1 Physics at different scales and renormalization group approach

1.1 Introductory remarks

Every experiment is restricted by a maximal momentum transfer which could be achieved or, equivalently, by a minimal distance which could be probed. Consider, for example, Newtonian mechanics, which describes the motion of the bodies at low momenta. In particular, the momentum transfer in the interaction process should be small enough so that the bodies can not penetrate each other and the atomic structure of these bodies can not be seen. Moreover, such an information *is not needed:* two bodies with a very different inner structure but with the same mechanical characteristics like the mass, the moments of inertia, etc, obey exactly the same equations of motion.

Imagine now that the momentum transfer in the experiment is increased so that the distance which can be studied (which is inversely proportional to the momentum transfer) becomes of the order of the size of an atom. Then, the theory which ignores the existence of atoms and describes a body in terms of few mechanical characteristics, can not be valid any more. It should be replaced by a more complicated theory that embeds inter-atomic interactions. At low momenta, the new theory will smoothly cross over into the old one. On the other hand, the new theory still ignores all physics on sub-atomic scales, like the existence of the nuclei which consist of the protons and neutrons which, in turn, consist of quarks and gluons and so on. In the theory, valid at the atomic scales, this sub-atomic information is not needed.

It is crystal-clear that this formal theoretical argument can be taken to any momentum/energy. The theory that describes interactions at a given energy and below, *should not* depend on the dynamics at higher energies. One may even loosely argue that such a behavior is a necessary condition of the existence of physics as an exact science. Namely, were this not the case, then, in order to describe the phenomena at the energies experimentally achievable at present, one would need information at all energies, which can not be obtained in principle.

In the context of field theory, the above discussion can be re-formulated as follows. Suppose, only the momentum transfers $|\mathbf{Q}| \leq \Lambda$ with a given Λ are available experimentally (the quantity Λ will be referred to as a "hard scale" hereafter). Consequently, only the distances $r \geq \Lambda^{-1}$ are probed. Since the Compton wavelength of a massive particle is proportional to the inverse of its mass, the relevant *degrees of freedom* for an *effective field theory*, which describes the processes with $|\mathbf{Q}| \leq \Lambda$, are only those particles whose masses are smaller than Λ. If in the *underlying theory*, which is valid at higher momentum transfers, there are particles with $M \gg \Lambda$, their presence *should not* be explicitly felt in the effective theory which is valid at the momentum transfers below Λ. In other words, in the effective theory these high-mass particles are *integrated out.*

Here one should also point out that, within field theory, the above arguments are very subtle. This happens because, e.g., in perturbation theory, the observables can be expressed in terms of Feynman loop diagrams where the integration is carried out to infinity. Consequently, contributions from all energies are present even in the low-energy observables. In this case, one should properly define, what one means under the statement that the low-energy/large distance

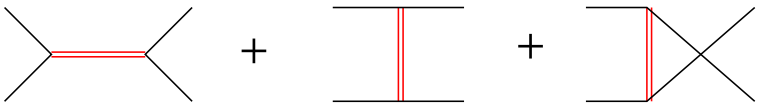

Figure 1: The tree-level scattering amplitude for the process $\phi\phi \to \phi\phi$ in the model described by the Lagrangian given in Eq. (1). Single and double lines correspond to the light and heavy fields, respectively.

physics *does not depend* on the details of the high-energy/short-distance dynamics. This is the central issue of Effective Field Theory (EFT).

In the Nature, there are numerous examples of effective field theories. Say, Chiral Perturbation Theory (ChPT) is a low-energy effective theory of QCD, the dynamics of the Standard model of electroweak interactions at low energy is described by QED plus the Fermi-theory of weak interactions, and so on. In this lecture course, we shall be explicitly interested in the above cases. However, before addressing the issue in full glory, we shall illustrate the general pattern first within a simple model.

1.2 Integrating out a heavy scale: the model at tree level

Let us consider a model described by the Lagrangian [1, 2]

$$\mathcal{L} = \frac{1}{2}\,(\partial\phi)^2 + \frac{1}{2}\,(\partial\Phi)^2 - \frac{m^2}{2}\,\phi^2 - \frac{M^2}{2}\,\Phi^2 - \frac{g}{2}\,\phi^2\Phi\,, \tag{1}$$

where ϕ, Φ denote the light and heavy fields with masses m, M, respectively[1]. We shall study the heavy mass limit $M \to \infty$ in this theory.

Consider the scattering process $\phi\phi \to \phi\phi$ at the energies $E \sim m \ll M$. The momenta of the initial (final) particles are p_1 and p_2 (p_3 and p_4). At tree level, the scattering amplitude is given by the diagrams depicted in Fig. 1, and is equal to

$$T_{\mathsf{tree}} = \frac{g^2}{M^2 - s} + \frac{g^2}{M^2 - t} + \frac{g^2}{M^2 - u}\,, \tag{2}$$

where $s = (p_1 + p_2)^2,\ t = (p_1 - p_3)^2,\ u = (p_1 - p_4)^2$ are usual Mandelstam variables. On the mass shell, these variables obey the relation $s + t + u = 4m^2$.

In the limit $M \to \infty$, the amplitude in Eq. (2) can be expanded in Taylor series:

$$\begin{aligned} T_{\mathsf{tree}} &= \frac{3g^2}{M^2} + \frac{g^2}{M^4}\,(s+t+u) + \frac{g^2}{M^6}\,(s^2+t^2+u^2) + \cdots \\ &= \frac{3g^2}{M^2} + \frac{4g^2m^2}{M^4} + \frac{g^2}{M^6}\,(s^2+t^2+u^2) + \cdots\,. \end{aligned} \tag{3}$$

[1] In general, there will be an additional linear term $c\Phi$ present in the Lagrangian in Eq. (1), which is needed to cancel tadpole diagrams with one external Φ-leg. Here, however, we work in tree approximation, where $c = 0$.

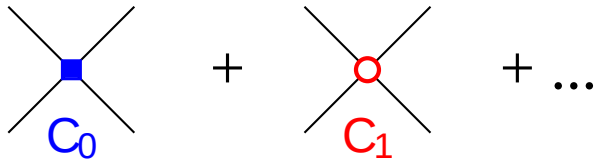

Figure 2: The tree-level scattering amplitude for the process $\phi\phi \to \phi\phi$ in the effective theory described by the Lagrangian given in Eq. (4). This amplitude can be obtained from the amplitude shown in Fig. 1 by contracting all heavy lines to a point.

At low energies, each subsequent term in this expansion is suppressed by a factor E^2/M^2 with respect to the previous one, where E is the characteristic energy of the light particles.

Our aim is to find a Lagrangian, which contains only ϕ-fields, and which reproduces the expansion of the amplitude in Eq. (3). In general, such an *effective* Lagrangian must contain an infinite tower of the quartic terms in the field ϕ

$$\mathcal{L}_{\mathsf{eff}} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 + C_0\phi^4 + C_1\phi^2\Box\phi^2 + C_2\phi^2\Box^2\phi^2 + \cdots . \tag{4}$$

Note that, at tree level, the mass parameters in both the underlying and effective Lagrangians are equal. As we shall see below, this is no more the case at one loop.

The tree-level amplitude, obtained from this Lagrangian, takes the form

$$\begin{aligned} T^{\mathsf{eff}}_{\mathsf{tree}} &= 24C_0 - 8C_1(s+t+u) + 8C_2(s^2+t^2+u^2) + \cdots \\ &= 24C_0 - 32m^2C_1 + 8C_2(s^2+t^2+u^2) + \cdots . \end{aligned} \tag{5}$$

This amplitude is shown in Fig. 2. Demanding $T^{\mathsf{eff}}_{\mathsf{tree}} = T_{\mathsf{tree}}$ leads to *matching conditions* which enable one to express the couplings of the effective theory in terms of the parameters of the underlying theory

$$\begin{aligned} 24C_0 - 32m^2C_1 &= \frac{3g^2}{M^2} + \frac{4g^2m^2}{M^4}, \\ 8C_2 &= \frac{g^2}{M^6}, \end{aligned} \tag{6}$$

and so on.

Note that the mass-shell matching does not allow one to determine the couplings C_0 and C_1 separately. According to Eq. (6), only the combination $24C_0 - 32m^2C_1$ can be determined from the matching condition. This is related to the accidental fact that (in this model only) all second-order terms can be eliminated by using the equations of motion. In order to prove this, note that

$$\phi^2\Box\phi^2 = 2\phi^3(\Box + m^2)\phi - 2m^2\phi^4 + 2\phi^2(\partial\phi)^2 . \tag{7}$$

On the other hand,

$$\phi^2(\partial\phi)^2 = \underbrace{\frac{1}{3}\,\partial^\mu(\phi^3\partial_\mu\phi)}_{=\,\mathsf{total\ derivative}} - \frac{1}{3}\phi^3(\Box + m^2)\phi + \frac{m^2}{3}\,\phi^4\,. \tag{8}$$

Using the equations of motion

$$(\Box + m^2)\phi = 4C_0\phi^3 + \cdots\,, \tag{9}$$

it is seen that $\phi^3(\Box + m^2)\phi$ is transformed into a sum of operators containing more than four fields and, therefore, do not contribute to the tree-level amplitude. Finally, the term proportional to ϕ^4 can be lumped together with the similar term in the Lagrangian. To summarize, the second-order terms can be completely eliminated from the Lagrangian – without losing generality, one may set $C_1 = 0$ everywhere.

