inflation; generation and evolution of perturbations

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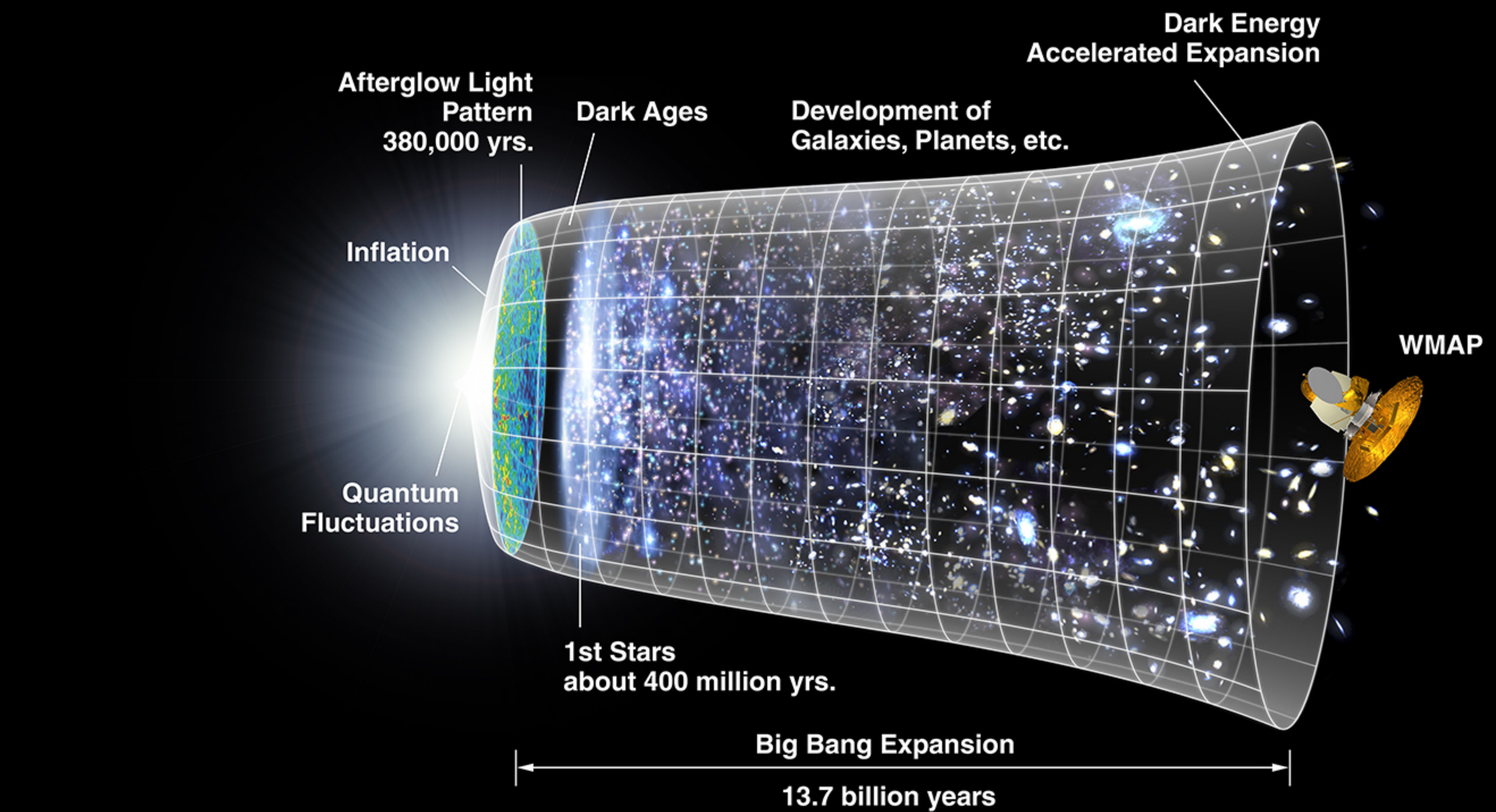
Overview

- **Problems of the standard model …**
- **… and how to solve them**
  - inflation with scalar fields
  - generation of perturbations
  - predictions and constraints
- **The evolution of cosmological perturbations**
  - cosmological perturbation theory
  - perturbation evolution
  - the (dark) matter power spectrum
  - special focus: BAO and neutrino masses
- **Summary**

Brief history of the Universe



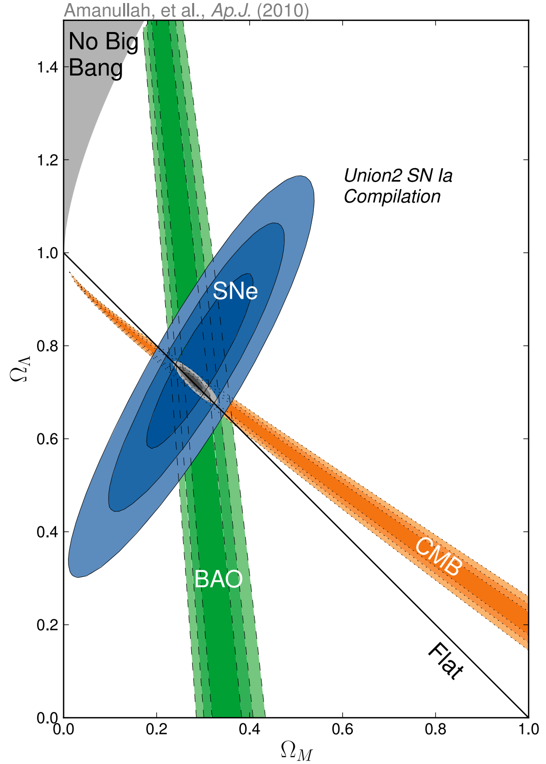
why is the world flat?

