Lecture 5

- renormalization group
- Applications to QED and QCD
- Anomalies in QFT

Renormalization Group

For the following considerations it will be convenient to introduce the one-particle irreducible truncated n-point functions defined by removing the propagators on the external legs and omitting the delta-function expressing the conservation of the four-momentum

$$(2\pi)^{2\omega}\delta^{(2\omega)}(\sum_{i=1}^{n} p_i)\Gamma^{(n)}(p_1,p_2,\cdots,p_n) = G_{1PI}^{(n)}(p_1,p_2,\cdots,p_n)\prod_{i=1}^{n} D_F(p_i)^{-1}$$

$$D_F(p) = \int d^{2\omega}x e^{ipx}\langle 0|T(\varphi(x)\varphi(0))|0\rangle$$

Here, $D_F(p)$ is the exact euclidean propagator. We can evaluate the Γ functions both starting from the original lagrangian written in terms of bare quantities or expressing the bare parameters in terms of the renormalized ones and renormalizing the fields $\varphi_B = Z_\varphi^{1/2}\varphi$

Therefore the following relation holds

$$\Gamma_B^{(n)}(p_1,p_2,\cdots,p_n;\lambda_B,m_B,\omega)=Z_\varphi^{-n/2}\Gamma^{(n)}(p_1,p_2,\cdots,p_n;\lambda,m,\mu,\omega)$$

Notice that $Z_\varphi^{-n/2}$ arises from n factors $Z_\varphi^{1/2}$ from the renormalization of the fields in the connected Green function and from n Z_φ^{-1} factors from the exact propagators. Notice also that the renormalized couplings must be though of as depending on the bare ones. Notice that $\Gamma_B^{(n)}$ does not depend on μ and therefore, requiring that the derivative with respect to μ of the right hand side vanishes, we get

$$\left[\mu\frac{\partial}{\partial\mu}+\mu\frac{\partial\lambda}{\partial\mu}\frac{\partial}{\partial\lambda}+\mu\frac{\partial m}{\partial\mu}\frac{\partial}{\partial m}-\frac{n}{2}\mu\frac{\partial\log Z_\varphi}{\partial\mu}\right]\Gamma^{(n)}(p_1,p_2,\cdots,p_n;\lambda,m,\mu,\omega)=0$$

The content of this relation is that a change of the scale μ can be compensated by a convenient change of the couplings and of the field normalization.

Defining the following quantities

$$\mu\frac{\partial\lambda}{\partial\mu}=\beta\left(\lambda,\frac{m}{\mu},\omega\right),\quad \frac{1}{2}\mu\frac{\partial\log Z_\varphi}{\partial\mu}=\gamma_d\left(\lambda,\frac{m}{\mu},\omega\right),\quad \frac{\mu}{m}\frac{\partial m}{\partial\mu}=\gamma_m\left(\lambda,\frac{m}{\mu},\omega\right)$$

the previous equation becomes

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+\gamma_m m\frac{\partial}{\partial m}-n\gamma_d\right]\Gamma^{(n)}(p_1,p_2,\cdots,p_n;\lambda,m,\mu,\omega)=0$$

It is possible to eliminate the μ-derivative looking at the scale dimensions of $\Gamma^{(n)}$

$$G_c^{(n)}(p_1,\cdots,p_n)=\underbrace{\int\prod_{i=1}^{n}d^{2\omega}x_i}_{-2n\omega}\, e^{i(p_1x_1+\cdots+p_nx_n)}\underbrace{\langle 0\,|\,T(\varphi(x_1)\cdots\varphi(x_n)\,|\,0\rangle_{\text{conn}}}_{n(\omega-1)}$$

$$\dim G_c^{(n)}=-2n\omega+n(\omega-1)=-n(\omega+1)$$

$$(2\pi)^{2\omega}\underbrace{\delta^{(2\omega)}(\sum_{i=1}^{n} p_i)}_{-2\omega}\Gamma^{(n)}(p_1,p_2,\cdots,p_n)=\underbrace{G_c^{(n)}(p_1,p_2,\cdots,p_n)}_{-n(\omega+1)}\underbrace{\prod_{i=1}^{n} D_F(p_i)^{-1}}_{-n(2(\omega-1)-2\omega)=2n}$$

$$\dim\Gamma^{(n)}=2\omega-n(\omega+1)+2n=n+(2-n)\omega$$

Using again

$$\epsilon=4-2\omega$$

$$\dim\Gamma^{(n)}=4-n+\frac{\epsilon}{2}(n-2)$$

Introducing a common scale s for the momenta, we can write (using Euler's theorem)

$$\left[\mu\frac{\partial}{\partial\mu}+s\frac{\partial}{\partial s}+m\frac{\partial}{\partial m}-(4-n+\frac{\epsilon}{2}(n-2))\right]\Gamma^{(n)}(sp_i;\lambda,m,\mu,\epsilon)=0$$

Eliminating the μ derivative and taking the limit for d = 4

$$\left[-s\frac{\partial}{\partial s}+\beta\frac{\partial}{\partial\lambda}+(\gamma_m-1)m\frac{\partial}{\partial m}-n\gamma_d+4-n\right]\Gamma^{(n)}(sp_i;\lambda,m,\mu)=0$$

From this equation we can evaluate the Γ functions at any value of the momenta, if we know it at some particular value. In order to use the renormalization group equations we need to know more about the structure of the functions β, γ_d and γ_m. We have seen that the counterterms are singular expressions in ε, therefore we will express them as a Laurent series

$$\lambda_B=\mu^{\epsilon}\left[a_0+\sum_{k=1}^{\infty}\frac{a_k}{\epsilon^k}\right],\quad m_B^2=m^2\left[b_0+\sum_{k=1}^{\infty}\frac{b_k}{\epsilon^k}\right],\quad Z_\varphi=\left[c_0+\sum_{k=1}^{\infty}\frac{c_k}{\epsilon^k}\right]$$

and let us consider the counterterms at one-loop level. Recalling that

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\left[\frac{2}{\epsilon} + G_1\right], \quad \delta m^2 = \frac{\lambda}{32\pi^2} m^2 \left[\frac{2}{\epsilon} + F_1\right]$$

we obtain the bare parameters

$$\lambda_B = \mu^\epsilon \left[\lambda + \frac{3\lambda^2}{32\pi^2}\left(\frac{2}{\epsilon} + G_1\right)\right] + O(\lambda^3),$$

$$m_B^2 = m^2 \left[1 + \frac{\lambda}{32\pi^2}\left(\frac{2}{\epsilon} + F_1\right)\right] + O(\lambda^2)$$

and, comparing with the Laurent expansion

(remember that at this order there is no wave function renormalization)

$$a_0 = \lambda + \frac{3\lambda^2}{32\pi^2} G_1, \quad a_1 = \frac{3\lambda^2}{16\pi^2},$$

$$b_0 = 1 + \frac{\lambda}{32\pi^2} F_1, \quad b_1 = \frac{\lambda}{16\pi^2},$$

$$c_0 = 1, \quad c_1 = 0$$

Notice that all the dependence on m/μ comes only in the finite terms. In fact, the divergent terms originate fron the UV behaviour where masses can be neglected.This is important since the RG equations are generally difficult to solve due to the mass dependence. On the other hand one can devise renormalization conditions where such dependence disappears (‘t Hooft and Weinberg).

