p_1^2+m^2}\right]\approx$$

$$\approx(2\pi)^{2\omega}\delta^{2\omega}(p_1+p_2)\frac{1}{p_1^2+m^2+\delta m^2-T_2}$$

with

$$\delta m^2-T_2=\frac{\lambda}{32\pi^2}m^2\left[F_1-\psi(2)+\log\frac{m^2}{4\pi\mu^2}\right]$$

finite. This mechanism works because the divergence corresponds to an operator (φ^2) already present in the lagrangian, otherwise we would have added to our lagrangian counterterms with different operatorial content. This is the typical case of non-renormalizable theory. The condition is the one that we have seen before, that is that the the couplings have mass dimensions greater or equal to zero.

Therefore at first order in the coupling we get the finite result

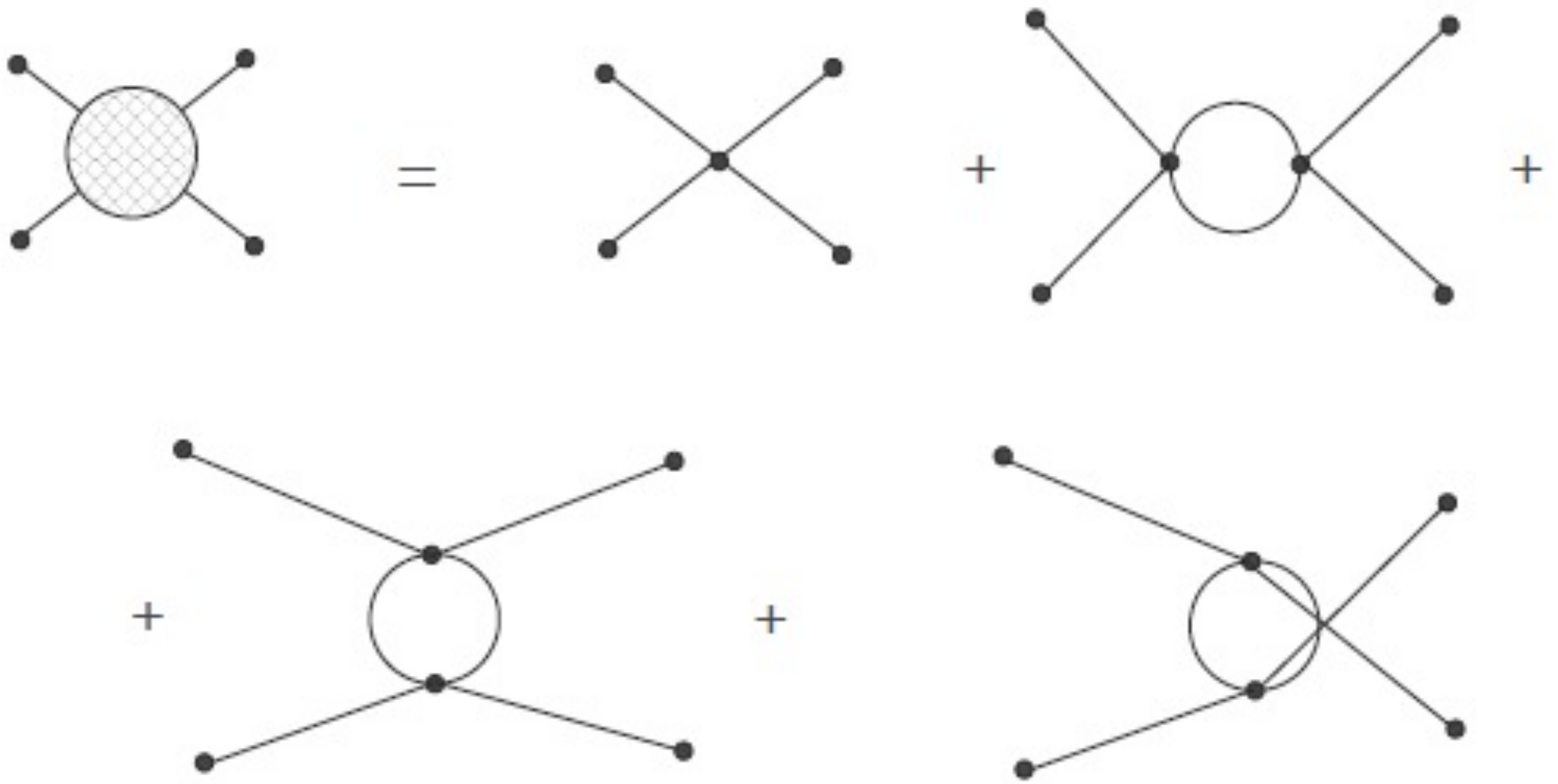
$$G_c^{(2)}(p_1,p_2)=(2\pi)^{2\omega}\delta^{2\omega}(p_1+p_2)\frac{1}{p_1^2+m^2\left(1+\frac{\lambda}{32\pi^2}\left[F_1-\psi(2)+\log\frac{m^2}{4\pi\mu^2}\right]\right)}$$

The quantity F_1 can be determined by identifying the physical mass with the position of the pole (in Minkowski space)

$$p_1^2=m_{phys}^2$$

Notice that in this particular case there is no wave function renormalization (δZ_φ) at one-loop.