Monday:

$$\frac{d}{dt}\underbrace{\left(\frac{\Omega-1}{\kappa}\right)}_{|\Omega-1| \quad (\kappa\neq 0)} = \frac{d}{dt}\frac{1}{\dot{a}^2} = -2\frac{\ddot{a}}{\dot{a}^3}$$

>0 for expanding universe filled with dust or radiation (and k ≠ 0)
-> the universe becomes “less flat”
-> Ω=1 is an unstable fix-point

Supernova Cosmology Project
Amanullah, et al., *Ap.J.* (2010)



following the evolution back in time, we find that (during radiation domination, i.e. before t_{eq})

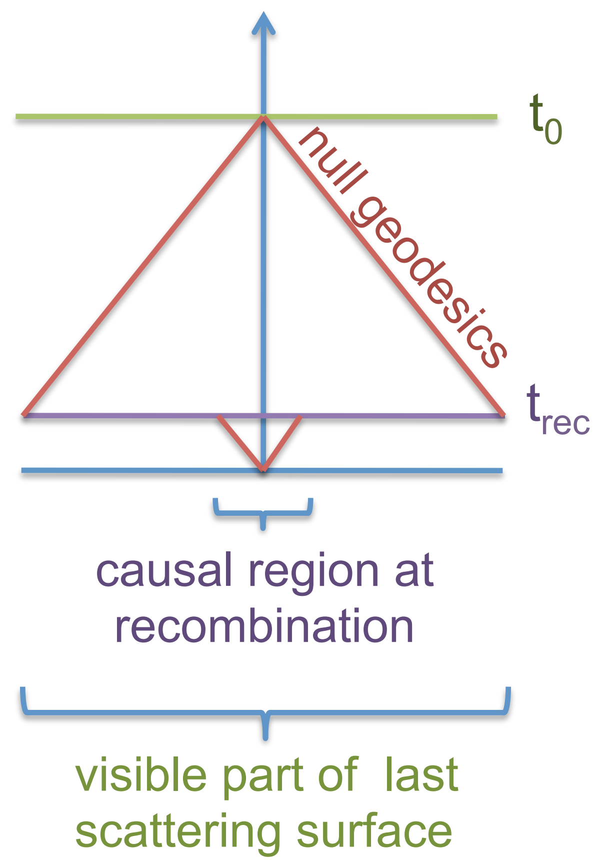
$$|\Omega(t)-1| \approx 10^{-4}\left(\frac{1\text{eV}}{T}\right)^2$$

BBN: T ≈ 1 MeV -> $|\Omega\text{-}1| < 10^{-16}$
Planck: T ≈ 10^{19} GeV -> $|\Omega\text{-}1| < 10^{-60}$

-> what fine-tuned the initial conditions?

why is the sky uniform?

- distance travelled by light: $$r = \int \frac{dt}{a(t)}$$ (= conformal time)
- distance to last scattering surface: $$r_0 = \int_{t_\mathrm{rec}}^{t_0} \frac{dt}{a(t)} \approx 3t_0$$
- distance travelled from big bang to recombination: $$r_c = \int_0^{t_\mathrm{rec}} \frac{dt}{a(t)}$$

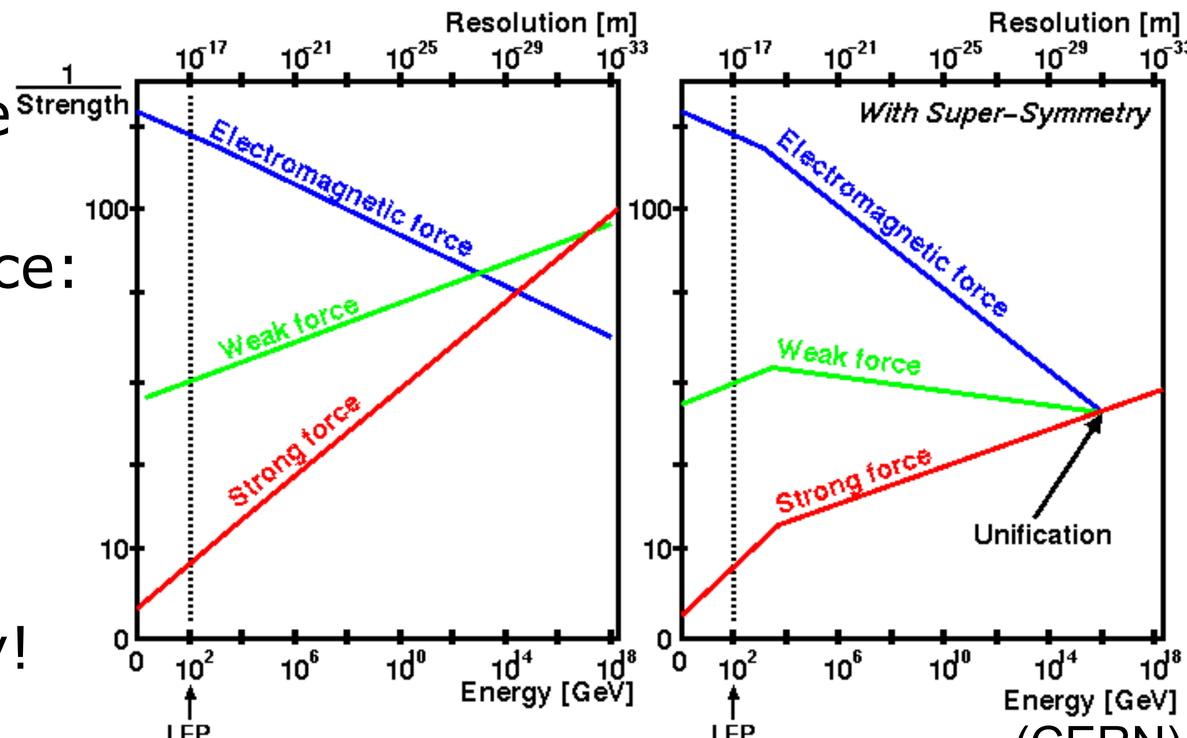


in general $r_c << r_0$, unless $a(t) \sim t^a$ with $a \geq 1 \Leftrightarrow w \leq -1/3$!

since $$a(t) \propto t^{2/(3+3w)}$$

where did all the monopoles go?