Renormalization Conditions

By definition, the Γ functions are finite. At first order in the loop expansion the 2 and 4 point functions are given by

$$\Gamma^{(2)}(p) = p^2 + m^2\left(1 + \frac{\lambda}{32\pi^2}\left(F_1 - \psi(2) + \log\frac{m^2}{4\pi\mu^2}\right)\right),$$

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4) = -\mu^{\epsilon}\lambda\left(1 - \frac{3\lambda}{32\pi^2}\left(\psi(1) + 2 - \log\frac{m^2}{4\pi\mu^2} - \frac{1}{3}A(s,t,u) - G_1\right)\right)$$

Of course, there are many different ways to define the renormalization. In any case the values chosen for the parameters m and λ will be determined by evaluating some physical quantity and by comparison with the experiments. Let us consider various possibilities

➡ (A) Renormalization at zero momentum (A(0,0,0)=6)

$$\Gamma^{(2)}(p)\Big|_{p^2\sim 0} \approx p^2 + m_A^2, \quad \Gamma^{(4)}(p_1,p_2,p_3,p_4)\Big|_{p_i=0} = -\mu^{\epsilon}\lambda_A$$

$$F_1^A = \psi(2) - \log\frac{m_A^2}{4\pi\mu^2}, \quad G_1^A = \psi(1) - \log\frac{m_A^2}{4\pi\mu^2}$$

This scheme is somewhat dangerous for massless particles since it gives rise to spurious IR divergences.

➡ (B) renormalization at an arbitrary scale

$$\Gamma^{(2)}(p) = p^2 + m_B^2, \quad p^2 = M^2, \quad \Gamma^{(4)}(p_1,p_2,p_3,p_4) = -\mu^{\epsilon}\lambda_B, \quad s=t=u=M^2$$

$$F_1^B = \psi(2) - \log\frac{m_B^2}{4\pi\mu^2}, \quad G_1^B = \psi(1) + 2 - \log\frac{m_B^2}{4\pi\mu^2} - \frac{1}{3}A(M^2,M^2,M^2)$$

However, the coefficients of the RG equations depend on the masses

The most convenient scheme is the one by ‘t Hooft and Weinberg:

➡ Take all the arbitrary functions equal to zero. Then, it is easy to determine the coefficients of the RG equations. As an example, consider (at the order considered here)

$$\lambda_B = \mu^\epsilon \left[\lambda + \frac{a_1(\lambda)}{\epsilon} \right]$$

all the coefficients depend only on the coupling. Differentiating with respect to μ,

$$0 = \epsilon \left[\lambda + \frac{a_1(\lambda)}{\varepsilon} \right] + \mu \frac{\partial \lambda}{\partial \mu} \left[1 + \frac{a_1'(\lambda)}{\varepsilon} \right], \quad a_k'(\lambda) = \frac{d a_k(\lambda)}{d\lambda}$$

We have $\beta = \mu \frac{\partial \lambda}{\partial \mu}$

finite at d = 4 and therefore we will expand it as

$$\beta = A + B\epsilon$$

Inserting in the previous equation

$$0=\epsilon\left[\lambda+\frac{a_1}{\epsilon}\right]+(A+B\epsilon)\left[1+\frac{a_1'}{\epsilon}\right]$$

we get

$$B+\lambda=0,\quad a_1+A+Ba_1'=0$$

and

$$B=-\lambda,\quad A=-\left(1-\lambda\frac{d}{d\lambda}\right)a_1 \Rightarrow \beta(\lambda)=-\left(1-\lambda\frac{d}{d\lambda}\right)a_1-\lambda\epsilon$$

$$\Downarrow$$

$$\beta(\lambda)=-\left(1-\lambda\frac{d}{d\lambda}\right)a_1$$

and, from $a_1=\frac{3\lambda^2}{16\pi^2}$ ➡ $$\mu\frac{\partial\lambda}{\partial\mu}=\beta(\lambda)=\frac{3\lambda^2}{16\pi^2}$$

Integration of

$$\mu \frac{\partial \lambda}{\partial \mu} = \beta(\lambda) = \frac{3\lambda^2}{16\pi^2}$$

defines how the coupling scale with μ (running coupling constant), and the knowledge of λ(μ) and m(μ) allows to integrate the RG equations. Let us now show how this works. We will show that the RG eqs can be written in the form (sum over i)

$$\frac{\partial}{\partial t} F(x_i, t) = \beta_i(x_i) \frac{\partial}{\partial x_i} F(x_i, t), \quad t = \log s$$

The general solution to this equation is obtained by using the method of "characteristics". That is, one considers the integral curves (the characteristic curves) defined by the ordinary differential equations

$$\frac{dx_i(t)}{dt} = \beta_i(x_i(t)), \quad x_i(0) = \bar{x}_i$$

Then, the general solution is

$$F(x_i,t) = \psi(x_i(t)). \quad \psi(\bar{x}_i) = F(\bar{x}_i,0)$$

as it can be checked immediately

$$\frac{\partial}{\partial t}\psi(x_i(t)) = \frac{dx_i(t)}{dt}\frac{\partial \psi}{\partial x_i} = \beta_i \frac{\partial \psi}{\partial x_i}$$

In our case there is a non-derivative term

$$\left[-s\frac{\partial}{\partial s} + \beta\frac{\partial}{\partial \lambda} + (\gamma_m - 1)m\frac{\partial}{\partial m} - n\gamma_d + 4 - n\right]\Gamma^{(n)}(sp_i;\lambda,m,\mu) = 0$$

that is, our equation is of the type

$$\frac{\partial}{\partial t}G(x_i,t) = \beta_i(x_i)\frac{\partial}{\partial x_i}G(x_i,t) + \gamma(x_i)G(x_i,t)$$

but this term can be eliminated by writing

$$G(x_i,t)=e^{\int_0^t dt'\gamma(x_i(t'))}F(x_i,t)$$

with F satisfying the previous equation, and we get

$$G(x_i,t)=e^{\int_0^t dt'\gamma(x_i(t'))}\psi(x_i(t))$$

In our case

$$\Gamma^{(n)}(sp_i;m,\lambda,\mu)=\Gamma^{(n)}(p_i;m(s),\lambda(s),\mu)s^{4-n}e^{-n\int_1^s \gamma_d(\lambda(s'))\frac{ds'}{s'}},$$

$$s\frac{\partial\lambda(s)}{\partial s}=\beta(\lambda(s))s,\quad \frac{\partial m(s)}{\partial s}=m(s)(\gamma_m(\lambda(s))-1)$$

This result tells us that when re-scaling the momenta, the amplitudes do not scale only with the trivial dimensional factor s^{4-n}, but that they show also a non trivial scaling, due to the anomalous dimension γ_d, necessary to compensate the variations of λ and m with the scale.

