Exercises
Advanced Quantum Field Theory
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For most of the exercises see
R. Casalbuoni, Lecture notes in
Advanced Quantum Field Theory,
http://theory.fi.infn.it/casalbuoni/lezioni99.pdf

Lecture 1

1) Evaluate the Feynman amplitude in the free case and prove that it is given by:

$$\langle q',t'|q,t\rangle = \left[\frac{m}{2\pi i(t'-t)}\right]^{\frac{1}{2}} e^{iS_{\rm cl}}$$

where

$$S_{\rm cl} = \frac{m}{2}\frac{(q'-q)^2}{(t'-t)}$$

is the classical action for the free particle (see lecture 1).

2)The wave function is determined in this formalism as the amplitude for going to a point starting from a fixed one

$$\psi(q',t') = \langle q',t' | 0,0 \rangle$$

In the free case evaluate the wave function in momentum space and show that

$$\psi(p,t) = e^{-iEt}, \quad E = \frac{p^2}{2m}$$

3) Show that the state $| q,t \rangle$

such that (the operator is in the Heisenberg representation)

$$q(t) | q,t \rangle = q | q,t \rangle$$

is given by (q is in the Schrodinger representation)

$$| q,t \rangle = e^{iHt} | q \rangle, \quad q | q \rangle = q | q \rangle$$

4) Prove the Feynman formula in configuration space given in page 13.

5) Evaluate the functional derivatives of the functionals defined in the lectures

$$F_1[\eta] = \int_{-\infty}^{+\infty} dx\, w(x)\eta(x) \qquad F_2[\eta] = e^{-\frac{1}{2}\int dx\eta^2(x)}$$

$$F_3[\eta] = \eta(x_0)$$

and of $$F_4[\eta] = e^{-\frac{1}{2}\int dxdyK(x,y)\eta(x)\eta(y)}$$

6) If $S[q] = \int_t^{t'} L(q,\dot{q})dt$ with $\delta q(t) = \delta q(t') = 0$

prove that $$\frac{\delta S}{\delta q(t)} = \frac{\partial L}{\partial q(t)} - \frac{d}{dt}\frac{L}{\partial \dot{q}(t)}$$

Lecture 2

1) Assuming that the set of particles $\beta = \gamma, p'$ prove the following equation using the procedure outlined in the main text (see lecture 2, pages 13-14)

$$\langle \beta; \text{out} | \alpha, p; \text{in} \rangle = \langle \beta - p; \text{out} | \alpha; \text{in} \rangle +$$

$$+ \frac{i}{\sqrt{Z}} \int d^4x f_{\vec{p}}(x) (\Box + m^2)_x \langle \gamma; \text{out} | \phi(x) | \alpha - p'; \text{in} \rangle +$$

$$+ \left(\frac{i}{\sqrt{Z}} \right)^2 \int d^4x \, d^4y \, f_{\vec{p}}(x) f^*_{\vec{p}'}(y) (\Box + m^2)_x (\Box + m^2)_y \langle \gamma; \text{out} | T(\phi(y)\phi(x)) | \alpha; \text{in} \rangle$$

2) For a free field, evaluate the momentum operator and show that

$$[P_\mu, \varphi(x)] = -i\partial_\mu \varphi(x), \quad [P_\mu, a(k)] = -k_\mu a(k)$$

where a(k) is the annihilation operator

3) Evaluate the expression for the action in normal coorfinates given at page 20 in Lecture 2.

4) Evaluate $G_0^{(4)}$ in the free case, see lecture 2, page 24.

5) Show that W[J] (page 26, lecture 2) generates the connected Green functions (see page 127 of Casalbuoni's lecture notes in the bibliography)

6) Evaluate δ_1 given in page 30, lecture 2

Lecture 3

1) Derive thee expression for the normalization factor N(θ,θ*) given in Lecture 3, page 7.

2) Prove the translational invariance of the Grassmann integral, ;ecture 3, page 10.

3) Show that a linear invertible change of the generators does not change the algebraic relations defining a Grassmann algebra (see lecture 3, page 11)

4) Show that a real antisymmetric matrix A can be put in the canonical form given il lecture 3, page 14, through an orthogonal transformation.

5) Using the result of the previous exercise prove the following equation

$$I = \int d\theta_n \cdots d\theta_1 e^{\theta^T A\theta/2} = \lambda_1 \cdots \lambda_{\frac{n}{2}} = \sqrt{\det A}$$

6) In a gauge theory consider the gauge invariant operator

$$F[A^\Omega] = F[A]$$

and its expectation value in a given gauge f.

$$\langle 0 | T^*(F[\mathbf{A}]) | 0 \rangle_f = \int D(A_\mu)\delta[f]\Delta_f e^{iS} F[A]$$

Show that this expression does not depend on the choice of the function f.

Lecture 4

1) Derive the Feynman rules for a non abelian gauge theory, using the functional formalism (see lecture 4, pages 4 and following.

2) In the context of regularization of the scalar theory derive the expressions given in Lecture 4 (pages 23-24) for T_2 and T_4.

3) Perform explicitly the valculations about the one-loop renormalization of the scalar theory discussed in Lecture 4, pages 30-35)

4) Perform the perturbative expansion at second order discussed Lecture 2 and derive the diagrammatic expansion given in Lecture 4, page 36 and the expressions of page 37.

Lecture 5

1) Derive the expression for the 2 and 4 points Γ functions at given in Lecture 5, page 9, and determine the arbitrary functions F and G in the renormalization schemes A and B in Lecture 5, page 10.

2) In the frameworkof QED derive the expression for the bare electric charge given in lecture 5, page 26 (assume the validity of the Ward identity $Z_1 = Z_2$.

3) Repeat the same exercise as in 2) for QCD, that is evaluate g_B.

4) Check that the Ward identity derived, at first order, in lecture 5, page 35 can be derived from the equal time commutation relation

$$[j_0^A(\vec{x},t),\varphi_i(\vec{y},t)] = -\delta^3(\vec{x}-\vec{y})(T^A)_{ij}\varphi_j(\vec{x},t)$$