- monopoles form when a simple group G is broken to sub-groups containing U(1)
- typical scenario: GUT with unification $\sim 10^{16}$GeV
- typically form one monopole per Hubble volume, mass $\sim 10^{16}$ GeV
- find $\Omega_{MP} \sim 10^{12}$ due to huge mass
- no long-range force: no annihilation
- $\rho \sim 1/a^3$ (like matter)
- can dilute them if a(t) grows quickly!



(CERN)

how to solve the problems

all the problems disappear if $\ddot{a} > 0$ for long enough!

Since $\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G_N}{3}(\rho + 3p)$ this needs $p < -\rho/3$

We have seen that for Λ : $p = -\rho$, but forever

-> we need a way to have evolving eq. of state

Solution: use a field … what kind of field? When in doubt, try a scalar field ☺

scalar fields in cosmology

GR + scalar field:

$$S = S_g + S_\phi = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G} + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right)$$

gravity e.o.m. (Einstein eq.):

$$\frac{\delta S[g_{\mu\nu},\phi]}{\delta g^{\mu\nu}} = 0 \qquad G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

entries in scalar field EM tensor (FLRW metric)

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

scalar field e.o.m. :

$$\frac{\delta S[g_{\mu\nu},\phi]}{\delta\phi} = 0 \qquad \ddot{\phi} + 3H\dot{\phi} + dV(\phi)/d\phi = 0$$

- this is the general method to compute Einstein eq., EM tensor and field e.o.m. from any action
- w=p/ρ for scalar fields can vary, as a function of V(ϕ)

the inflaton eq. of state

$$\left.\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{aligned}\right\}$$

$\dot{\phi}$ small -> p ≈ -ρ, w ≈ -1 (**slow roll**)

$\dot{\phi}$ large -> p ≈ ρ, w ≈ +1

=> slow roll is just what we need

slow-roll approximation: $3H\dot{\phi} = -V' \quad H^2 = \frac{1}{3m_P^2}V$

slow-roll parameters: $\epsilon(\phi) \equiv \frac{m_P^2}{2}\left(\frac{V'}{V}\right)^2 \quad \eta(\phi) \equiv m_P^2 \frac{V''}{V}$

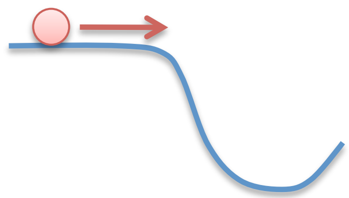
$\epsilon \ll 1\,, \quad |\eta| \ll 1$ for slow-roll -> flat pot.

SR approx => $\dot{\phi}^2 = \frac{2}{3}\epsilon V \Rightarrow p = \left(\frac{2}{3}\epsilon - 1\right)\rho \;\longrightarrow \ddot{a} > 0 \leftrightarrow \epsilon < 1$

(first order in ε)

prototypical inflation models

- small field



e.g. $V = V_0\,[1-(\phi/\mu)^\alpha]$, $\alpha = 2,4,\ldots$
original inflation: 1st order phase transition -> exit problem

- chaotic / large field

e.g. $V = m^2\phi^2$ or
$V \sim \phi^4$
also eternal inflation models

- hybrid / multifield

- curvaton, N-flation, cyclic models, …

-> large number of inflation scenarios
-> not all work // initial conditions generally problematic

the duration of inflation

"number of e-foldings": N ~ ln(a)

$$\frac{d}{dt}\left(\frac{\Omega - 1}{\kappa}\right) = \frac{d}{dt}\frac{1}{\dot{a}^2} = -2\frac{\ddot{a}}{\dot{a}^3}$$ and SR: $$a(t) = \exp(Ht), \quad H = \sqrt{\frac{\Lambda}{3}}$$

⇒ $|\Omega-1| \sim 1/a^2$ during slow roll inflation

⇒ we need 20 (BBN) to 70 (Planck-scale) e-foldings to achieve necessary flatness (typically 40-60)

⇒ also sufficient to solve horizon problem and to dilute monopoles

(for horizon problem: need $N_{inf} \sim N_{post\text{-}inf}$, which also is between 30 e-foldings (BBN) and 60 e-foldings (GUT))

example

chaotic inflation: $V(\phi) = \frac{1}{2} m^2 \phi^2$

slow-roll equations: $3H\dot{\phi} + m^2\phi = 0$ $\quad H^2 = \frac{m^2}{6m_P^2}\phi^2$

slow-roll parameters: $\epsilon = \eta = \frac{2m_P^2}{\phi^2}$ -> $|\phi_f| = \sqrt{2} m_P$

# of e-foldings: $N = \int_{a_i}^{a_f} \frac{da}{a} = \int_{t_i}^{t_f} H dt = \ldots = -\frac{1}{m_P^2} \int_{\phi_i}^{\phi_f} \frac{V}{V'} d\phi = \frac{\phi_i^2}{4m_P^2} - \frac{1}{2}$

solution of SR eqn's $\phi(t) = \phi_i - \sqrt{\frac{3}{2}} m m_P t$

$$a(t) = a_i \exp\left\{ \frac{m}{\sqrt{6} m_P} \left(\phi_i t - \frac{m m_P}{\sqrt{6}} t^2 \right) \right\}$$

reheating the universe

after many e-foldings of inflation, the universe is very empty and cold; but we want a radiation-dominated universe!?

-> **reheating**: convert energy in inflaton field to radiation!