Properties of the renormalization group equations

Integrating

$$\mu \frac{d\lambda}{d\mu} = \frac{3\lambda^2}{16\pi^2}$$

we get

$$\lambda(\mu) = \lambda_s \frac{1}{1 - \frac{3\lambda_s}{16\pi^2} \log \frac{\mu}{\mu_s}}, \quad \lambda_s = \lambda(\mu_s)$$

with μ_s a reference scale. From this expression we see that starting from μ_s , λ increases with μ. Suppose that we start with a small λ, such that the perturbative expansion holds

However increasing μ we will need to add more and more terms to the expansion due to the increasing of λ. Therefore this theory allows a perturbative expansion only at small mass scales, or equivalently at large distances and asymptotic states have a meaning.

The increasing of λ is related to the sign of the β-function. In this theory and for QED (see later) the β-function is positive.

However if β would be negative, the theory would become perturbative at large mass scales or at small distances.

In this case one could solve the RG equations at large momenta by using the perturbative expansion and evaluate all the coefficients. However, the coupling would increase at large distances and this would create a problem for an approach based on the in- and out- states.

This is in fact, as we shall see, the situation in QCD.
The fields useful to describe the dynamics at short distances (the quark fields) cannot describe the asymptotic states, which should be rather described by bound states of quarks as mesons and baryons.

From our expression
$$\lambda(\mu) = \lambda_s \frac{1}{1 - \frac{3\lambda_s}{16\pi^2} \log \frac{\mu}{\mu_s}}$$

we see that the coupling, starting at the reference scale explodes at the scale

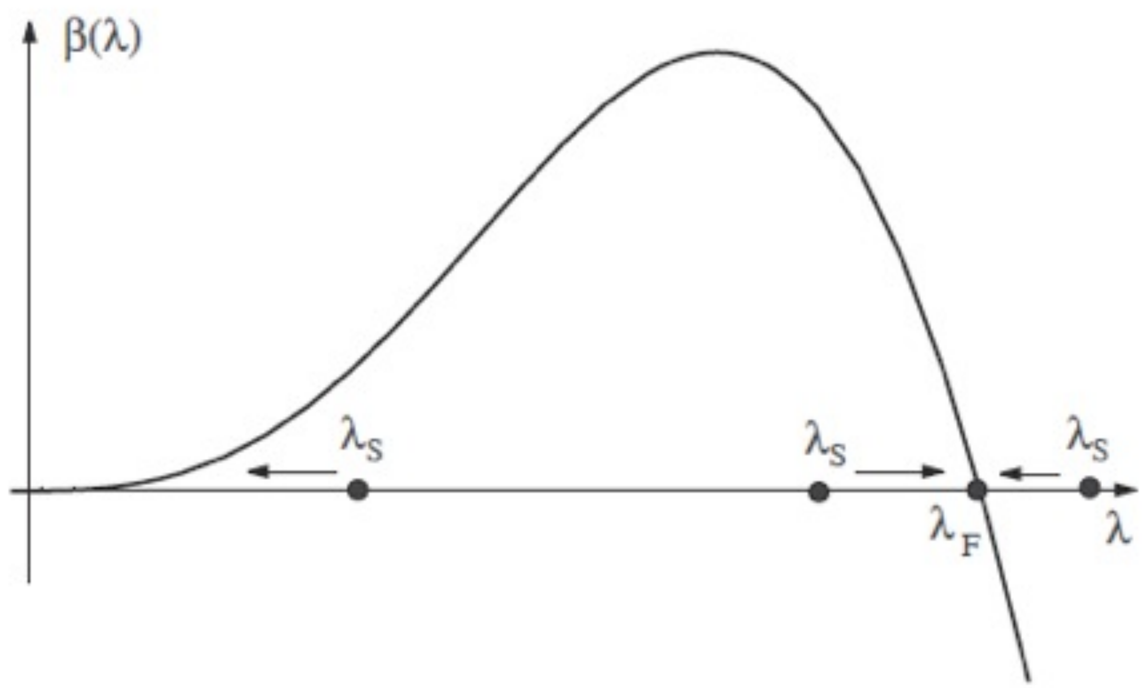
$$\mu = \mu_s e^{\frac{16\pi^2}{3\lambda_s}}$$

This is called the Landau pole and it exists also in QED.

We will consider now a few possible scenarios

➡ If $\beta(\lambda)$ is positive for any λ, than the running coupling is always increasing, and it will become infinite at some value of μ. If this happens for a finite μ we say that there is a Landau pole at that scale.

➡ Suppose that $\beta(\lambda)$ is positive at small λ and that it becomes negative vanishing at $\lambda = \lambda_F$.



The point λ_F is called a fixed point since, starting with the initial condition $\lambda = \lambda_F$, λ remains fixed at that value.

To study the behaviour of λ around the fixed point, let us expand β around its zero