Several remarks are in order:

i) The effective theory is equivalent to the underlying theory at tree level. The problems come in when we start to consider the effective field theory as a *field theory,* generating loops with the Lagrangian given in Eq. (4). The underlying theory is a superrenormalizable theory (the single coupling constant g has the dimension of mass), whereas the resulting effective theory contains a tower of non-renormalizable vertices. It is not clear, how the equivalence of these two theories can be formulated beyond the tree level.

ii) This issue is related to the previous one. The tree-level amplitude, calculated in the effective theory, violates the unitarity bound. In order to see this, define the partial-wave amplitudes

$$\begin{aligned} T_l^{\mathsf{eff}}(s) &= \frac{1}{32\pi\sqrt{s}}\int_{-1}^{1} d\cos\theta\, T^{\mathsf{eff}}(s,\cos\theta)\, P_l(\cos\theta)\,, \\ T^{\mathsf{eff}}(s,\cos\theta) &= 16\pi\sqrt{s}\sum_{l=0}^{\infty}(2l+1)T_l^{\mathsf{eff}}(s)P_l(\cos\theta)\,, \end{aligned} \tag{10}$$

where $P_l(\cos\theta)$ denote the conventional Legendre polynomials. The unitarity relation for the partial-wave amplitudes gives

$$\mathrm{Im}\, T_l^{\mathsf{eff}}(s) \geq p\,|T_l^{\mathsf{eff}}(s)|^2\,, \qquad p = \sqrt{\frac{s}{4} - m^2}\,, \tag{11}$$

where the inequality turns to equality below the first inelastic threshold $s = (4m)^2$, where the processes like $\phi\phi \to \phi\phi\phi\phi$ are no more energetically allowed.

From Eq. (11) it is immediately seen that the real part of the amplitude obeys the so-called *unitarity bound:*

$$|\mathrm{Re}\, T_l^{\mathsf{eff}}(s)| \leq \frac{1}{2p}\,. \tag{12}$$

The above bound is violated by the tree amplitude given in Eq. (5). For example, in the partial wave with $l = 0$ the tree-level amplitude is equal to

$$\mathrm{Re}\, T^{\mathsf{eff}}_{0,\mathsf{tree}}(s) = 24\tilde{C}_0 + \frac{32}{3}\, C_2 \left(\frac{s}{4} - m^2\right)^2 + \cdots\,, \qquad \tilde{C}_0 = C_0 - \frac{4m^2}{3}\, C_1\,. \tag{13}$$

This expression saturates the unitarity bound at

$$s_{\mathsf{M}} = 4M^2 \sqrt{\frac{16\pi - 3\tilde{g}^2}{4\tilde{g}^2/3}} + O(1)\,, \qquad \tilde{g} = \frac{g}{M}\,. \tag{14}$$

The large-M limit in the underlying theory is performed so that the dimensionless quantity $\tilde{g}$ stays finite – otherwise, the leading coupling C_0 could not be finite. Consequently, the quantity s_{M} is of order of M^2. If $s > s_{\mathsf{M}}$, loops are necessary in order to render the tree-level amplitude unitary. In turn, this means that the loops must be of the same of order of magnitude as the tree amplitude, heralding the trouble in the perturbative expansion.

In reality, if s is of order of $s_{\mathsf{M}} \sim M^2$, the effective theory can not be applied any more, and one should resort to the perturbative expansion in the underlying theory, which is superrenormalizable and where the amplitude decreases as s^{-1} at large values of s. It is said the the underlying theory provides a *Wilsonian ultra-violet (UV) completion* of the effective theory at the scales of order M.

iii) There is a well-known example, which exactly follows the path outlined in this toy model. In the Standard Model, the interactions between left-handed charged currents are mediated by $W^\pm$ bosons with the mass $M_W \simeq 80$ GeV. If the momentum transfer in a process is much smaller than M_W, the propagator of $W^\pm$ takes the form

$$D_{\mu\nu}(p) = \frac{g_{\mu\nu} - p_\mu p_\nu / M_W^2}{M_W^2 - p^2} \to \frac{g_{\mu\nu}}{M_W^2}\,, \qquad \text{if} \quad p^2 \ll M_W^2\,. \tag{15}$$

In this limit, the flavor-changing weak interactions are described by a local four-fermion Hamiltonian

$$H_{\mathsf{eff}} = \frac{G_F}{\sqrt{2}}\, J^\mu J^\dagger_\mu\,, \qquad \frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}\,, \tag{16}$$

where G_F denotes the Fermi coupling, and g is the $SU(2)$-gauge coupling in the Standard model. Further, the charged currents are given by a sum of hadronic and leptonic parts:

$$J_\mu = \sum_{ij} \bar{U}_i \gamma_\mu (1 - \gamma^5) V_{ij} D_j + \sum_\ell \bar{\nu}_\ell \gamma_\mu (1 - \gamma^5) \ell\,, \tag{17}$$

where U and D are the up- and down-quark fields (the indices i, j correspond to the generations), and V_{ij} denotes the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix.

As seen, the (dimensionful) coupling G_F in the effective theory is expressed at tree level in terms of the parameters g and M_W of the underlying theory.

Historical note: The violation of the unitarity was the first hint that the Fermi-theory is incomplete. The efforts to resolve this problem have culminated in the creation of the Standard Model of the electroweak interactions. For more information, see, e.g. [3].

1.3 The model at tree level: path-integral formalism

Consider the generating functional of the theory described by the Lagrangian in Eq. (1):

$$Z(j,J) = \int d\phi d\Phi \, \exp\left\{ i \int d^4x (\mathcal{L}(\phi,\Phi) + j\phi + J\Phi) \right\}, \tag{18}$$

where $j(x)$, $J(x)$ denote the classical external sources for the fields $\phi(x), \Phi(x)$, respectively. The Green's functions are obtained by functional differentiating Z with respect to these sources (once per each external leg), and putting $j = J = 0$ at the end.

Since we are interested here in the Green's functions of the light field only, we may put $J = 0$ and consider the quantity $Z(j) \doteq Z(j, J = 0)$. Performing a shift of the integration variable

$$\Phi \to \Phi - \frac{g}{2} (\Box + M^2)^{-1} \phi^2 \,, \tag{19}$$

it is possible to rewrite the generating functional in the following form:

$$\begin{aligned} Z(j) &= \int d\phi d\Phi \, \exp\left\{ i \int d^4x \left(-\frac{1}{2} \Phi(\Box + M^2)\Phi + \frac{g^2}{8} \phi^2 (\Box + M^2)^{-1} \phi^2 \right.\right. \\ &- \left.\left. \frac{1}{2}\phi(\Box + m^2)\phi + j\phi \right) \right\}. \end{aligned} \tag{20}$$

The integration over the variable Φ in the first term gives a constant that can be included into the normalization. Expanding now the second term in the argument, we get:

$$\frac{g^2}{8} \phi^2 (\Box + M^2)^{-1} \phi^2 = \frac{g^2}{8M^2} \left(\phi^4 - \phi^2 \frac{\Box}{M^2} \phi^2 + \phi^2 \frac{\Box^2}{M^4} \phi^2 + \cdots \right). \tag{21}$$

Comparing this expansion with Eq. (4), we may immediately read off

$$C_0 = \frac{g^2}{8M^2} \,, \qquad C_1 = -\frac{g^2}{8M^4} \,, \qquad C_2 = \frac{g^2}{8M^6} \,, \qquad \cdots \,, \tag{22}$$

and the result in Eq. (6) is reproduced.

It is legitimate to ask, why the above result is valid only at tree level, even if, formally, no approximations have been made so far. The answer to this question is that the Taylor expansion of the integrand in the path integral is not justified – the answer of the integral changes as a result of this expansion. On the other hand, at tree level, the path integral is equal just to the value of the integrand at the classical trajectory. Consequently, in this case, the expansion is justified, since the integration over ϕ is no longer involved.

A final remark is in order. It is easy to see that, before Taylor-expanding, the theory with the effective Lagrangian, containing only ϕ fields, is formally equivalent to the underlying theory to all orders in perturbation theory. The effective theory contains a vertex $\phi^2(\Box + M^2)^{-1}\phi^2$ and is thus non-local. Its high-energy behavior is, however, damped by the inverse D'Alembertian and corresponds to that of the original superrenormalizable theory. The expansion makes a local effective Lagrangian out of the non-local one, but at a cost of a worse behavior at high momenta. It is clear that the expansion breaks down at the momenta of the order of M, and we are back to the underlying theory.

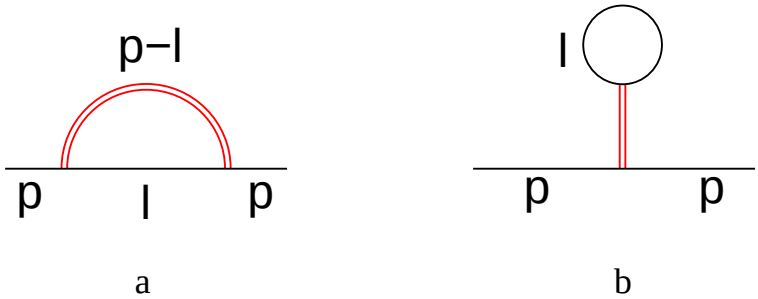

Figure 3: The self-energy of the light particle at one loop in the model described by the Lagrangian given in Eq. (1). Single and double lines correspond to the light and heavy fields, respectively.