- after end of inflation: inflaton oscillates at bottom of potential -> will decay into other particles if coupling non-zero. Usually modelled as dissipative term $\Gamma\dot{\phi}$

  "cold" inflation: $\Gamma < H$ during inflation, ρ_ϕ -> ρ_γ when $\Gamma \sim H$

  at that time: $\Gamma^2 \approx H^2 = \frac{1}{3m_P^2}\rho_\phi \Rightarrow \rho_\gamma \approx 3m_P^2\Gamma^2$, ~10^{10} GeV

  (warm inflation: $\Gamma \sim H$ always -> smooth transition)

- however: oscillating inflaton -> oscillating effective mass of coupled fields -> parametric resonance, "**pre-heating**"

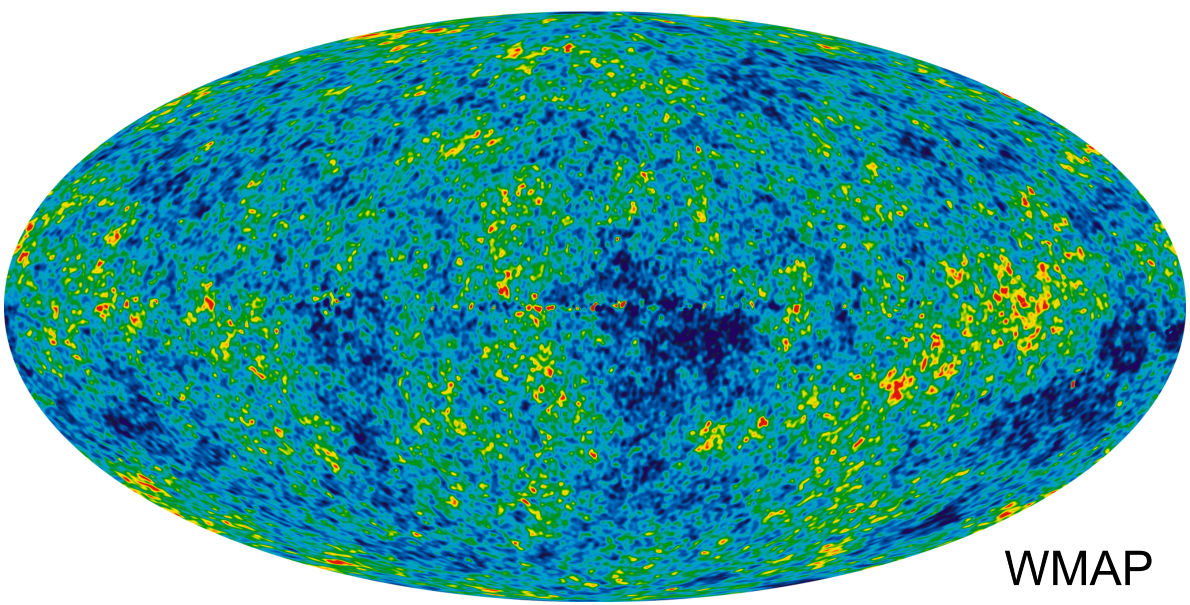
  e.g. coupling $-\frac{1}{2}g^2\chi^2\phi^2$ -> eom $\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + g^2\Phi^2\sin^2(m_\phi t)\right)\chi_k = 0$

(lots of nice particle physics to be found here! ☺)

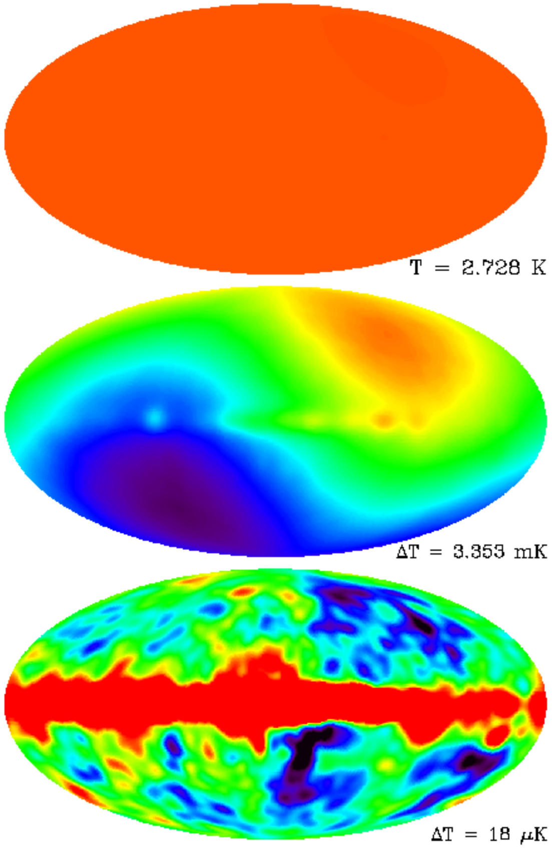
anisotropies in the CMB

Actually, there is another problem in standard cosmology:

- galaxies would not have formed yet from thermal fluctuations alone
- we see fluctuations at high redshift directly in the CMB [-> Friday]



WMAP



COBE

the smallest and the largest

Inflation has another amazing property:

- SR inflation ~ de Sitter space-time -> horizon
- horizon -> Hawking radiation -> particle creation!

⇒ inflation should create perturbations! ☺

⇒ quantum fluctuations are stretched to huge scales and become classical curvature perturbations

⇒ ***the largest structures in the universe are due to quantum fluctuations!!!???***

- what kind of perturbations?
- can we see them?
- what do they tell us about inflation?

inflationary perturbations

- write ϕ(x,t) = ϕ(t)+δϕ(x,t)
- linearize eom for δϕ, V′(ϕ+δϕ) -> V′(ϕ)+δϕV′′(ϕ)
- Fourier-expansion with creation & annihilation op's for δϕ

$$\ddot{w}(k,t) + 3H\dot{w}(k,t) + \left(\frac{k^2}{a^2} + 3\eta H^2\right) w(k,t) = 0$$

- Horizon: k/a=H: neglect η << 1 during SR (-> corrections)

$$w(k,t) = -H\frac{k\tau - i}{k}e^{-ik\tau}$$

- compute fluctuation spectrum

can be neglected

$$\int d^3x \langle 0|\delta\phi^2(x,t)|0\rangle \rightarrow \int \frac{d^3k}{(2\pi)^3 2k}|w(k,t)|^2 = \int \frac{dk}{k}\left(\frac{H}{2\pi}\right)^2 (1 + k^2\tau^2)$$

$k^3\, P_{\delta\phi}(k)$: scale invariant spectrum!