$$\beta(\lambda) \approx (\lambda - \lambda_F)\beta'(\lambda_F)$$

from which

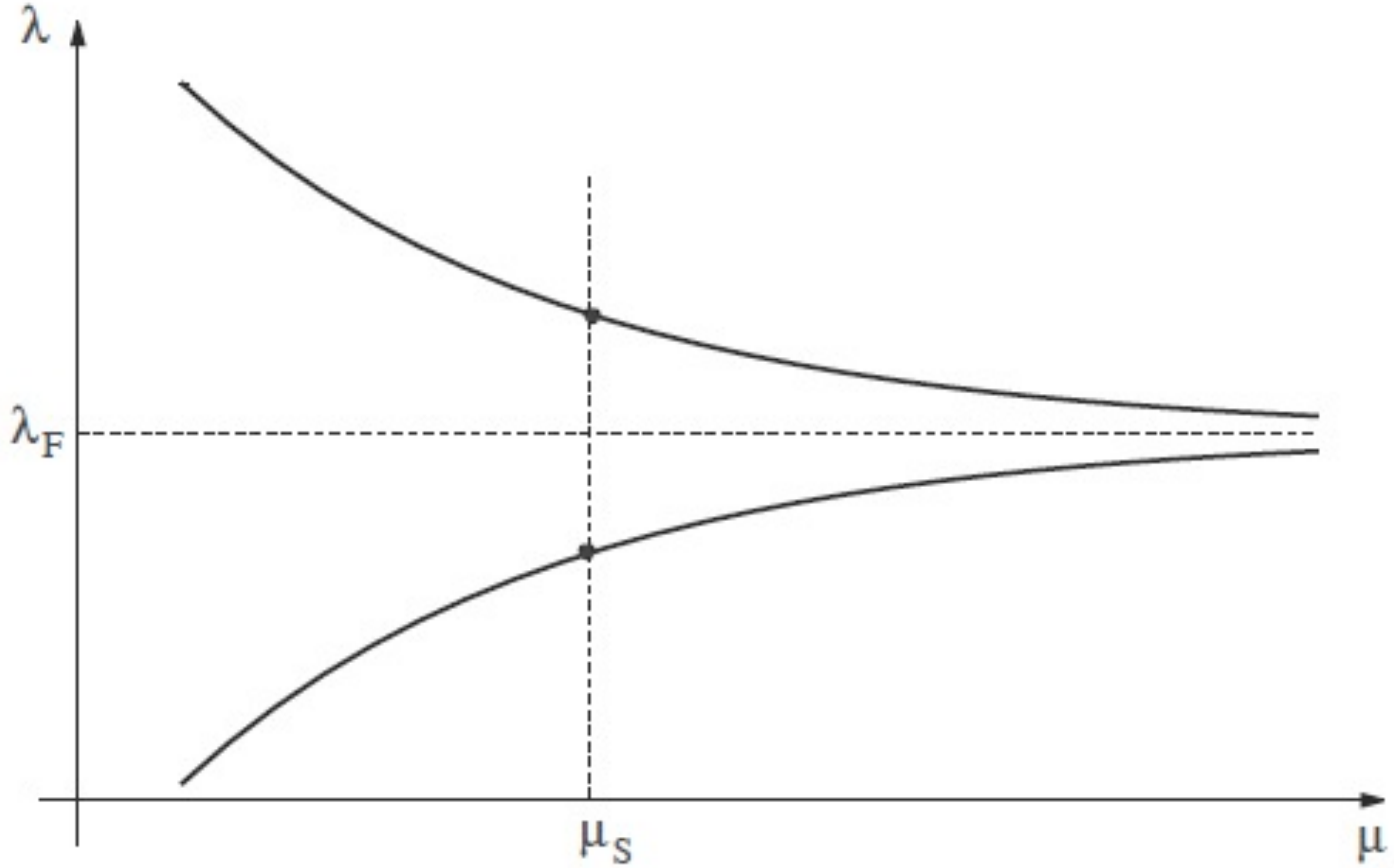
$$\mu \frac{d\lambda}{d\mu} = (\lambda - \lambda_F)\beta'(\lambda_F) + \cdots$$

and integrating

$$\frac{\lambda - \lambda_F}{\lambda_s - \lambda_F} = \left(\frac{\mu}{\mu_s}\right)^{\beta'(\lambda_F)}$$

The sign of $\beta'(\lambda_F)$ plays here an important role. In the present case the sign is negative since $\beta(\lambda) > 0$ for $\lambda < \lambda_F$ and $\beta(\lambda) < 0$ for $\lambda < \lambda_F$.

For large values of μ we have that $\lambda \rightarrow \lambda_F$ independently on the initial value λ_S. For this reason λ_F is referred to as a ultraviolet (UV) fixed point.



The large scale behavior of a field theory of this type depends on the value of λ_F. If $\lambda_F << 1$ and $\lambda_S < \lambda_F$ the theory is always in the perturbative regime. Otherwise, starting from $\lambda_S > \lambda_F$ the theory will become perturbative at large scales (see figure)

Notice that $\lambda = 0$ is a fixed point since $\beta(0) = 0$. However this is an infrared (IR) fixed point, since the coupling goes to zero for $\mu \to 0$ independently on the initial value λ_S. We say also that the fixed point is repulsive since λ goes away from the fixed point for increasing μ, whereas a UV fixed point is said to be attractive.

➡ Let us now consider the case of for $\beta(\lambda) < 0$ small values of λ and decreasing monotonically. For instance, suppose $\beta(\lambda)=-a\,\lambda^2$ with $a > 0$. Integrating the equation for the running coupling we get

$$\lambda(\mu) = \lambda_s \frac{1}{1 + a\lambda_s \log \dfrac{\mu}{\mu_s}}$$

In this case λ is a decreasing function of the scale and goes to zero at infinity, that is $\lambda = 0$ is a UV fixed point. This property is known as "asymptoyic freedom" and it holds for non-abelian gauge theories

Non-abelian gauge theories are the only 4-dimensional theories enjoying this property. In higher dimensions other theories are asymptotically free. For instance, in d = 6 the theory $\lambda\varphi^3$ is asymptotically free.

Notice that for these theories the running coupling has a pole, at a smaller scale than μ_s

$$\frac{\mu}{\mu_s} = e^{-\frac{1}{a\lambda_s}}$$

and the coupling increases at large distances.

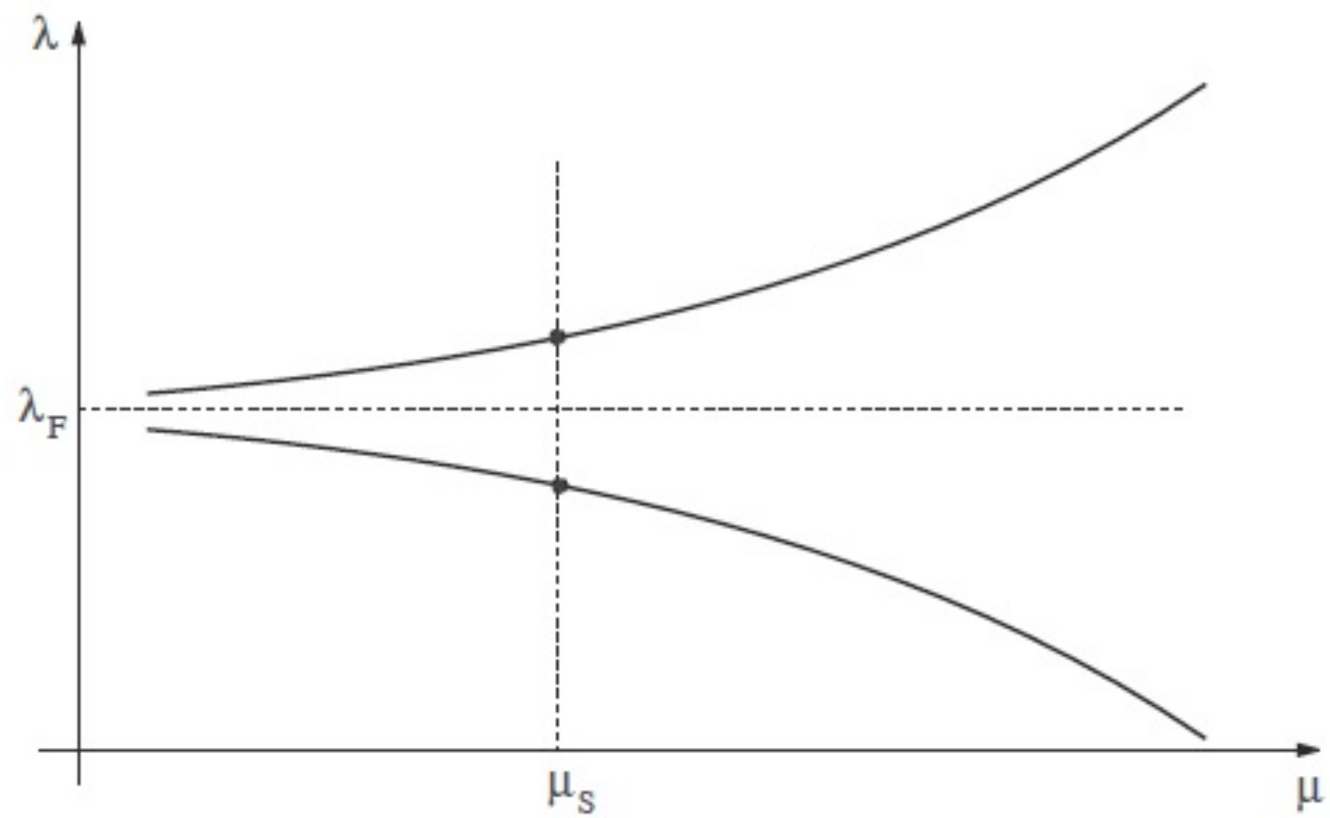
➡ Finally let us consider the case of $\beta(\lambda) < 0$ for small values of λ, becoming zero at λ_F and then positive. Then $\beta'(\lambda_F) > 0$. By expanding $\beta(\lambda)$ around λ_F we get (by the same analysis as before)