We will now consider the 4-point function given by the following sum of Feynman's diagrams



Introducing the Mandelstam variables

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_1 + p_4)^2$$

one gets

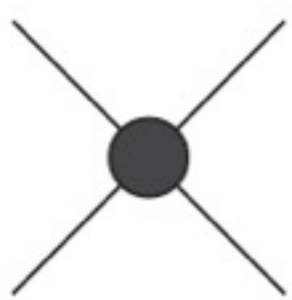
$$G_c^{(4)}(p_1,p_2,p_3,p_4)=(2\pi)^{2\omega}\delta^{(2\omega)}(p_1+p_2+p_3+p_4)\prod_{k=1}^{4}\frac{1}{p_k^2+m^2}\times$$

$$\times(-\mu^{4-2\omega})\lambda\left[1-\frac{3\lambda}{32\pi^2}\left(\frac{2}{\epsilon}+\psi(1)+2-\log\frac{m^2}{4\pi\mu^2}-\frac{1}{3}A(s,t,u)\right)\right]$$

$$A(s,t,u)=\sum_{z=s,t,u}\sqrt{1+\frac{4m^2}{z}}\log\frac{\sqrt{1+\frac{4m^2}{z}}+1}{\sqrt{1+\frac{4m^2}{z}}-1}$$

The divergence can be cancelled by the counterterm

$$\frac{\delta\lambda}{4!}\mu^{4-2\omega}\varphi^4=\frac{1}{4!}\mu^{4-2\omega}\frac{3\lambda^2}{32\pi^2}\left[\frac{2}{\epsilon}+G_1\left(\omega,\frac{m^2}{\mu^2}\right)\right]\varphi^4$$



$$=-(\mu^2)^{2-\omega}\delta\lambda$$

In fact we get

$$G_c^{(4)}(p_1,p_2,p_3,p_4)=(2\pi)^{2\omega}\delta^{(2\omega)}(p_1+p_2+p_3+p_4)\prod_{k=1}^{4}\frac{1}{p_k^2+m^2}\times$$

$$\times(-\mu^{2-4\omega})\lambda\left[1-\frac{3\lambda}{32\pi^2}\left(-G_1+\psi(1)+2-\log\frac{m^2}{4\pi\mu^2}-\frac{1}{3}A(s,t,u)\right)\right]$$