1.4 Light particle mass at one loop

The self-energy of the light particle in the underlying theory at one loop is described by two diagrams shown in Fig. 3. We shall calculate them by using dimensional regularization. The contribution of the diagram in Fig. 3a is given by

$$\Sigma_a(p^2) = g^2 \int_l \frac{1}{m^2 - l^2} \frac{1}{M^2 - (p-l)^2}\,, \tag{23}$$

where we have used the shorthand notation

$$\int_l f(l) \doteq \int \frac{d^D l}{(2\pi)^D i}\, f(l)\,. \tag{24}$$

In the above expression, D denotes the number of space-time dimensions. In physical dimensions, $D \to 4$. In addition, in all propagators, the usual causal prescription $\text{mass}^2 \to \text{mass}^2 - i\epsilon$, $\epsilon \to 0^+$ is implicit.

Performing the integral with the use of the Feynman trick

$$\frac{1}{AB} = \int_0^1 \frac{dx}{(xA + (1-x)B)^2}\,, \tag{25}$$

as $D \to 4$, we obtain:

$$\Sigma_a(p^2) = -2g^2 L - \frac{g^2}{16\pi^2} \int_0^1 dx\, \ln \frac{xm^2 + (1-x)M^2 - x(1-x)p^2}{\mu^2}\,, \tag{26}$$

where μ denotes the scale of the dimensional regularization, and

$$L = \frac{\mu^{D-4}}{16\pi^2} \left(\frac{1}{D-4} - \frac{1}{2} \left(\Gamma'(1) + \ln 4\pi \right) \right). \tag{27}$$

Here, $\Gamma(z)$ stands for the Γ-function and $\Gamma'(1) = -\gamma$, where $\gamma = 0.577215665\ldots$ denotes Euler's constant. Integrating over the variable x, we obtain

$$\begin{aligned}\Sigma_a(p^2) &= -2g^2L - \frac{g^2}{16\pi^2}\left\{\frac{1}{2}\left(1-\frac{M^2-m^2}{p^2}\right)\ln\frac{m^2}{\mu^2} + \frac{1}{2}\left(1+\frac{M^2-m^2}{p^2}\right)\ln\frac{M^2}{\mu^2}\right.\\ &- \left.\frac{\lambda^{1/2}}{2p^2}\left(\ln\frac{\frac{1}{2}\left(1-\frac{M^2-m^2}{p^2}\right)-\frac{\lambda^{1/2}}{2p^2}}{\frac{1}{2}\left(1-\frac{M^2-m^2}{p^2}\right)+\frac{\lambda^{1/2}}{2p^2}} - \ln\frac{-\frac{1}{2}\left(1+\frac{M^2-m^2}{p^2}\right)-\frac{\lambda^{1/2}}{2p^2}}{-\frac{1}{2}\left(1+\frac{M^2-m^2}{p^2}\right)+\frac{\lambda^{1/2}}{2p^2}}\right) - 2\right\}, \quad (28)\end{aligned}$$

where

$$\lambda \doteq \lambda(p^2, m^2, M^2)\,, \qquad \lambda(x,y,z) = x^2+y^2+z^2-2xy-2yz-2zx\,, \quad (29)$$

denotes the Källén triangle function. Expanding this expression at large M, we obtain

$$\begin{aligned}\Sigma_a(p^2) &= -2g^2L - \frac{g^2}{16\pi^2}\left(-1+\ln\frac{M^2}{\mu^2}\right)\\ &- \frac{g^2}{16\pi^2M^2}\left(m^2\ln\frac{M^2}{\mu^2} - m^2\ln\frac{m^2}{\mu^2} - \frac{p^2}{2}\right) + O(M^{-4})\,, \quad (30)\end{aligned}$$

where the notation $O(M^{-4})$ includes also the terms at $O(M^{-4}\ln^k M^2)$. We shall consistently adhere to this notation below.

The calculations in case of the diagram in Fig. 3b (the "tadpole") can be done analogously. The result is given by

$$\Sigma_b(p^2) = \frac{g^2}{2M^2}\int_l \frac{1}{m^2-l^2} = \frac{g^2m^2}{M^2}L - \frac{g^2m^2}{32\pi^2M^2}\left(1-\ln\frac{m^2}{\mu^2}\right). \quad (31)$$

Adding these two expressions together, we finally get:

$$\begin{aligned}\Sigma_a(p^2)+\Sigma_b(p^2) &= -2g^2L - \frac{g^2}{16\pi^2}\left(-1+\ln\frac{M^2}{\mu^2}\right) + \frac{g^2m^2}{M^2}L\\ &- \frac{g^2}{16\pi^2M^2}\left(m^2\ln\frac{M^2}{\mu^2} - \frac{3m^2}{2}\ln\frac{m^2}{\mu^2} - \frac{1}{2}(p^2-m^2)\right) + O(M^{-4})\,. \quad (32)\end{aligned}$$

Next, we wish to reproduce this result within the effective theory. We shall, namely, try to answer the following question: The effective Lagrangian in Eq. (4) reproduces all Green's functions of the underlying theory in the tree approximation. Can also the results of the *loop calculations* in the underlying theory be reproduced by the loops in the effective theory, using the same Lagrangian? The answer to this question is *no,* as will become clear from our calculation at one loop. In order to reproduce the results up-to-and-including $O(M^{-2})$, it

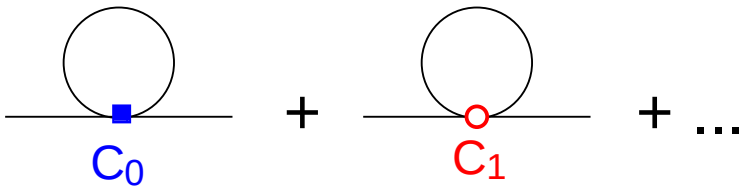

Figure 4: The self-energy of the light particle in the effective theory described by the Lagrangian given in Eq. (4). C_0 multiplies the vertex with no derivatives, C_1 with two derivatives, and so on.

is sufficient to include only the leading term with no derivatives, proportional to C_0, in the effective Lagrangian (cf. Eq. (4) and Eq. (22))

$$\mathcal{L} = \frac{1}{2}\,(\partial\phi)^2 - \frac{m^2}{2}\,\phi^2 + C_0\,\phi^4 + O(M^{-4})\,. \tag{33}$$

In the effective theory, up-to-and-including $O(M^{-2})$, only the first diagram in Fig. 4, proportional to $C_0 = g^2/(8M^2)$, contributes. The result is given by:

$$\Sigma_{\mathsf{eff}}(p^2) = 12C_0 \int_l \frac{1}{m^2 - l^2} + O(M^{-4}) = 24C_0 m^2\, L_{\mathsf{eff}} - \frac{3C_0 m^2}{4\pi^2}\left(1 - \ln\frac{m^2}{\mu_{\mathsf{eff}}^2}\right) + O(M^{-4})\,. \tag{34}$$

In the above expression, μ_{eff} denotes the dimensional regularization scale in the effective theory, which need not be the same, and L_{eff} is determined from Eq. (33), with the replacement $\mu \to \mu_{\mathsf{eff}}$.

As one sees from the above equations, $\Sigma_a + \Sigma_b \neq \Sigma_{\mathsf{eff}}$ at one loop. One may now ask the question, how one could modify the effective theory so that the above relation were restored? It is easy to see that, to this end, the effective Lagrangian should be supplemented by the counterterms responsible for the mass and wave function renormalization:

$$\begin{aligned}
\mathcal{L}_{\mathsf{eff}} \quad &\to \quad \mathcal{L}_{\mathsf{eff}} + \frac{A}{2}\,(\partial\phi)^2 - \frac{B}{2}\,\phi^2\,, \\
A \quad &= \quad \frac{g^2}{32\pi^2 M^2} + O(M^{-4})\,, \\
B \quad &= g^2 \quad \left(2L_{\mathsf{eff}} + \frac{1}{16\pi^2}\left(\ln\frac{M^2}{\mu_{\mathsf{eff}}^2} - 1\right)\right) \\
&\quad + \quad \frac{g^2 m^2}{M^2}\left(2L_{\mathsf{eff}} + \frac{1}{16\pi^2}\left(\ln\frac{M^2}{\mu_{\mathsf{eff}}^2} - 1\right)\right) + O(M^{-4})\,.
\end{aligned} \tag{35}$$

Further, let us calculate the physical mass of the light particle which coincides with the pole in the two-point function. In the underlying theory, at one loop, the physical mass m_P can be determined by the solution of the equation

$$m^2 - m_{\mathsf{P}}^2 - (\Sigma_a(m_{\mathsf{P}}^2) + \Sigma_b(m_{\mathsf{P}}^2)) = 0\,. \tag{36}$$

From this equation to the lowest order in g we obtain

$$m_{\mathsf{P}}^2 = m_{\mathsf{r}}^2(\mu) + \frac{g^2}{16\pi^2}\left(-1 + \ln\frac{M^2}{\mu^2}\right) + \frac{g^2 m_{\mathsf{r}}^2(\mu)}{16\pi^2 M^2}\left(\ln\frac{M^2}{\mu^2} - \frac{3}{2}\ln\frac{m_{\mathsf{r}}^2(\mu)}{\mu^2}\right) + O(M^{-4})\,, \tag{37}$$

where $m_{\mathsf{r}}(\mu)$ denotes the *running mass* in the underlying theory, which in the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme, which is defined through the subtraction of the divergent pieces proportional to L:

$$m_{\mathsf{r}}^2(\mu) = m^2 + 2g^2 L - \frac{g^2 m^2}{M^2}\,L\,. \tag{38}$$

Note that, at this order, it is still not necessary to consider the loop corrections of other parameters of the theory, g and M.