$$\frac{\lambda - \lambda_F}{\lambda_s - \lambda_F} = \left(\frac{\mu}{\mu_s}\right)^{\beta'(\lambda_F)}$$

Therefore

$$\lambda \to \lambda_F, \quad \text{when} \quad \mu \to 0$$

whereas it goes away from λ_S for increasing μ. Therefore λ_F is an IR fixed point



It is possible to determine the other RG eqs coefficients by following the same procedure, but we will skip the derivation.

Application to QED

Consider the QED action in D = 2ω dimensions (ignoting the gauge fixing)

$$S_{2\omega} = \int d^{2\omega}x \left[\bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi - e\bar{\psi}\hat{A}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right]$$

since

$$\dim[\bar{\psi}\hat{A}\psi] = \omega - 1 + 2\left(\omega - \frac{1}{2}\right) = 3\omega - 2$$

we have

$$\dim[e] = 2 - \omega$$

and define

$$e_{new} = e_{old}(\mu)^{\omega-2} = e_{old}\mu^{-\epsilon/2}$$

As before we decompose the original lagrangian as follows

$$L_p = \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi - e\mu^{\epsilon/2}\bar{\psi}\hat{A}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

$$L_{ct} = i\delta Z_2 \bar{\psi}\gamma_\mu \partial^\mu \psi - \delta m \bar{\psi}\psi - e\delta Z_1 \mu^{\epsilon/2}\bar{\psi}\hat{A}\psi - \frac{\delta Z_3}{4}F_{\mu\nu}F^{\mu\nu}$$

giving rise to

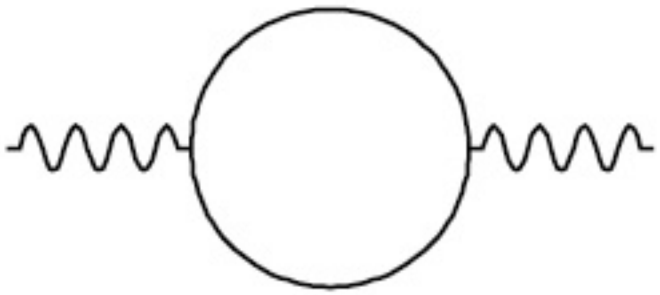
$$L = i(1+\delta Z_2)\bar{\psi}\gamma_\mu \partial^\mu \psi - (m+\delta m)\bar{\psi}\psi - $$
$$-e(1+\delta Z_1)\mu^{\epsilon/2}\bar{\psi}\hat{A}\psi - \frac{1+\delta Z_3}{4}F_{\mu\nu}F^{\mu\nu}$$

After rescaling the fields one gets the definition of the bare electric charge as

$$e_B = \mu^{\epsilon/2} e \frac{(1+\delta Z_1)}{(1+\delta Z_2)(1+\delta Z_3/2)} = \mu^{\epsilon/2} e(1-\delta Z_3/2)$$

where we have used the Ward identity saying $\delta Z_1 = \delta Z_2$

Therefore the bare electric charge is determined by the wave function renormalization of the photon, corresponding to the diagram



$$\delta Z_3 = -\frac{e^2}{6\pi^2 \epsilon}$$

$$e_B = \mu^{\epsilon/2} e \left(1 + \frac{e^2}{12\pi^2 \epsilon} \right)$$

Using the same notations as in the scalar case we see that

$$a_1 = \frac{e^3}{12\pi^2}$$ (the coefficient of 1/ε)

and using $$\beta(e) = \mu \frac{\partial e}{\partial \mu} = -\frac{1}{2}\left(1 - e\frac{d}{de} \right) a_1 \Rightarrow \beta(e) = \frac{e^3}{12\pi^2}$$

Integrating the equation for the running coupling

$$\mu \frac{\partial e(\mu)}{\partial \mu} = \beta(e) = \frac{e^3}{12\pi^2}$$

we get

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi} \log \frac{Q^2}{\mu^2}}$$

showing a Landau pole at

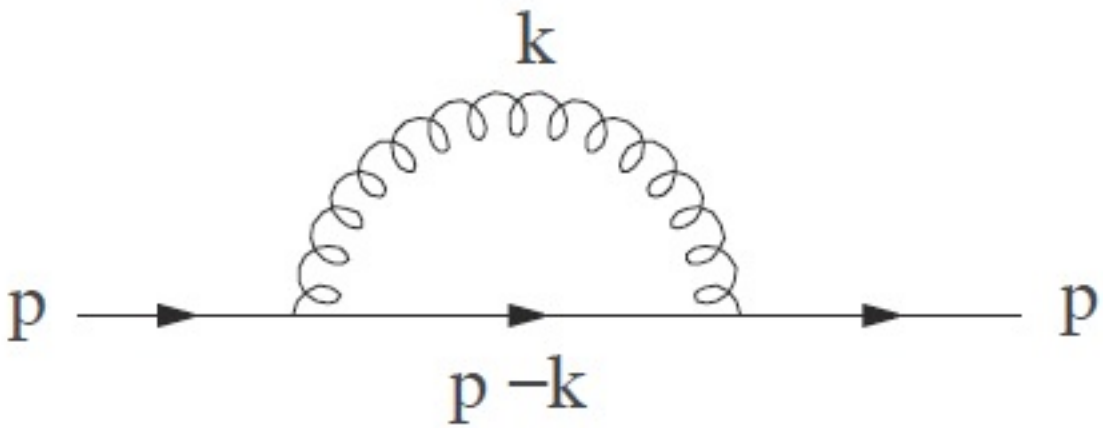
$$Q_{LP} = \mu e^{3\pi/2\alpha(\mu)}$$

Application to QCD

We can evaluate the running of the coupling in QCD using the same methods seen before. The relation between the bare and the renormalized coupling is the same as in QED