- phases of $\delta\phi_k$ random -> Gaussian fluctuations

cosmological perturbations

- inflaton will decay, but perturbations are frozen into metric
- can use Poisson eq. $\Delta\Phi = 4\pi G\, \delta\rho_\phi$ for grav. pot. Φ
- in terms of curvature perturbation *R*:

$$k^3 P_{\mathcal{R}}(k) = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \propto \frac{V}{\epsilon} \quad \text{(eval. at k = aH)}$$

- power-law ansatz $k^3 P \sim (k/k^*)^{n-1}$ -> $n-1 = (d \ln P)/(d \ln k)$

$$n_s - 1 = -6\epsilon + 2\eta$$

- **nearly scale invariant**, models make different predictions!

- There is another degree of freedom: **gravitational waves!**
- accelerated expansion will ***necessarily*** also create a gravitational wave background!

$$P_g(k) \propto \left(\frac{H}{2\pi}\right)^2 \qquad n_g = -2\epsilon \qquad r = \frac{T}{S} \approx 12.4\epsilon = -6.2 n_g$$

$V \sim \phi^2$ example continued

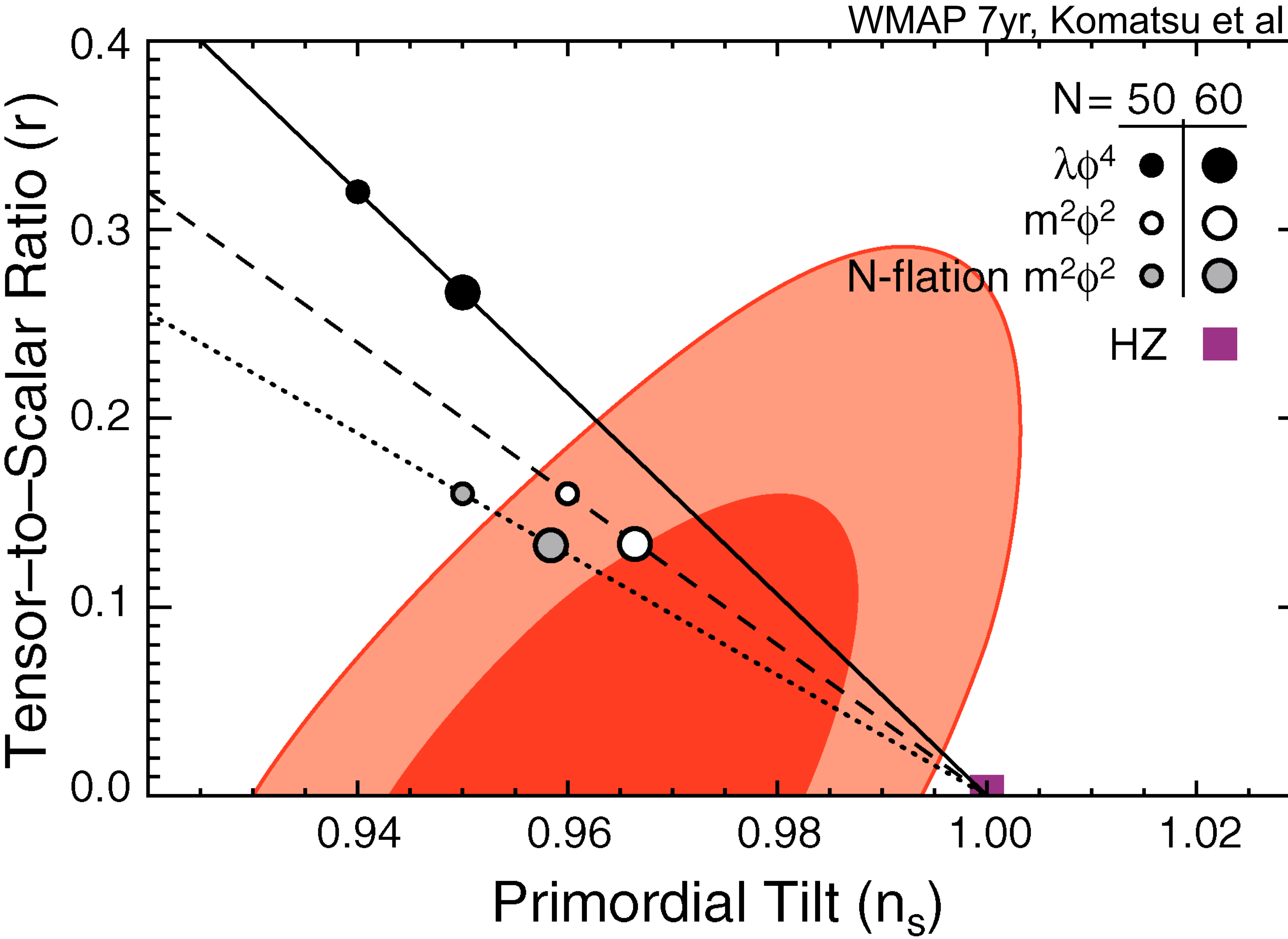
slow-roll parameters: $\epsilon = \eta = \dfrac{2m_P^2}{\phi^2}$

# of e-foldings: $N \approx \dfrac{\phi_i^2}{4m_P^2}$

so $n_s - 1 = -6\varepsilon + 2\eta = -4\varepsilon = 8m_P^2/\phi = -2/N \approx -0.04$

$r = T/S \approx 12.4\varepsilon = -3.1\,(n_s - 1)$ -> potentially observable

constraints on inflation



generic predictions of inflation

- universe large and nearly flat

  -> **okay**
- nearly (but not quite) scale-invariant spectrum of adiabatic perturbations

  -> **okay** [killed defects]
- (nearly) Gaussian perturbations

  -> **okay** [deviations -> constrain models]
- perturbations on all scales, including super-horizon

  -> **okay** [kills all “causal” sources of perturb.]
- primordial gravitational waves **HOT TOPIC**

  -> nothing yet -> “smoking gun” for acc. exp.

beyond SR inflation

- **single-field slow roll inflation**: nearly scale invariant adiabatic Gaussian perturbations
- more general models: can create
  - **non-Gaussianity**
  - **isocurvature perturbations**
  - **features in the power spectrum**
- realistic (multi-field) models often form **cosmic strings** at the end of inflation
- if detected, such signatures would give important information on fundamental physics of inflation!