Since we have modified the effective Lagrangian to ensure that the Green functions in the underlying and the effective theories coincide, the poles in both theories will be at the same place[2]. The physical mass, calculated in the effective theory, is given by the solution of the following equation

$$m^2 + B - (1+A)m_{\mathsf{P}}^2 - \Sigma_{\mathsf{eff}}(m_{\mathsf{P}}^2) = 0\,, \tag{39}$$

and takes the form

$$m_{\mathsf{P}}^2 = m_{\mathsf{r,eff}}^2 + \frac{3g^2 m_{\mathsf{r,eff}}^2}{32\pi^2 M^2}\left(1 - \ln\frac{m_{\mathsf{r,eff}}^2(\mu_{\mathsf{eff}})}{\mu_{\mathsf{eff}}^2(\mu_{\mathsf{eff}})}\right) + O(M^{-4})\,, \tag{40}$$

where $m_{\mathsf{r,eff}}(\mu_{\mathsf{eff}})$ denotes the running mass in the effective field theory, which is related to the bare mass in the following manner:

$$m_{\mathsf{r,eff}}^2(\mu_{\mathsf{eff}}) = m_{\mathsf{eff}}^2 - \frac{3g^2 m_{\mathsf{eff}}^2}{M^2}\,L_{\mathsf{eff}}\,, \tag{41}$$

and, finally, the bare mass can be read from the effective Lagrangian

$$\begin{aligned}
\mathcal{L}_{\mathsf{eff}} &= \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 + \frac{A}{2}(\partial\phi)^2 - \frac{B}{2}\phi^2 + \text{quartic terms} \\
&= \frac{1}{2}Z_{\mathsf{eff}}(\partial\phi)^2 - \frac{m_{\mathsf{eff}}^2}{2}Z_{\mathsf{eff}}\phi^2 + \text{quartic terms}\,, \\
Z_{\mathsf{eff}} &= 1 + A\,, \qquad m_{\mathsf{eff}}^2 = \frac{m^2 + B}{1 + A}\,.
\end{aligned} \tag{42}$$

[2]Strictly speaking, only the matching of the *observables* in two theories (i.e., the masses and the S-matrix elements) is required. The two-point function is not an observable. So, in principle, one could leave the wave function renormalization constant Z_{eff} free. However, not much will change in our discussion of the physical mass, if we lift the restriction on this constant.

Since observables (the physical masses) should be the same in the underlying theory and in the effective theory, this finally gives the relation between the running masses in both theories:

$$\begin{aligned} m^2_{\mathsf{r,eff}}(\mu_{\mathsf{eff}}) &= m^2_{\mathsf{r}}(\mu) + \frac{g^2}{16\pi^2}\left(-1 + \ln\frac{M^2}{\mu^2}\right) \\ &+ \frac{g^2 m^2_{\mathsf{r}}(\mu)}{16\pi^2 M^2}\left(\ln\frac{M^2}{\mu^2} - \frac{3}{2}\left(1 + \ln\frac{\mu^2_{\mathsf{eff}}}{\mu^2}\right)\right) + O(M^{-4})\,. \end{aligned} \tag{43}$$

As we see, the *running masses* in both theories are not the same beyond tree approximation. Moreover, these masses *run differently* with respect to the scale variations:

$$\begin{aligned} \mu\frac{dm^2_{\mathsf{r}}(\mu)}{d\mu} &= \frac{g^2}{8\pi^2} - \frac{g^2 m^2_{\mathsf{r}}(\mu)}{16\pi^2 M^2} \\ \mu_{\mathsf{eff}}\frac{dm^2_{\mathsf{r,eff}}(\mu_{\mathsf{eff}})}{d\mu_{\mathsf{eff}}} &= \frac{3g^2 m^2_{\mathsf{r,eff}}(\mu_{\mathsf{eff}})}{16\pi^2 M^2} + O(M^{-4})\,. \end{aligned} \tag{44}$$

The above *renormalization group (RG)* equations can be obtained by differentiating the expression for the physical mass with respect to the scale and setting this derivative to zero, because the physical mass does not depend on the scale. Moreover, it should be pointed out that even the scale μ is present in Eq. (43), the running mass in the effective theory, $m_{\mathsf{r,eff}}(\mu_{\mathsf{eff}})$, in fact, does not depend on this scale. This statement can be straightforwardly checked by using the first of the equations in Eq. (44). This happens because Eq. (43) was obtained from the matching to the physical observable, which has to be scale-independent.

Concluding remarks are in order:

i) As we have seen, matching the two Lagrangians at tree level does not mean that the loops calculated with these Lagrangians will also match. The difference, however, can be taken away completely by the renormalization. This means that both theories are physically equivalent. This statement constitutes the content of the decoupling theorem [4], see later.

ii) Matching enables one to express the parameters of the effective theory in terms of the parameters of the underlying theory. What makes sense is the relation between the finite quantities: e.g., between the running masses and the couplings.

iii) Both sets of the running parameters depend on their own scale (μ and μ_{eff}, respectively). The parameters of the effective theory do not depend on the underlying scale μ, if they can be determined from the matching to physical observables.

iv) Note that in the relation given by Eq. (43), all logarithms containing the light mass cancel. This is the manifestation of the general pattern, which states that the couplings of the effective theory do not contain non-analytic behavior that emerges at the light scales. All of this non-analytic behavior has to be reproduced by the loops in the effective theory. On the contrary, the parameters of the effective theory encode the short-distance dynamics

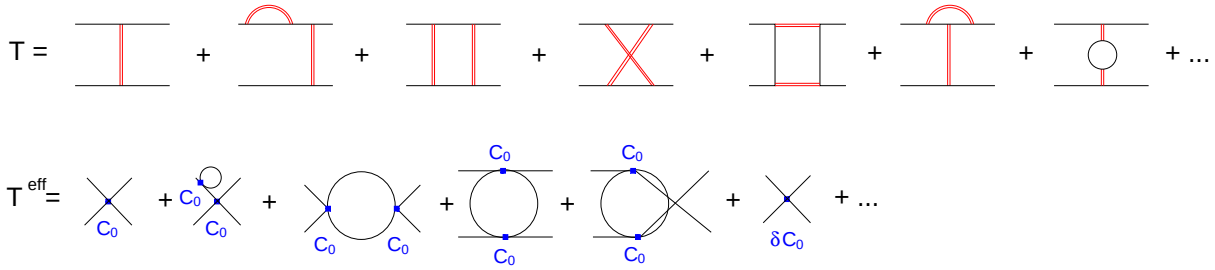

Figure 5: Representative set of the diagrams, contributing to the $\phi\phi \to \phi\phi$ amplitudes in the underlying and the effective theories.

and thus depend on the light mass, at most, in the polynomial form. For consistency, here one assumes that the scales μ, μ_{eff} are also "hard." On the other hand, reducing μ_{eff} down to the "light" scale, the couplings will no more depend analytically on this scale. We shall observe this phenomenon explicitly in ChPT.

v) As we know, the dimensionful coupling constant g is of order of the heavy mass M in the large-M limit. The running mass in the effective theory is not protected from large loop corrections (e.g., by some symmetries) and, according to Eq. (43), is driven up to the heavy scale, unless some fine tuning is enforced. This phenomenon is closely related to the *hierarchy problem* in Grand Unified Theories.

1.5 Matching of the quartic coupling at one loop

Matching of the $\phi\phi \to \phi\phi$ scattering amplitudes at one loop proceeds analogously. First of all, one has to calculate the scattering amplitude in the underlying theory and in the effective theory. A representative set of the diagrams is shown in Fig. 5. The matching condition is:

$$T = T^{\mathsf{eff}} \,. \tag{45}$$

It is seen that, in a result of this matching condition, the quartic couplings in the tree-level effective Lagrangian, given by Eq. (4), are modified according to $C_i \to C_i + \delta C_i$. This is shown schematically in Fig. 5.