We could fix G_1 by requiring that at some kinematical point, for instance

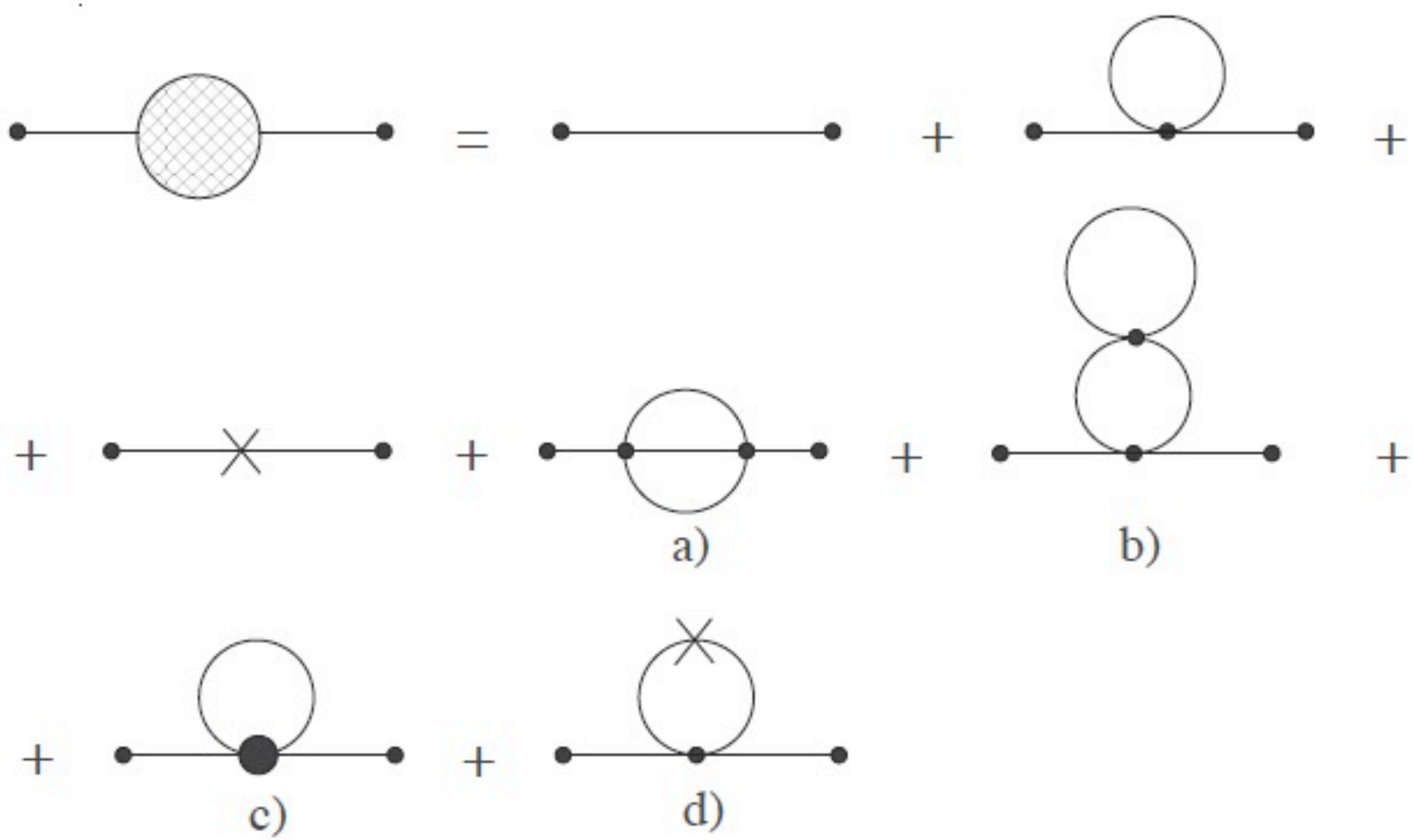
$$s=4m_{phys}^2,\quad t=u=0$$

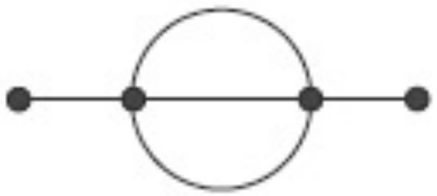
that the amplitude is equal to its tree value

$$G_c^{(4)}(p_1,p_2,p_3,p_4)=(2\pi)^{2\omega}\delta^{(2\omega)}(p_1+p_2+p_3+p_4)\prod_{k=1}^{4}\frac{1}{p_k^2+m^2}(-\mu^{2-4\omega})\lambda$$

Two-loops renormalization in $\lambda\varphi^4$

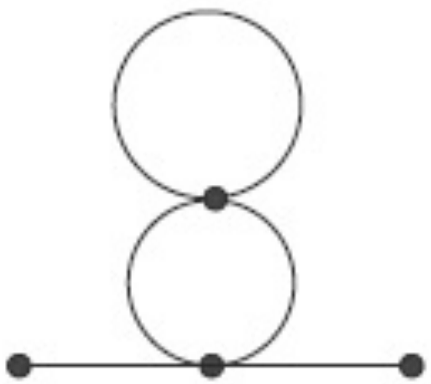
Without entering in details of the calculation we want to show how the renormalization program works at two loops. We will examine only the two-point function. Its complete expansion up to two loops, including also the counterterm contributions is given in figure





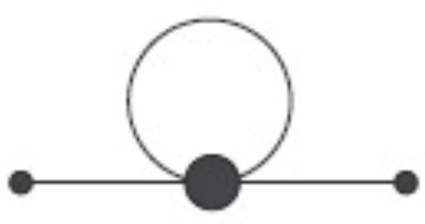
a)

$$\frac{\lambda^2}{6(16\pi^2)^2}\left[\frac{1}{2\epsilon}p^2+\frac{6m^2}{\epsilon^2}+\frac{6m^2}{\epsilon}\left(\frac{3}{2}+\psi(1)-\log\frac{m^2}{4\pi\mu^2}\right)\right]$$