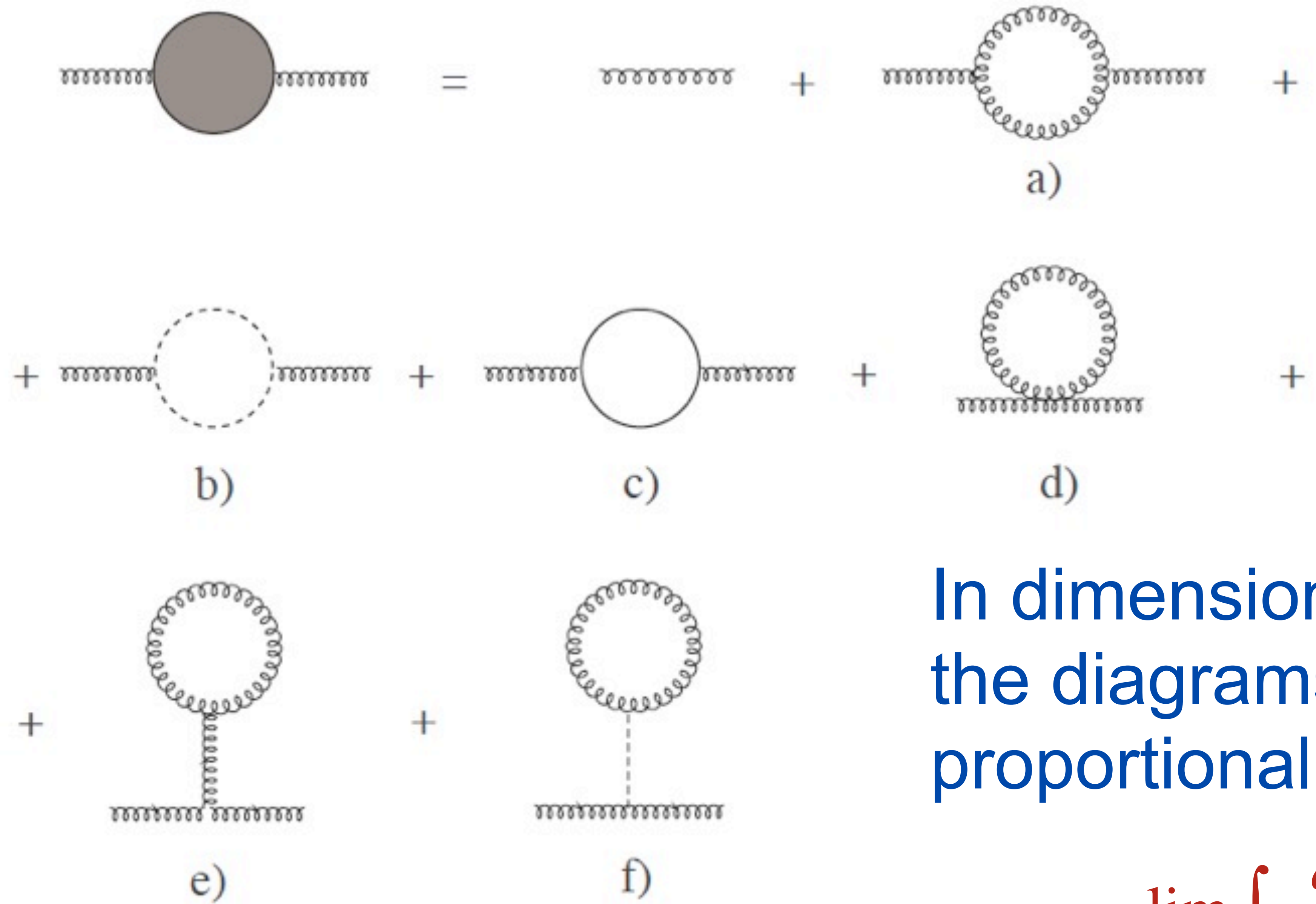
$$g_B = \mu^{\epsilon/2} g \frac{Z_1}{Z_2 Z_3^{1/2}}$$

with the same definitions of the renormalization factors as in QED, i.e., Z_1 is the renormalization of the coupling, Z_2 of the fermion and Z_3 of the gluon. However, in this case the identity $Z_1 = Z_2$ is not valid. So all the three corrections must be evaluated. Starting from the self-energy of the fermion



$$Z_2 = 1 - \frac{g^2}{8\pi^2\epsilon} C_2(F),$$

$$C_2(F) = \frac{N^2 - 1}{2N} \text{ for fermions in SU(N)}$$



In dimensional regularization the diagrams d) e) and f) are proportional to

$$\lim_{a\to 0}\int \frac{d^{2\omega}k}{k^2-a^2} =$$

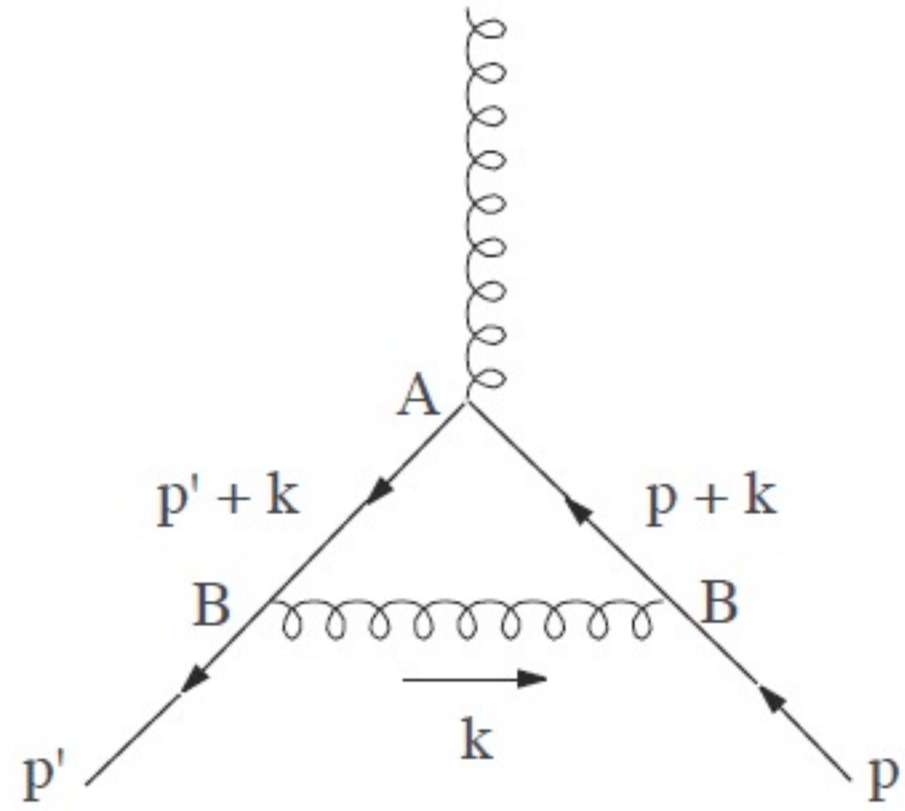
$$= \lim_{a\to 0}(-i)\pi^{\omega}\Gamma(1-\omega)(a^2)^{\omega-1} = 0$$