Since the one-loop contributions to the scattering amplitude in the effective theory (see Fig. 5) are divergent, the modified C_i should also contain divergent parts

$$C_i = \nu_i L_{\mathsf{eff}} + C_i^{\mathsf{r}}(\mu_{\mathsf{eff}}) \,, \tag{46}$$

where the coefficients ν_i determine the running of the renormalized couplings $C_i^{\mathsf{r}}(\mu_{\mathsf{eff}})$ with respect to the scale μ_{eff}:

$$\mu_{\mathsf{eff}} \frac{dC_i^{\mathsf{r}}(\mu_{\mathsf{eff}})}{d\mu_{\mathsf{eff}}} = -\frac{\nu_i}{16\pi^2} \,. \tag{47}$$

Matching enables one to express the *renormalized* couplings $C_i^{\mathsf{r}}(\mu_{\mathsf{eff}})$ in terms of the fundamental parameters of the underlying theory. Comparing with Eq. (22) which contains matching at tree level, and using dimensional arguments, we get

$$C_i^{\mathsf{r}}(\mu_{\mathsf{eff}}) = (-)^i \frac{g^2}{8M_{\mathsf{r}}^{2(i+1)}(\mu_{\mathsf{eff}})} \left\{ 1 + \kappa_i \frac{g^2}{16\pi^2 M_{\mathsf{r}}^2(\mu_{\mathsf{eff}})} \right\}, \tag{48}$$

where $M_{\mathsf{r}}(\mu_{\mathsf{eff}})$ is the renormalized heavy mass in the underlying theory, and the dimensionless constants κ_i can depend only on the dimensionless arguments $m_{\mathsf{r}}/M_{\mathsf{r}}$ and $\mu_{\mathsf{eff}}/M_{\mathsf{r}}$ (without loss of generality and in order to ease notations, we took here $\mu = \mu_{\mathsf{eff}}$). In Eq. (48), in addition, we took into account the fact that in the underlying (superrenormalizable) theory the coupling g is not renormalized, and we used g instead of g_r everywhere. Moreover, as became clear from the discussion in the section 1.4, the coupling constants, determined from the matching, can not contain infrared singularities at $m_{\mathsf{r}} \to 0$, since these singularities are the same in the underlying and in the effective theory, canceling each other in the matching condition. An example for this is the cancellation of all $\ln(m_{\mathsf{r}}^2)$-terms in the matching of the two-point functions, see the section 1.4. Consequently, the κ_i can only be a polynomial in the variable $m_{\mathsf{r}}^2/M_{\mathsf{r}}^2$ and the whole dependence on this variable can be eventually eliminated by using the equations of motion (EOMs) in the Lagrangian, see section 1.2. On the contrary, the dependence on the second variable $\mu_{\mathsf{eff}}/M_{\mathsf{r}}$ is non-analytic – in perturbation theory, logarithms $\ln(\mu_{\mathsf{eff}}/M_{\mathsf{r}})$ usually appear[3].

Carrying out the matching at one loop is straightforward but quite boring, since a large number of Feynman diagrams have to be calculated. Below, we shall demonstrate, how the same goal, with considerably less effort, can be achieved within the path-integral formalism. To this end, we evaluate the generating functional, given in Eq. (20) at one loop by using the saddle-point technique. In the beginning, we carry out the integration over the field Φ (this integration gives an uninteresting constant, which can be included in the normalization of the path integral). Further, we expand the action functional in this integral around the *classical* solution for the field ϕ, writing $\phi = \phi_c + \xi$. Here, the field ξ denotes the quantum fluctuation around the classical solution ϕ_c, which obeys the following equation of motion

$$\begin{aligned} 0 &= (\Box + m^2)\phi_c(x) + j(x) + \frac{g^2}{2} \int d^4y\, \phi_c(x) D_M(x-y) \phi_c^2(y) \\ &= (\Box + m^2)\phi_c(x) + j(x) + \frac{g^2}{2M^2}\, \phi_c^3(x) + \cdots , \end{aligned} \tag{49}$$

where

$$D_M(x-y) = \langle x|(\Box + M^2)^{-1}|y\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{M^2 - p^2} . \tag{50}$$

[3]Using the EOMs is justified, since the S-matrix elements, which are used in the matching condition, do not change. One should bear in mind, however, that the off-shell behavior of Green's function changes, if the EOMs are used.

Retaining the terms up to the second order in the expansion over ξ, and taking into account the fact that $d\phi = d\xi$, the generating functional in Eq. (20) can be rewritten as follows

$$\begin{aligned} Z(j) &= \int d\xi \exp\left\{ i \int d^4x \left(-\frac{1}{2}\,\phi_c(\Box + m^2)\phi_c + \frac{g^2}{8}\phi_c^2(\Box + M^2)^{-1}\phi_c^2 + j\phi_c \right) \right\} \\ &\times \exp\left\{ i \int d^4x d^4y \left(-\frac{1}{2}\,\xi(x)H(x-y)\xi(y) + O(\xi^3) \right) \right\}, \end{aligned} \tag{51}$$

where

$$\begin{aligned} H(x-y) &= (\Box + m^2 + S(x))\delta^4(x-y) - \Lambda(x-y)\,, \\ S(x) &= -\frac{g^2}{2}\,(\Box + M^2)^{-1}\phi_c^2(x)\,, \\ \Lambda(x-y) &= g^2\phi_c(x)\langle x|(\Box + M^2)^{-1}|y\rangle\phi_c(y)\,. \end{aligned} \tag{52}$$

Note that there are no terms linear in ξ, because ϕ_c is the solution of the equation of motion that makes the action functional stationary.

Evaluating the Gaussian integral over ξ in a standard manner, we obtain

$$Z(j) = \exp\left\{ i \int d^4x \left(-\frac{1}{2}\,\phi_c(\Box + m^2)\phi_c + \frac{g^2}{8}\phi_c^2(\Box + M^2)^{-1}\phi_c^2 + j\phi_c \right) + iS_{\mathsf{eff}} \right\}, \tag{53}$$

where

$$\begin{aligned} S_{\mathsf{eff}} &= \frac{i}{2}\,\mathrm{Tr}\,\ln\left((\Box + m^2 + S) - \Lambda\right) = \frac{i}{2}\,\mathrm{Tr}\,\ln(\Box + m^2) + \frac{i}{2}\,\mathrm{Tr}\left((\Box + m^2)^{-1}S\right) \\ &- \frac{i}{4}\,\mathrm{Tr}\left((\Box + m^2)^{-1}S(\Box + m^2)^{-1}S\right) - \frac{i}{2}\,\mathrm{Tr}\left((\Box + m^2)^{-1}\Lambda\right) \\ &+ \frac{i}{2}\,\mathrm{Tr}\left((\Box + m^2)^{-1}S(\Box + m^2)^{-1}\Lambda\right) - \frac{i}{4}\,\mathrm{Tr}\left((\Box + m^2)^{-1}\Lambda(\Box + m^2)^{-1}\Lambda\right) + O(g^6) \\ &= T_0 + T_1 + T_2 + T_3 + T_4 + T_5 + O(g^6)\,, \end{aligned} \tag{54}$$

Here, "Tr" denotes the trace of the operator in the coordinate space, i.e.,

$$\mathrm{Tr}\,A = \int d^4x\,\langle x|A|x\rangle\,. \tag{55}$$

Further, note that T_0 is an uninteresting constant, which can be included in the normalization of the path integral. T_1 and T_3 are quadratic in the field ϕ_c and contribute to the renormalization of the two-point function of the light field. We have studied this issue in detail is section 1.4. The remaining terms T_2, T_4 and T_5, which contribute to the renormalization of the quartic

couplings, can be rewritten as

$$
\begin{aligned}
T_2 &= \frac{g^4}{16}\int d^4x d^4y d^4u d^4v\,(-iD(u-v)D(v-u)D_M(u-x)D_M(v-y))\,\phi_c^2(x)\phi_c^2(y)\,,\\
T_4 &= \frac{g^4}{4}\int d^4x d^4y d^4u d^4v(-iD(v-u)D(u-y)D_M(y-v)D_M(u-x))\,\phi_c^2(x)\phi_c(y)\phi_c(v)\,,\\
T_5 &= \frac{g^4}{4}\int d^4x d^4y d^4u d^4v\,(-iD(v-u)D_M(u-x)D(x-y)D_M(y-v))\,\phi_c(x)\phi_c(y)\phi_c(u)\phi_c(v)\,,
\end{aligned}
\tag{56}
$$

where the light particle propagator $D(x-y)$ is given by the same formula (50), with the replacement $M \to m$. Schematically, the three quantities T_2, T_4, T_5 are depicted in Fig. 6.

Let us now consider the strategy of the matching at one loop. First, we recall that the matching condition is altered by loop corrections, because the heavy particles are present in the loops and the Taylor expansion in the inverse powers of the heavy mass can not be straightforwardly carried out in the Feynman integrals. Namely, let us denote $T_M\{T_i\},\ i=2,4,5$, the quantities T_i, evaluated from the same diagrams shown in Fig. 6, but with the Taylor-expanded heavy particle propagator

$$
\frac{1}{M^2-l^2} \to \frac{1}{M^2} + \frac{l^2}{M^4} + \cdots \tag{57}
$$

(the symbol "T_M" stands for the procedure of the Taylor expansion in the inverse powers of the heavy mass[4]). Then, the difference

$$
\Delta T = \sum_{i=2,4,5} (T_i - T_M\{T_i\}) \tag{58}
$$

should be compensated by adjusting the quartic coupling constants. This gives us the desired matching condition for these couplings.

From Fig. 6 we immediately conclude that T_2 will not affect the matching condition, because it does not contain heavy particles in the loops. Consequently,

$$
T_2 - T_M\{T_2\} = 0\,. \tag{59}
$$

T_4 and T_5 will, however, affect the matching condition. Let us start with the quantity T_4. The vertex diagram, which is the part of T_4 (see Fig. 6), is given by

$$
-iD(v-u)D(u-y)D_M(y-v) = \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\,e^{-ip_1(v-u)-ip_2(u-y)}\Gamma_v(p_1,p_2)\,, \tag{60}
$$

[4]Note that, graphically, the operation T_M amounts to contracting heavy propagators to one point. In a result, the diagrams, describing $\phi\phi \to \phi\phi$ scattering in the effective theory arise from the diagrams T_2, T_4 and T_5, shown in Fig. 6.