HOT TOPIC

evolution of the perturbations

- From inflation we have a nearly scale invariant spectrum of perturbations…
  - how will they evolve?
  - what do we observe today?

-> **matter power spectrum**

- compute evolution of density perturbations of the dark matter and baryons

-> **CMB power spectrum**

- compute evolution of the perturbations in the radiation

k-space, power spectra

We tend to use ‘k’-space (Fourier space):

- only perturbations have spatial dependence, so that linear differential eqn’s -> ODE’s in time
- ‘scales’ instead of ‘location’

physical wavelength vs comoving wave number: $\lambda = \frac{2\pi a(t)}{k}$

Fluctuations are random

- need a statistical description -> power spectrum
- power spectra: $P(k) = \langle|\text{perturbations}(k)|^2\rangle$
- <…> : average over realisations (theory) or over independent directions or volumes (observers)
- Gaussian fluctuations -> P(k) has full information

perturbation theory

basic method:

- set $g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu}$ $\quad T_\mu^\nu = \bar{T}_\mu^\nu + \delta T_\mu^\nu$
- stick into Einstein and conservation equations
- linearize resulting equation (order 0 : "background evol.")

⇒ two 4x4 symmetric matrices -> 20 quantities

⇒ we have 4 extra reparametrization d.o.f. -> can eliminate some quantities ("gauge freedom")

⇒ at linear level, perturbations split into "scalars", "vectors" and "tensors", we will mostly consider scalar d.o.f.

$$ds^2 = -(1+2\psi)dt^2 + a^2(1-2\phi)dx^2$$

⇒ do it yourself as an exercise ☺

scalar perturbation equations

Einstein equations:
r.h.s. summed over "stuff" in universe

δ = δρ/ρ density contrast
V divergence of velocity field

$$k^2\phi = -4\pi G a^2 \sum_i \rho_i \left(\delta_i + 3Ha\frac{V_i}{k^2}\right)$$

$$k^2(\phi - \psi) = 12\pi G a^2 \sum_i (1+w_i)\rho_i\sigma_i$$

conservation equations:
one set for each type (matter, radiation, DE, …)

$$\delta_i' = 3(1+w_i)\phi' - \frac{V_i}{Ha^2} - \frac{3}{a}\left(\frac{\delta p_i}{\rho_i} - w_i\delta_i\right)$$

$$V_i' = -(1-3w_i)\frac{V_i}{a} + \frac{k^2}{Ha}\left(\frac{\delta p_i}{\rho_i} + (1+w_i)(\psi - \sigma_i)\right)$$

w, δp, σ: determines physical nature, e.g. cold dark matter: w=δp=σ=0

$$\delta_m' = 3\phi' - \frac{V_m}{Ha^2} \qquad V_m' = -\frac{V_m}{a} + \frac{k^2}{Ha}\psi$$

perturbation evolution

We can eliminate V and obtain a second order eqn for δ,

$$\ddot{\delta}_i = -\alpha_i H \dot{\delta}_i + \left(\mu_i H^2 - \frac{c_{s,i}^2 k^2}{a^2} \right) \delta_i$$

α_i, μ_i depend on w_i, c_s^2 is sound speed (<-> δp), 1/3 for radiation, 0 for matter

- α-term: **expansion damping**, may suppress growth
- last term: gravitational collapse vs pressure support

  -> will prevent growth if c_s k > Ha   -> **sound horizon**

  -> with H2 = 8πGρ/3 we have the **Jeans length** λ_J = cs/(√Gρ)

- straightforward to analyze behaviour of matter, radiation, etc as function of scale (horizon, Jeans-length) and of background evolution (radiation or matter dominated).

perturbation evolution

period	scale	CDM	radiation	baryons
$t < t_{eq}$	$k < aH$	grows $\sim a^2$	grows $\sim a^2$	grows $\sim a^2$
$t > t_{eq}$	$k < aH$	grows $\sim a$	grows $\sim a$	grows $\sim a$
$t < t_{eq}$	$k > aH$	~ constant (ln a)	oscillates	oscillates
$t_{eq} < t < t_{dec}$	$k > aH$	grows $\sim a$	oscillates	oscillates
$t_{dec} < t$	$k > aH$	grows $\sim a$	free-streams	grows $\sim a$

CDM: inside horizon grows only after matter-radiation equality -> scale imprinted in power spectrum where power-law will change!

radiation: oscillates, then free-streams after decoupling -> oscillations remain imprinted in power spectrum -> acoustic oscillations in CMB!

baryons: oscillate with photons until decoupling, then fall into CDM potential wells -> small imprint of acoustic oscillations also in matter power spectrum -> BAO

initial conditions and P(k)

HOT TOPIC

(& growth rate, redshift-space distortions, non-linear growth)