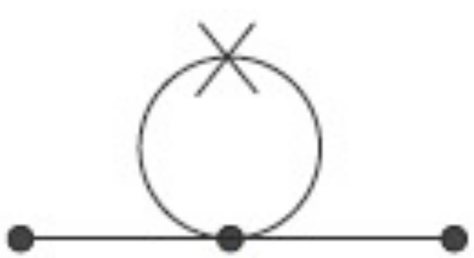
b)

$$-\frac{\lambda^2}{4(16\pi^2)^2}\left[\frac{4}{\epsilon^2}+\frac{2}{\epsilon}\left(\psi(2)+\psi(1)-2\log\frac{m^2}{4\pi\mu^2}\right)\right]$$



c)

$$\frac{3\lambda^2}{4(16\pi^2)^2}m^2\left[\frac{4}{\epsilon^2}+\frac{2}{\epsilon}\left(\psi(2)+G_1-\log\frac{m^2}{4\pi\mu^2}\right)\right]$$



d)

$$\frac{\lambda^2}{4(16\pi^2)^2}m^2\left[\frac{4}{\epsilon^2}+\frac{2}{\epsilon}\left(\psi(1)+F_1-\log\frac{m^2}{4\pi\mu^2}\right)\right]$$

Adding the 4 contributions together

$$-\frac{\lambda^2}{12(16\pi^2)^2}\frac{p^2}{\epsilon}+m^2\frac{2\lambda^2}{(16\pi^2)^2}\frac{1}{\epsilon^2}+m^2\frac{\lambda^2}{2(16\pi^2)^2}\frac{1}{\epsilon}\left[F_1+3G_1-1\right]$$

This is divergent but at the second order in the coupling. We make it finite by modifyng the mass counterterm at the second order

$$\frac{1}{2}\left(\delta m_1^2+\delta m_2^2\right)\varphi^2$$

$$\delta m_2^2=m^2\frac{\lambda^2}{2(16\pi^2)^2}\left[\frac{4}{\epsilon^2}+\frac{1}{\epsilon}(F_1+3G_1-1)+F_2\left(\omega,\frac{m^2}{\mu^2}\right)\right]$$

There is still a divergence in p^2 which is eliminated through the wave function renormalization

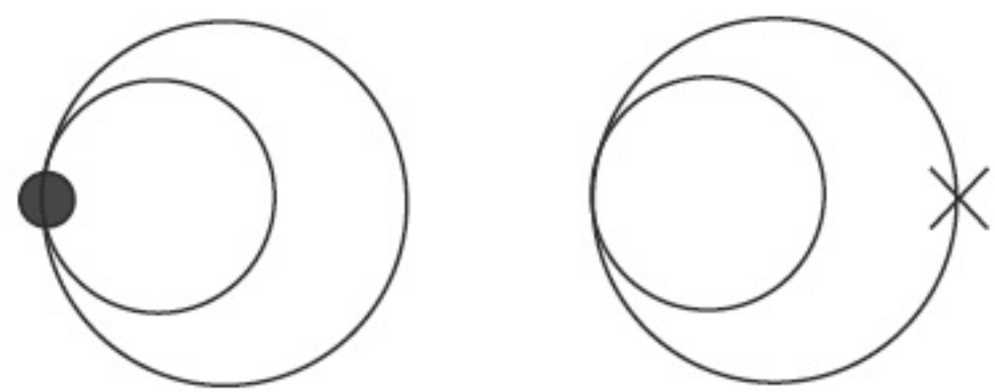
$$\frac{1}{2}\delta Z \partial_\mu \varphi \partial^\mu \varphi, \quad \delta Z = -\frac{\lambda^2}{24(16\pi^2)^2}\left[\frac{2}{\epsilon} + H_2(\omega, \frac{m^2}{\mu^2})\right]$$

The new (and the old) arbitrary functions are determined by the condition that the propagator around the physical mass is given by

$$\frac{1}{p^2 - m_{\mathrm{phys}}^2}$$

fixing two conditions, the position of the pole and the residuum (wave function renormalization)

The previous procedure can be easily extended to higher orders in the expansion in the coupling constant. What makes the theory renormalizable is the fact that the new divergences can all be disposed off by modifying the counterterms δm^2 and λ. This depends from the fact that the divergent terms are constant or quadratic in the external momentum, and therefore they can be reproduced by operators as φ^2, φ^4 and $(\partial\varphi)^2$. The non trivial part of the renormalization program is just the last one. In fact, consider the diagrams



where we have taken out all the possible external lines. These diagrams contain 1/ε terms coming from the counterterms times log p^2 terms from the loop integration. In the previous two-loop calculation we had no such terms but we got their strict relatives.

$$\log(m^2/4\pi\mu^2)$$

If present the log p^2 terms would be a real disaster since they are non-local in configuration space and as such they cannot be absorbed by local operators. However if one makes all the required subtractions, it is possible to show that the unwanted terms cancel out.