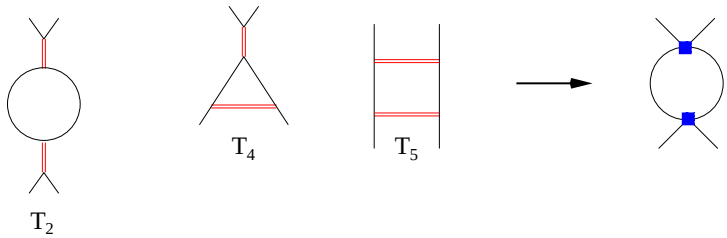

Figure 6: A schematic representation of T_2, T_4 and T_5. The solid and double lines denote the light and heavy fields, respectively. On the right, the one loop graph in the effective theory, which is obtained from T_2, T_4, T_5 by contracting the heavy propagators, is shown.

where

$$\Gamma_v(p_1,p_2) = \int_l \frac{1}{(m^2-(p_1+l)^2)}\frac{1}{(m^2-(p_2+l)^2)}\frac{1}{(M^2-l^2)}\,. \tag{61}$$

Note that the second heavy propagator, $D_M(u-x)$, which is outside the loop, can be expanded in inverse powers of M without much ado.

We are interested in the quantity $R_v(p_1,p_2) = \Gamma_v(p_1,p_2) - T_M\{\Gamma_v(p_1,p_2)\}$. Since the quantity $R_v(p_1,p_2)$ should be a low-energy polynomial in the small momenta p_1, p_2, one may expand it in the Taylor series

$$R_v(p_1,p_2) = R_v(0,0) + p_1^\mu \frac{\partial}{\partial p_1^\mu} R_v(p_1,p_2)\bigg|_{p_1,p_2=0} + p_2^\mu \frac{\partial}{\partial p_2^\mu} R_v(p_1,p_2)\bigg|_{p_1,p_2=0} + \cdots\,. \tag{62}$$

Note that, in the effective Lagrangian, this expansion translates into the derivative expansion in the light fields. In order to perform matching at lowest order in the inverse heavy mass M, it suffices to retain the first term in this expansion. Generalization to higher orders is straightforward.

Calculating $\Gamma_v(0,0)$, we get

$$\begin{aligned}\Gamma_v(0,0) &= \int_l \frac{1}{(m^2-l^2)^2}\frac{1}{M^2-l^2} = \frac{1}{16\pi^2}\int_0^1 \frac{dx\,x}{xm^2+(1-x)M^2} \\ &= \frac{1}{16\pi^2}\left(\frac{1}{m^2-M^2} - \frac{M^2}{(m^2-M^2)^2}\ln\frac{m^2}{M^2}\right) \\ &= -\frac{1}{16\pi^2 M^2}\left(1+\ln\frac{m^2}{M^2}\right) + O(M^{-4})\,. \end{aligned} \tag{63}$$

On the other hand,

$$T_M\{\Gamma_v(0,0)\} = \int_l \frac{1}{(m^2-l^2)^2}\left\{\frac{1}{M^2}+O(M^{-4})\right\} = \frac{2}{M^2}L_{\mathsf{eff}} - \frac{1}{16\pi^2 M^2}\ln\frac{m^2}{\mu_{\mathsf{eff}}^2} + O(M^{-4})\,. \tag{64}$$

Subtracting these two expressions, we finally obtain

$$R_v(0,0) = \frac{2}{M^2}\, L_{\mathsf{eff}} - \frac{1}{16\pi^2 M^2}\left(1 + \ln\frac{\mu_{\mathsf{eff}}^2}{M^2}\right) + O(M^{-4})\,. \tag{65}$$

As expected, the non-analytic terms, proportional to $\ln m^2$, cancel in this difference. Substituting now this expression into Eqs. (60) and (56), we finally obtain

$$T_4 - T_M\{T_4\} = \frac{g^4}{4M^2}\, R_v(0,0) \int d^4x\, \phi_c^4(x) + O(M^{-6})\,. \tag{66}$$

The quantity T_5 can be treated analogously. Here, we need the expression of the box integral at zero momenta (see Fig. 6)

$$\begin{aligned}
\Gamma_b(0,0) &= \int_l \frac{1}{(m^2-l^2)^2}\,\frac{1}{(M^2-l^2)^2} = \frac{1}{16\pi^2}\int_0^1 \frac{dx\, x(1-x)}{(xm^2+(1-x)M^2)^2} \\
&= \frac{1}{16\pi^2}\,\frac{-2(M^2-m^2)+(M^2+m^2)\ln(M^2/m^2)}{(M^2-m^2)^3} \\
&= \frac{1}{16\pi^2 M^4}\left(-2+\ln\frac{M^2}{m^2}\right) + O(M^{-6})\,.
\end{aligned} \tag{67}$$

The same integral, with the Taylor-expanded heavy propagator, is equal to

$$T_M\{\Gamma_b(0,0)\} = \int_l \frac{1}{(m^2-l^2)^2}\,\frac{1}{(M^2)^2} = -\frac{2}{M^4}\, L_{\mathsf{eff}} - \frac{1}{16\pi^2 M^4}\,\ln\frac{m^2}{\mu_{\mathsf{eff}}^2}\,. \tag{68}$$

From these equations we obtain

$$R_b(0,0) = \Gamma_b(0,0) - T_M\{\Gamma_b(0,0)\} = \frac{2}{M^4}\, L_{\mathsf{eff}} + \frac{1}{16\pi^2 M^4}\left(-2-\ln\frac{\mu_{\mathsf{eff}}^2}{M^2}\right) + O(M^{-6})\,. \tag{69}$$

Finally, from Eq. (56) we obtain

$$T_5 - T_M\{T_5\} = \frac{g^4}{4}\, R_b(0,0) \int d^4x\, \phi_c^4(x)\,. \tag{70}$$

From Eqs. (66) and (70) we may now read off the matching of the low-energy constant C_0 at one loop

$$\begin{aligned}
C_0 &= \frac{g^2}{8M^2} + \frac{g^4}{4M^2}\, R_v(0,0) + \frac{g^4}{4}\, R_b(0,0) + O(M^{-6}) \\
&= \frac{g^2}{8M^2} + \frac{g^4}{M^4}\, L_{\mathsf{eff}} - \frac{g^4}{64\pi^2 M^4}\left(3 + 2\ln\frac{\mu_{\mathsf{eff}}^2}{M^2}\right) + O(M^{-6})\,.
\end{aligned} \tag{71}$$

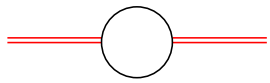

Figure 7: Renormalization of the heavy mass at one loop. The solid and double lines denote the light and heavy fields, respectively.

In order to arrive at the final result, one has to express everything in Eq. (71) in terms of the renormalized couplings. As already mentioned, g is not renormalized. The quantity M^2 should be, however, renormalized, see Fig. 7

$$M^2 = M_r^2(\mu_{\mathsf{eff}}) - g^2 L_{\mathsf{eff}}\,, \tag{72}$$

where, without loss of generality, one may assume that the scales in the underlying and effective theories coincide: $\mu = \mu_{\mathsf{eff}}$.

Substituting this expression into Eq. (71), we finally obtain

$$C_0 = \frac{9g^4}{8M_r^4}\,L_{\mathsf{eff}} + \frac{g^2}{8M_r^2} - \frac{g^4}{64\pi^2 M_r^4}\left(3 + 2\ln\frac{\mu_{\mathsf{eff}}^2}{M_r^2}\right) + O(M_r^{-6}) = \nu_0 L_{\mathsf{eff}} + C_0^r(\mu_{\mathsf{eff}})\,. \tag{73}$$

It is seen that $C_0^r(\mu_{\mathsf{eff}})$ can be written in form of Eq. (48). Reading off the coefficient κ_0, one gets

$$\kappa_0 = -6 - 4\ln\frac{M^2}{\mu_{\mathsf{eff}}^2} + O(M_r^{-2})\,. \tag{74}$$

The coefficient κ_0 does not depend on the light mass m at this order. This is, however, not true in general to all orders in the expansion in the inverse powers of M_{r}, unless the EOMs are used.

Finally, from Eq. (73) one can straightforwardly ensure that the renormalized coupling constant at this order obeys the well-known RG equation in the ϕ^4 theory

$$\mu_{\mathsf{eff}}\frac{dC_0^r}{d\mu_{\mathsf{eff}}} = -\frac{9}{2\pi^2}\,(C_0^r)^2\,. \tag{75}$$

Last but not least, it should be noted that the effective Lagrangian beyond tree level contains terms with 6,8,... ϕ-fields as well. These are needed, in particular, to cancel the divergences in the loop diagrams of the effective theory of the type shown in Fig. 8. Such terms emerge as a result of using the EOMs in the quartic terms as well.