Inflationary spectrum: $\delta(k,t_{enter}) \sim k^{n/2-2}$

scales entering before t_{eq}: $\lambda<\lambda_{eq}$

$$\delta_\lambda(t) \simeq \delta_\lambda(t_{enter})(a/a_{eq})$$

scales entering after t_{eq}: $\lambda>\lambda_{eq}$

$$\delta_\lambda(t) = \delta_\lambda(t_{enter})(a/a_{enter})$$

$$= \delta_\lambda(t_{enter})(a/a_{eq})(a_{eq}/a_{enter})$$

horizon: $t_{enter} = \lambda\, a_{enter} \sim \lambda t_{enter}^{2/3}$

$$\rightarrow (a_{eq}/a_{enter}) = (\lambda_{eq}/\lambda)^2$$

in terms of k:

scales entering before teq:

$$|\delta_k(t)|^2 \propto k^{n-4}(a/a_{eq})^2$$

scales entering after teq:

$$|\delta_k(t)|^2 \propto k^{n}(a/a_{eq})^2$$

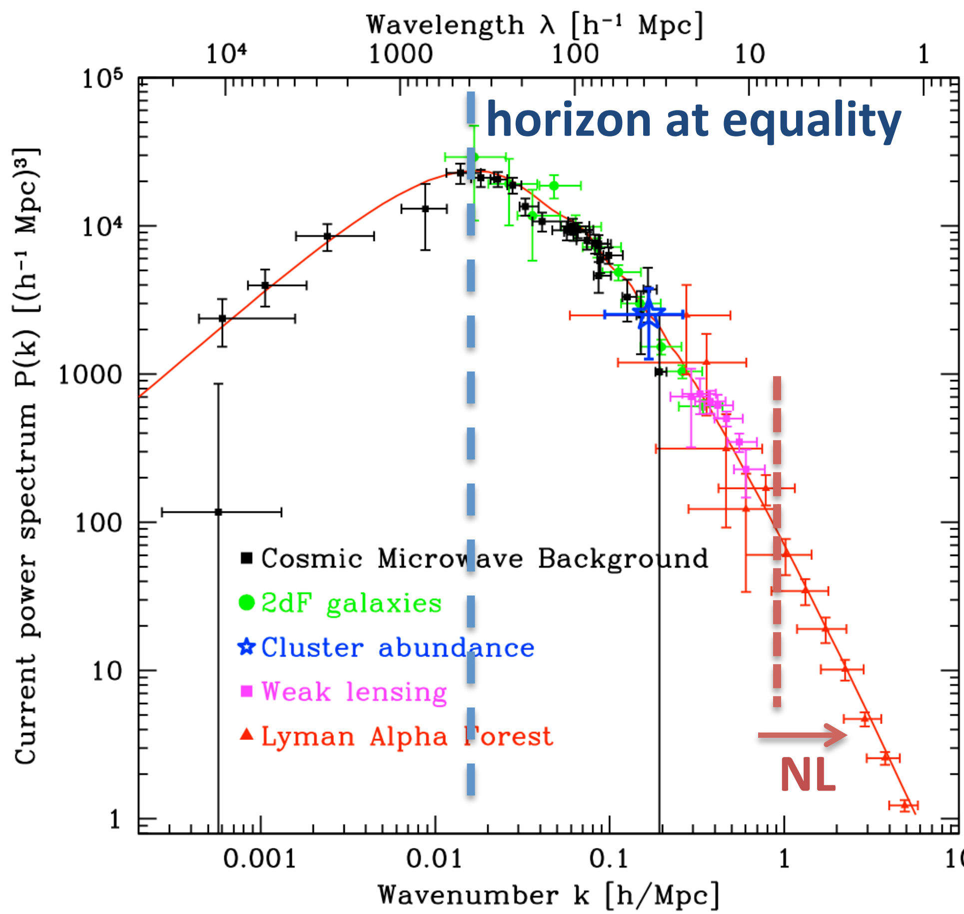
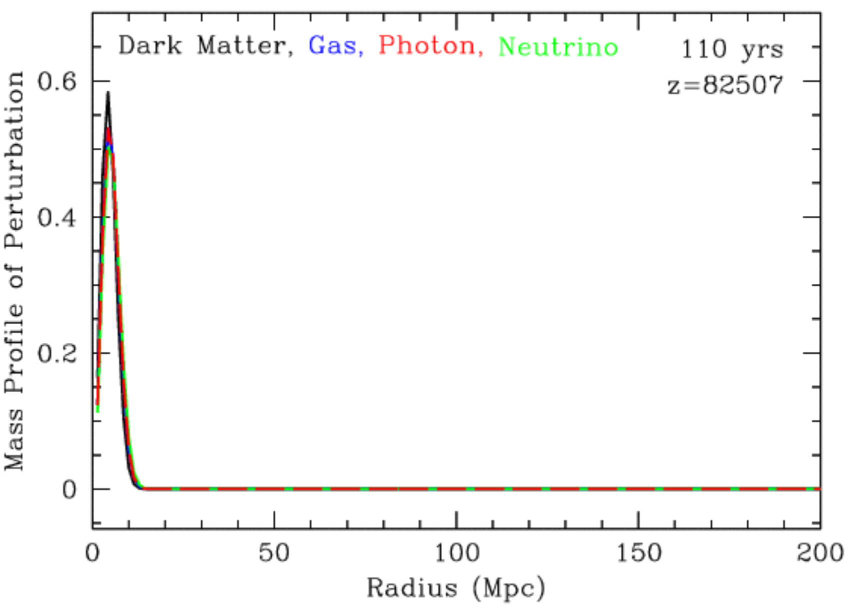