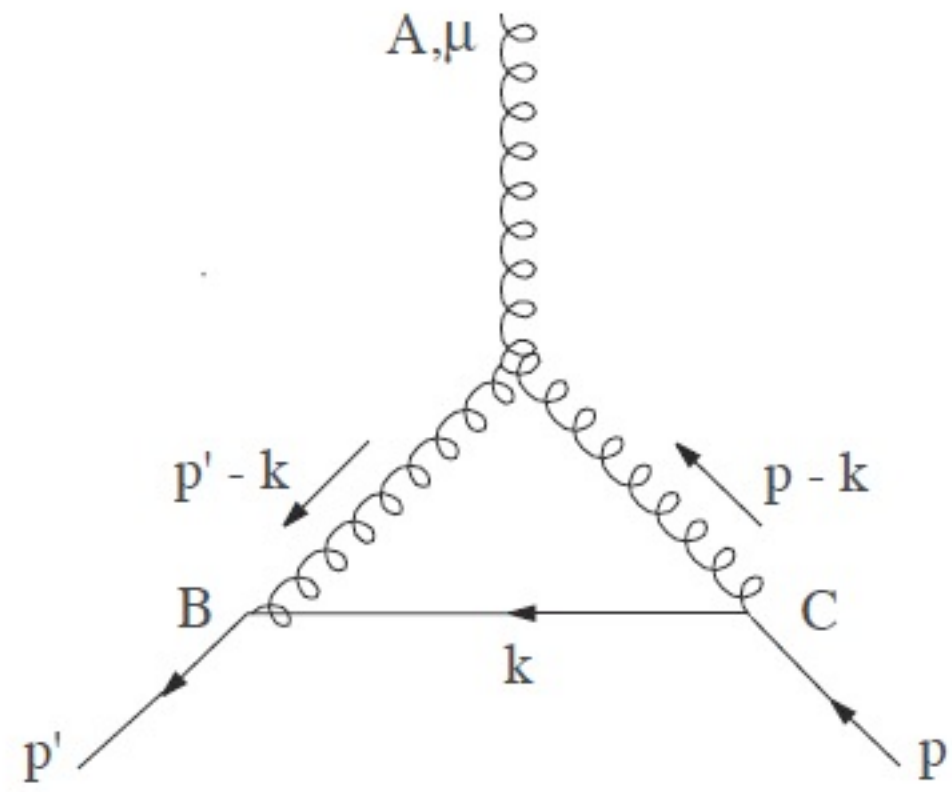
for $\omega > 1$

$$Z_3 = 1 + \frac{g^2}{8\pi^2\epsilon}\left(\frac{5}{3}C_2(G) - \frac{2}{3}n_F\right),$$

$$f^{ADC}f^{DBC} = -C_2(G)\delta_{AB}, \quad C_2(G) = N,$$

$$Tr(T^A T^B) = \frac{1}{2}n_F\delta_{AB}, \quad n_f = \#\,\text{fermion fundamental rprs.}$$





$$Z_1 = 1 - \frac{g^2}{8\pi^2\epsilon}\left(C_2(F) + C_2(G)\right)$$

Adding everything together

$$g_B = \mu^{\epsilon/2} g(1 + \Delta Z_1 - \Delta Z_2 - \frac{1}{2}\Delta Z_3) = \mu^{\epsilon/2}\left(g + \frac{a_1}{\epsilon}\right)$$

$$a_1 = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{2}{3}n_F\right)$$

Recalling the definition of the beta-function

$$\beta(g) = \mu \frac{\partial g(\mu)}{\partial \mu} = -\frac{1}{2}\left(1 - g\frac{d}{dg}\right)a_1 = -\frac{1}{2}(a_1 - 3a_1) = a_1$$

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{2}{3}n_F\right)$$

In QCD the gauge group is SU(3) and $C_2(G) = 3$

$$\beta_{QCD}(g) = -\frac{g^3}{16\pi^2}\left(11 - \frac{2}{3}n_F\right),$$

$$\beta_{QCD}(g) < 0 \text{ for } n_F < \frac{33}{2}$$

and the theory is asymptotically free.

Anomalies in QFT

We have assumed that all the symmetries valid in the classical case are preserved at the quantum level. In the path integral formalism it is clear that a symmetry of the action corresponds to a quantum symmetry only if also the functional measure of integration is invariant. We first review the Ward identities (equivalent to a classycal symmetry) in the path integral formalism. Consider an action invariant under the global symmetry

$$\phi_i \to \phi_i + \delta\phi_i, \quad \delta\phi_i(x) = -i\epsilon_A (T^A)_{ij}\phi_j(x)$$

Then, consider the same transformation but with ε = ε(x). We get

$$\delta S = \int d^4x \left[-i \frac{\partial L}{\partial \phi_i} \epsilon_A (T^A)_{ij} \phi_j - i \frac{\partial L}{\partial \phi_{i,\mu}} \epsilon_A (T^A)_{ij} \partial_\mu \phi_j - i \frac{\partial L}{\partial \phi_{i,\mu}} (T^A)_{ij} \phi_j \, \partial_\mu \epsilon_A \right]$$

Since the action is invariant under the global transformation, the sum of the first two terms vanishes. Therefore

$$\delta S = \int d^4x \, \partial^\mu \epsilon_A j_\mu^A, \quad j_\mu^A = -i \frac{\partial L}{\partial \phi_{i,\mu}} (T^A)_{ij} \phi_j \text{ (Noether's current)}$$

Consider now the generating functional

$$Z[\eta] = \int D(\phi)^{iS[\phi] + i \int d^4x \, \eta_i \phi_i}$$

and perform the change of variable, assuming the invariance of the measure

$$\phi_i \rightarrow \phi_i + \delta\phi_i = \phi_i - i\epsilon_A(x)(T^A)_{ij}\phi_j$$

$$Z[\eta] = \int D(\phi) e^{iS[\phi] + i \int d^4x \, \eta_i \phi_i} e^{i \int d^4x (\partial^\mu \epsilon_A j_\mu^A + \epsilon_A \eta_i (T^A)_{ij} \phi_j)}$$

The first order in the expansion must vanish

$$0 = \int D(\phi) e^{iS[\phi]+i\int d^4x\,\eta_i\phi_i} \left[-i\partial^\mu j_\mu^A + \eta_i (T^A)_{ij}\phi_j \right]$$

This generates all the Ward identities differentiating w.r.t. to η and putting η = 0. At the lowest order

$$\partial^\mu \langle 0 | j_\mu^A(x) | 0 \rangle = 0$$

At first order

$$\partial_x^\mu \langle 0 | T(j_\mu^A(x)\phi_i(y)) | 0 \rangle = -\delta^4(x-y)\langle 0 | (T^A)_{ij}\phi_j(y) | 0 \rangle$$