1.6 Dependence of the effective couplings on the heavy mass

In a simple model considered in the previous sections, the heavy mass M sets the hard scale, above which the structure of the theory changes. For this reason, it is interesting to learn, how the parameters of the low-energy theory depend on the heavy mass. First, let us consider

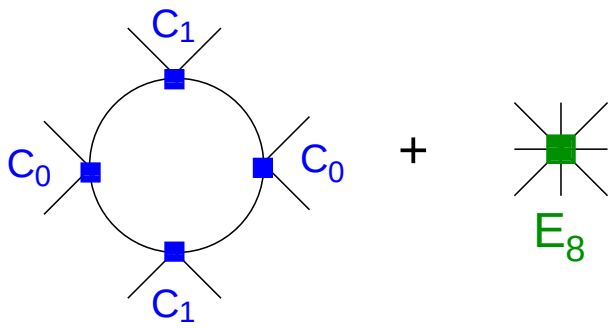

Figure 8: The renormalization of the Green's function with eight external legs in the effective theory. In order to cancel the divergence, a local term with eight ϕ-fields is introduced in the Lagrangian. The pertinent coupling is denoted by E_8.

the effective Lagrangian at tree level. One may judge about the leading behavior of these couplings in the limit $M \to \infty$ on the basis of the dimensionality of these couplings alone. Only the effective mass of the light particle has a positive mass dimension. The coupling C_0 is dimensionless, and the couplings C_i with $i > 0$ have negative mass dimension. On dimensional grounds, the leading behavior in M in the latter couplings should be proportional to $g^2/M^{2(i+1)} \propto M^{-2i}$. Consequently, the couplings $C_i,\ i > 0$, fall off as negative powers of M, as $M \to \infty$. The dimensionless couplings are defined as $\tilde{C}_i = C_i M^{2i}$. The couplings $\tilde{C}_i$ are said to be of *natural size* if these are of order one.

According to the mass dimension, the couplings are referred to as the *relevant* (positive mass dimension), *marginal* (dimensionless) and *irrelevant* (negative mass dimension). It is seen that at low energies that corresponds to the limit of a very large M, the contribution from the irrelevant couplings to the Green's functions is suppressed by powers of the large mass M.

Does the situation change beyond the tree level? Let us next consider the insertion of the irrelevant couplings in the loops. For simplicity, consider one loop in the effective theory with the insertion of two irrelevant vertices multiplied by the couplings C_i and C_j, see Fig. 9. The product of these two couplings falls off as $M^{-2(i+j)}$. Further, the mass dimension of the diagram in Fig. 9 is equal to 0. So, in order to make up the required mass dimension, the above factor should be multiplied by $\mathsf{mass}^{2(i+j)}$, where mass denotes any available mass scale in the effective theory: external momenta, effective mass or the regulator mass in the loops.

The discussion is particularly simple in the dimensional regularization. The diagram in Fig. 9 is given by the expression

$$I_{ij} = \frac{\tilde{C}_i \tilde{C}_j}{M^{2(i+j)}} \int_l \frac{N(l;\{p_i\})}{(m^2 - l^2)(m^2 - (P-l)^2)}\,, \tag{76}$$

where the tree-level couplings $\tilde{C}_i = C_i M^{2i}$ are dimensionless and stay finite as $M \to \infty$. Further, $p_1, \cdots, p_4$ are the external momenta with $P = p_1 + p_2$, and the numerator N, which has the mass dimension $2(i+j)$, depends on the integration momentum l, the external momenta and on the light mass m (at this order, one may replace running effective mass of a light particle $m_{\mathsf{r,eff}}$

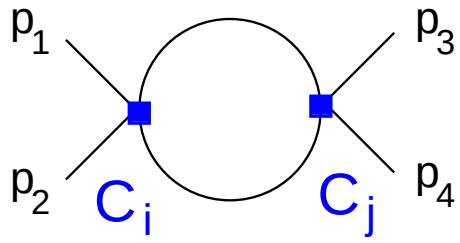

Figure 9: Insertion of two irrelevant vertices into the one-loop diagram. p_1, p_2 and p_3, p_4 are the incoming and outgoing momenta, respectively. The total momentum is $P = p_1 + p_2 = p_3 + p_4$.

by m). After integration, the dependence on the scale μ_{eff} appears. However, in dimensional regularization the dependence on the scale μ_{eff} is logarithmic and thus safe (i.e., the power of M in front of the integral is not changed through the multiplication by a logarithm). The loops with the insertions of the irrelevant couplings are also irrelevant in the limit $M \to \infty$. Thus, irrelevant couplings can be eliminated from the theory at one loop level as well.

The argumentation is a bit more subtle in arbitrary regularization (say, cutoff regularization), where the powers of the large regulator scale Λ_{cut} can appear. This situation also emerges if we have a multi-scale problem, with heavy particles appearing in the effective field theory loops together with light particles (example: pion-nucleon scattering in ChPT). According to the dimensional counting, the maximal power of Λ_{cut} is contained in the maximally UV-divergent piece of the integral I_{ij} in Eq. (76), which does not depend on the external momenta $p_1, \cdots p_4$. Denoting this maximally divergent piece by $\tilde{I}_{ij}$, we have

$$\tilde{I}_{ij} = \frac{\tilde{C}_i \tilde{C}_j}{M^{2(i+j)}} \int^{\Lambda_{\mathsf{cut}}} \frac{d^4 l}{(2\pi)^4 i} \frac{l^{2(i+j)}}{(m^2 - l^2)^2} \sim \frac{\tilde{C}_i \tilde{C}_j \Lambda_{\mathsf{cut}}^{2(i+j)}}{M^{2(i+j)}} \,. \tag{77}$$

In other words, this term is no more suppressed since $\Lambda_{\mathsf{cut}} \sim M$. Note, however, that the above contribution does not depend on the external momenta and has exactly the same form as the contribution coming at tree level from the marginal vertex with the coupling C_0. Consequently, the whole contribution $\tilde{I}_{ij}$ can be removed by the renormalization of C_0 which we of course are free to perform. We arrive at the same conclusion as earlier, within dimensional regularization: the contributions from the irrelevant couplings are irrelevant at one loop as well. Thus, our results, as expected, do not depend on the regularization used. The above arguments can be readily generalized for any number of insertions in the diagrams with arbitrary number of loops.

1.7 Renormalization group flow

The heuristic arguments given in the previous section were still restricted to the discussion of individual Feynman diagrams. In this section, we shall provide a more general proof. A systematic method is based on integrating out high-frequency modes in the generating functional of the theory. The discussion below closely follows the one in Ref. [5], see also [6, 7].

We do not want to focus on any particular model. To this end, we shall interpret M merely as some hard scale of the theory, after which the unknown physics starts, be this

a new particle with a mass M, non-local effects, or whatever. Further, in order to make the arguments maximally transparent, below we shall use the momentum cutoff instead of dimensional regularization. Consider, for simplicity, a theory with a single scalar field ϕ. The Euclidean generating functional for the *renormalized Green functions* in momentum space is given by

$$Z(J) = \int \big[d\phi\big]_M \exp\bigg\{\int \frac{d^4p}{(2\pi)^4}\left(-\frac{1}{2}\,\phi(p)(p^2+m^2)\phi(-p) + J(p)\phi(-p)\right) + S_{\mathsf{int}}(\phi)\bigg\}\,, \qquad (78)$$

where $\big[d\phi\big]_M$ denotes the path integral measure with a cutoff on the high-frequency modes with $p \sim M$. This shorthand notation should be interpreted as follows: calculating Eq. (78) in perturbation theory, a cutoff at the momentum scale of order M is introduced in all Feynman graphs (an explicit form of the cutoff does not play a role). Further, S_{int} is the interaction part of the action functional that contains the *bare* coupling constants $C_i(M)$ corresponding to the cutoff at a scale M. In addition, in the *renormalizable theory,* it is possible to choose $C_i(M)$ so that the generating functional remains finite as $M \to \infty$. This is not possible in the non-renormalizable theory, but everything is perfectly defined if the cutoff stays finite.

Now, let us ask ourselves, how the couplings C_i depend on the cutoff. To this end, we introduce an *effective field theory scale* Λ_{eff}, which obeys the condition $m \ll \Lambda_{\mathsf{eff}} \ll M$, and consider a smooth change of a cutoff from M to Λ_{eff}. Namely, we define a scale $\Lambda_{\mathsf{eff}} \le \Lambda \le M$ and consider the Euclidean path integral

$$\begin{aligned} Z(J,\Lambda) &= \int \big[d\phi\big]_\Lambda \exp\bigg\{\int \frac{d^4p}{(2\pi)^4}\left(-\frac{1}{2}\,\phi(p)(p^2+m^2)\phi(-p) + J(p)\phi(-p)\right) + S_{\mathsf{int}}(\phi,\Lambda)\bigg\} \\ &= \int d\phi \exp(S(\phi,\Lambda))\,. \end{aligned} \qquad (79)$$

Here, we are only interested in the low-frequency modes, so we assume that

$$J(p) = 0 \qquad \text{for} \quad p^2 > \Lambda_{\mathsf{eff}}^2\,. \qquad (80)$$

The point is that, in order to ensure that the renormalized Green functions do not depend on Λ, the effective action $S(\phi,\Lambda)$ should obey certain *RG flow equations.* In other words, the effective couplings that enter $S(\phi,\Lambda)$, should depend on Λ in a manner that compensates the explicit Λ-dependence coming from the cutoff. For example, the mass parameters at two scales in the pure ϕ^4-theory with the only coupling C_0 at one loop are related by

$$\begin{aligned} m^2(\Lambda) &= m^2(M) + 12C_0 \int_l^M \frac{1}{m^2(M)+l^2}\bigg|_{\mathsf{Eucl.}} - 12C_0 \int_l^\Lambda \frac{1}{m^2(M)+l^2}\bigg|_{\mathsf{Eucl.}} \\ &= m^2(M) + \frac{3C_0}{4\pi^2}\left(M^2 - \Lambda^2 - m^2(M)\ln\frac{M^2}{\Lambda^2} + \cdots\right), \\ \Lambda\frac{d}{d\Lambda}\,m^2(\Lambda) &= -\frac{3C_0}{2\pi^2}\,\Lambda^2\left(1 + O\left(\frac{m^2}{\Lambda^2}\right)\right), \end{aligned} \qquad (81)$$

where the momentum integrals are evaluated in Euclidean space. Note that, in order to obtain the above equation, the tadpole diagram (the first diagram in Fig. 4) has been evaluated. In general, at a scale Λ, the effective Lagrangian includes the contribution from all momenta $\Lambda < p < M$, which emerged through the loops, when the cutoff was set to M. Thus, the Lagrangian at lower scales gets very complicated. For example, even if the theory does not contain non-renormalizable operators at $\Lambda = M$, the flow equation tells us that they will necessarily emerge at lower scales. From dimensional reasons, these operators $\phi^2\Box\phi^2,\ \phi^2\Box^2\phi^2, \cdots$ will be suppressed by the respective powers of Λ if $m \ll \Lambda$ still holds. The first-order differential equations, analogous to one given in Eq. (81), emerge for the couplings $C_i(\Lambda)$ entering the quantity $S_{\mathsf{int}}(\phi, \Lambda)$. These are nothing but the conventional RG equations.