figure from Tegmark & Zaldarriaga

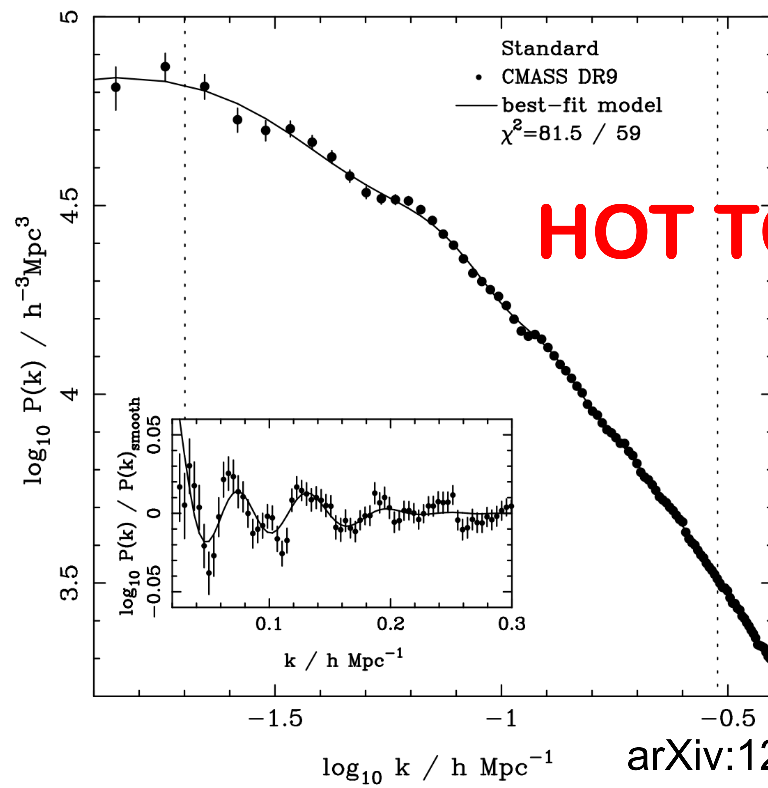
BAO

- On sub-horizon scales, the baryon-photon fluid oscillates until t_{dec}
- After t_{dec}, the photons free-stream away, and the baryons fall into the potential wells of the cold dark matter
- But the CDM also falls a bit into the baryon potential wells
- This imprints the oscillations also into the matter power spectrum

-> Baryonic Acoustic Oscillations feature -> **standard ruler!**



animation by Eisenstein, uses cmbfast



HOT TOPIC

arXiv:1203.6594

neutrino free-streaming

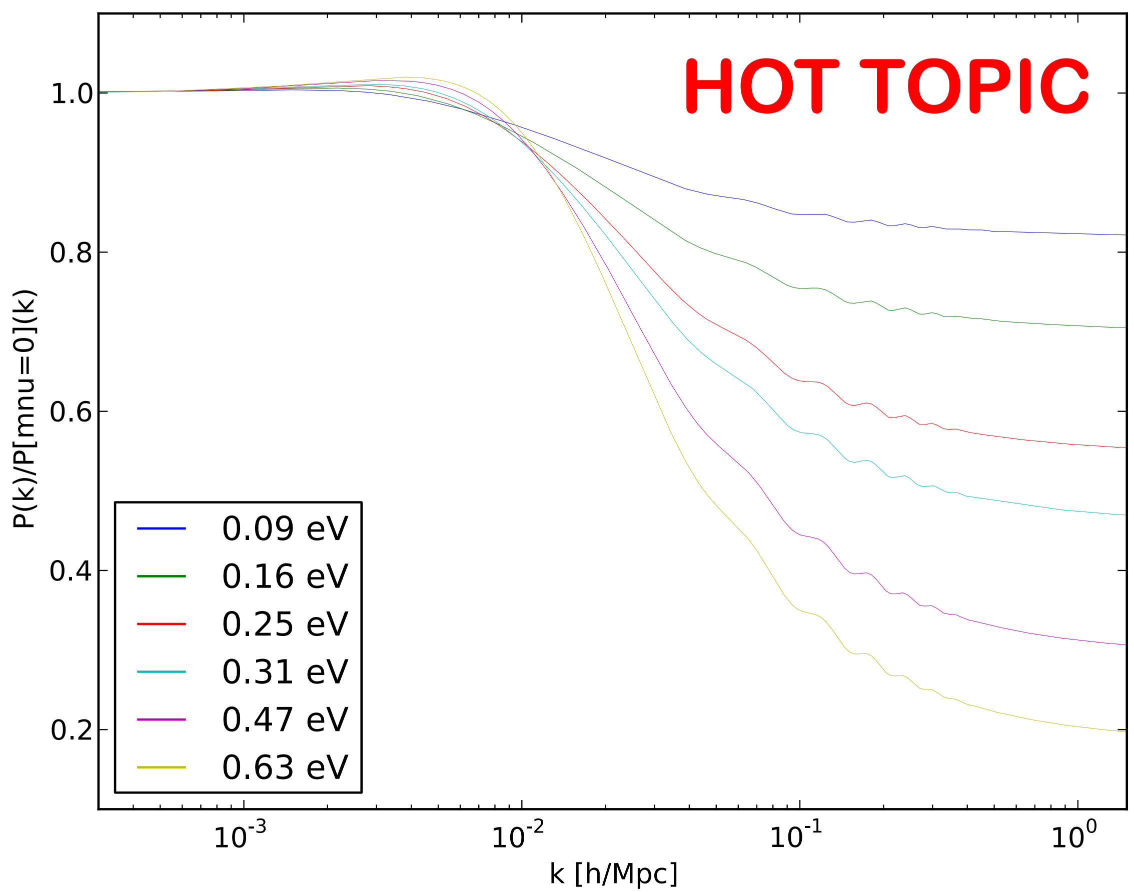
- Massive neutrinos change background evolution
- Neutrinos also free-stream out of overdensities
- this suppresses clustering on small scales

Large-scale structure surveys are looking for this change in P(k)!

Will soon probe the inverted hierarchy

By 2025 Euclid should probe $m_\nu \approx 0.02$ eV

the hunt is on!



Summary

- **standard cosmology is incomplete**
- **inflation solves some problems**
  - flattens the universe, dilutes monopoles, extends horizon
  - creates the right kind of perturbations (usually)
  - very hard to avoid (in some way or other)
  - smoking gun would be primordial gravitational waves
- **perturbations and their evolution**
  - many observables are based on perturbations …
  - … and the perturbations carry much information!
  - main scale in matter perturbations: equality
  - also BAO: sound horizon at decoupling (<-> CMB)