At the order N

$$\partial_x^\mu \langle 0 | T(j_\mu^A(x)\phi_{i_1}(x_1)\cdots\phi_{i_N}(x_N)) | 0 \rangle =$$

$$= \sum_{p=1}^{N} \delta^4(x-x_p)\langle 0 | T(\phi_{i_1}(x_1)\cdots(-(T^A)_{i_p j}\phi_j(x_p))\cdots\phi_{i_N}(x_N)) | 0 \rangle$$

Now consider a zero mass fermion interacting with an abelian gauge field which, for simplicity, will be considered as external, and consider the functional

$$Z = \int D\psi D\bar{\psi} e^{i\int d^4x \bar{\psi} i\gamma_\mu D^\mu \psi}, \quad D_\mu = \partial_\mu + igA_\mu$$

The action is invariant under the global transformation

$$\psi \to e^{i\alpha\gamma_5}\psi \text{ (chiral symmetry)}$$

with a Noether current (classically conserved)

$$J_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$$

Therefore

$$\delta S = \int d^4x\, \alpha(x)\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi)$$

Consider left and right eigenvectors of the Dirac operator

$$i\gamma_\mu D^\mu \phi_m(x) = \lambda_m \phi_m(x),$$

$$\tilde{\phi}_m(i\gamma_\mu \overleftarrow{D}^\mu) \equiv -iD_\mu \tilde{\phi}_m(x)\gamma^\mu = \lambda_m \tilde{\phi}_m(x)$$

For zero gauge field the eigenvalues are

$$k_\mu \to \lambda_m^2 = k_0^2 - |\vec{k}|^2 \Rightarrow -k_4^2 - |\vec{k}|^2 = -k_E^2$$

We expand in this basis the Dirac field

$$\psi(x) = \sum_m a_m \phi_m(x), \quad \bar{\psi}(x) = \sum_m b_m \tilde{\phi}_m(x)$$

with Grassmann coefficients a_m and b_m. Changing basis (barring a possible constant)

$$D\psi D\bar{\psi} = \prod_m da_m db_m$$

we can evaluate the effect of the chiral transformation on the coefficients a_m and b_m

$$\psi'(x) = (1 + i\alpha(x)\gamma_5)\psi(x) = \sum_m a'_m \phi_m(x)$$

Using the orthogonality

$$\int d^4x\, \phi_m^\dagger \phi_n = \delta_{mn}$$

$$a'_m = \int d^4x\, \phi_m^\dagger(x)\psi'(x) = a_m + \sum_n \int d^4x\, \phi_m^\dagger(x) i\alpha(x)\gamma_5 \phi_n(x) a_n =$$

$$= a_m + \sum_n C_{mn} a_n, \quad C_{mn} = \int d^4x\, \phi_m^\dagger(x) i\alpha(x)\gamma_5 \phi_n(x)$$

Changing variables (notice that we wre in the Grassman case)

$$D\psi' D\bar{\psi}' = \frac{1}{det\,|\,I\,|^2} D\psi D\bar{\psi}, \quad det\,|\,I\,| = e^{Tr\log(1+C)} \approx e^{Tr\,C}$$

$$\log det\,|\,I\,| \approx Tr\,C = i\int d^4x\,\alpha(x)\sum_n \phi_n^{\dagger}(x)\gamma_5\phi_n(x)$$

The trace of γ5 over the Dirac indices is zero, but one has to be careful since the trace is taken all over the Hilbert space. We will regularize it in the euclidean region in the following way

$$\sum_n \phi_n^{\dagger}(x)\gamma_5\phi_n(x) = \lim_{M\to\infty}\sum_n \phi_n^{\dagger}(x)\gamma_5\phi_n(x)e^{\lambda_n^2/M^2} =$$

$$\lim_{M\to\infty}\sum_n \phi_n^{\dagger}(x)\gamma_5 e^{(i\gamma_\mu D^\mu)^2/M^2}\phi_n(x) = \lim_{M\to\infty}\langle x\,|\,tr\left[\gamma_5 e^{(i\gamma_\mu D^\mu)^2/M^2}\right]|\,x\rangle$$

We have also

$$(\gamma_\mu D^\mu)^2 == D^2 - \frac{i}{2}\sigma_{\mu\nu}[D^\mu, D^\nu]_- = D^2 + \frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}$$

To get a contribution to the trace over the Dirac indices we need a term with 4 gamma-matrices

$$\sum_n \phi_n^{\dagger}(x)\gamma_5\phi_n(x) = \lim_{M\to\infty} tr\left[\gamma_5\frac{1}{2!}\left(-\frac{g}{2M^2}\sigma_{\mu\nu}F^{\mu\nu}\right)^2\right]\langle x\,|\,e^{-\Box/M^2}\,|\,x\rangle$$

Evaluating the matrix element

$$\langle x | e^{-\Box/M^2} | x \rangle = \lim_{x\to y} \int \frac{d^4k}{(2\pi)^4} e^{k^2/M^2} e^{-ik(x-y)} = i \int \frac{d^4k_E}{(2\pi)^4} e^{-k_E^2/M^2} = i \frac{M^4}{16\pi^2}$$

we get

$$\sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x) = \lim_{M\to\infty} tr\left[\gamma_5 \frac{1}{2!}\left(-\frac{g}{2M^2}\sigma_{\mu\nu}F^{\mu\nu}\right)^2\right]\langle x | e^{-\Box/M^2} | x \rangle =$$

$$= \frac{ig^2}{8\cdot 16\pi^2} \underbrace{tr[\gamma_5\sigma_{\mu\nu}\sigma_{\rho\lambda}]}_{4i\epsilon_{\mu\nu\rho\lambda}} F^{\mu\nu}F^{\rho\lambda} = -\frac{g^2}{32\pi^2}\epsilon_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda}$$

from which

$$\det |I| = e^{-i\int d^4x\,\alpha(x)\frac{g^2}{32\pi^2}\epsilon_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda}}$$

$$\Downarrow$$

$$Z = \int D\psi D\bar\psi\, e^{i\int d^4x\,\bar\psi i\gamma_\mu D^\mu\psi}\, e^{i\int d^4x\,\alpha(x)(\partial_\mu(\bar\psi\gamma^\mu\gamma_5\psi) + \frac{g^2}{16\pi^2}\epsilon_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda})}$$

After changing variable (as we did before), we get

$$Z = \int D\psi D\bar{\psi} e^{i\int d^4x \bar{\psi} i\hat{D}\psi} e^{i\int d^4x \alpha(x)(\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) + \frac{g^2}{16\pi^2}\epsilon_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda})}$$

implying that the axial current is not conserved in the quantum case

$$\partial_\mu j_5^\mu = -\frac{g^2}{16\pi^2}\epsilon_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda}$$