Below, we are going to prove the following crucial statement:

> Even if the different theories may look very different at the hard scale $\Lambda = M$, the difference vanishes, when going to lower scales $\Lambda = \Lambda_{\mathsf{eff}}$.

In order to understand what happens, consider first a toy example with two couplings: a marginal one, $C_0(\Lambda)$, and an irrelevant one, $C_1(\Lambda)$, putting, for simplicity, all other couplings to zero. Utilizing dimensional arguments, it can be shown that, generally, the RG equations for these couplings take the form:

$$\begin{aligned} \Lambda\frac{dC_0}{d\Lambda} &= \beta_0(C_0, \Lambda^2 C_1)\,, \\ \Lambda\frac{dC_1}{d\Lambda} &= \Lambda^{-2}\beta_1(C_0, \Lambda^2 C_1)\,, \end{aligned} \tag{82}$$

where β_0, β_1 denote the pertinent β-functions. Introducing now the dimensionless constants $\tilde{C}_0 = C_0$, $\tilde{C}_1 = \Lambda^2 C_1$, we may rewrite the RG equations as

$$\begin{aligned} \Lambda\frac{d\tilde{C}_0}{d\Lambda} &= \beta_0(\tilde{C}_0, \tilde{C}_1)\,, \\ \Lambda\frac{d\tilde{C}_1}{d\Lambda} - 2\tilde{C}_1 &= \beta_1(\tilde{C}_0, \tilde{C}_1)\,. \end{aligned} \tag{83}$$

The above equations define the *renormalization group flow* for the couplings $\tilde{C}_i(\Lambda)$, provided their values are fixed at some point:

$$\tilde{C}_i(\Lambda)\Big|_{\Lambda=M} = \tilde{C}_i^{(0)}\,, \qquad i = 0, 1\,. \tag{84}$$

Let us now imagine that the pair $(\bar{C}_0, \bar{C}_1)$ is a solution of the above equation. Consider small deviations $\varepsilon_i = \tilde{C}_i - \bar{C}_i$, for which the RG equations linearize

$$\begin{aligned} \Lambda\frac{d\varepsilon_0}{d\Lambda} &= \overline{\frac{d\beta_0}{d\tilde{C}_0}}\varepsilon_0 + \overline{\frac{d\beta_0}{d\tilde{C}_1}}\varepsilon_1\,, \\ \Lambda\frac{d\varepsilon_1}{d\Lambda} - 2\varepsilon_1 &= \overline{\frac{d\beta_1}{d\tilde{C}_0}}\varepsilon_0 + \overline{\frac{d\beta_1}{d\tilde{C}_1}}\varepsilon_1\,, \end{aligned} \tag{85}$$

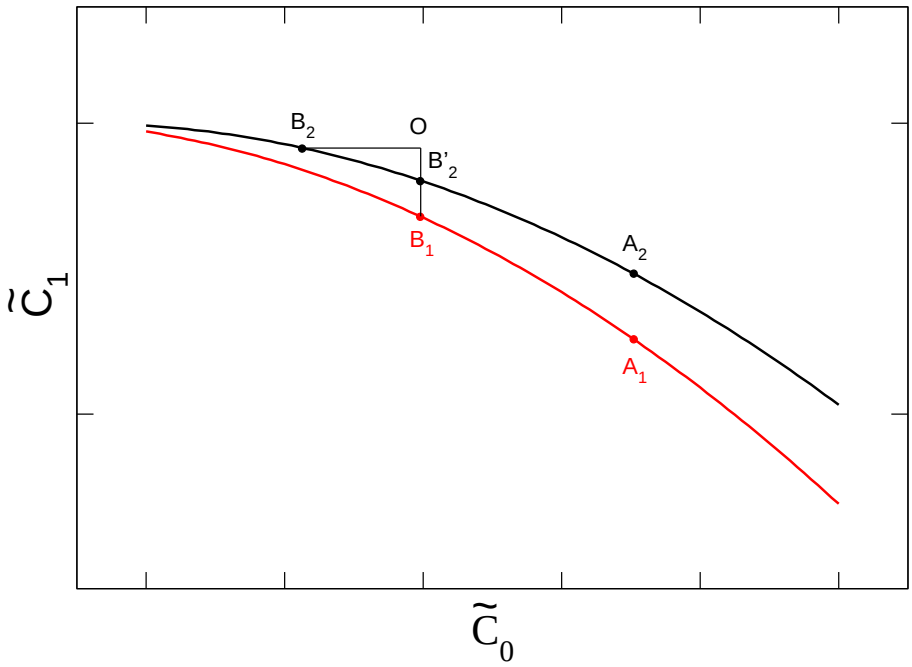

Figure 10: Two neighboring trajectories on the $\tilde{C}_0, \tilde{C}_1$ plane (see the text for more details).

where the bar means that the partial derivatives are evaluated at $(\bar{C}_0, \bar{C}_1)$. The term $-2\varepsilon_1$ in Eq. (85) will cause a damping of the variation of the deviations in the ε_1-direction. This will be demonstrated in the following.

Further, let us take some initial large value of $\Lambda = M$ and the initial pair $(\varepsilon_0, \varepsilon_1)$. This corresponds to the point A_1 in the $(\varepsilon_0, \varepsilon_1)$-plane, see Fig. 10. Now decrease the value of Λ so that it comes down to the low-energy region, that corresponds to moving along the trajectory, defined by the RG equations, from point A_1 to point B_1.

Now, let us take a slightly different initial values of the couplings at $\Lambda = M$, corresponding to the point A_2. RG evolution brings this point to B_2 along the neighboring trajectory. However, as seen from Fig. 10, there is a point B_2' on the same trajectory, whose distance to B_1 is minimal. Up to the first-order terms in ε_i, it is possible to write

$$\zeta_1 = |B_2' B_1| = |OB_1| - |OB_2'| \simeq \varepsilon_1 - \varepsilon_0 \frac{d\bar{C}_1/d\Lambda}{d\bar{C}_0/d\Lambda} \,. \tag{86}$$

Using Eqs. (83) and (85), it can be straightforwardly shown that ζ_1 obeys the following RG equation:

$$\Lambda \frac{d\zeta_1}{d\Lambda} - 2\zeta_1 = \left(\overline{\frac{\partial \beta_1}{\partial \tilde{C}}}_1 + \overline{\frac{\partial \beta_0}{\partial \tilde{C}}}_0 - \Lambda \frac{d}{d\Lambda} \ln \bar{\beta}_0 \right) \zeta_1 \,. \tag{87}$$

The solution of this equation at $\Lambda = \Lambda_{\mathsf{eff}}$ is given by:

$$\zeta_1(\Lambda_{\mathsf{eff}}) = \zeta_1(M) \left(\frac{\Lambda_{\mathsf{eff}}^2}{M^2} \right) \left(\frac{\beta_0(M)}{\beta_0(\Lambda_{\mathsf{eff}})} \right) \exp\left\{ \int_M^{\Lambda_{\mathsf{eff}}} \frac{dM'}{M'} \left(\overline{\frac{\partial \beta_1}{\partial \tilde{C}}}_1 (M') + \overline{\frac{\partial \beta_0}{\partial \tilde{C}}}_0 (M') \right) \right\} . \tag{88}$$

It is seen that if the couplings are sufficiently small, so that the integrand in Eq. (88) is small, and if β_0 does not vary very fast, then the leading suppression comes from the factor $(\Lambda_{\mathsf{eff}}^2/M^2)$. This means that all RG trajectories approach each other in the infrared, and there is one essential parameter left instead of two: the value of $\tilde{C}_1$ is predicted, given the value of $\tilde{C}_0$. Moreover, the value of the parameter $\tilde{C}_0$ is also not independent: it just marks the place, where one is on a single trajectory in the infrared – in other words, $\tilde{C}_0$ can be traded for the scale Λ_{eff}.

Imagine now that we vary the initial scale M and, correspondingly, the initial value of the coupling constant $\tilde{C}_0(M)$, leaving Λ_{eff} and $\tilde{C}_0(\Lambda_{\mathsf{eff}})$ fixed. Since in the infrared all trajectories converge to a single one, we again conclude that $\tilde{C}_1(\Lambda_{\mathsf{eff}})$ can be expressed in terms of $\tilde{C}_0(\Lambda_{\mathsf{eff}})$, up to the terms of order $(\Lambda_{\mathsf{eff}}^2/M^2)$. Stated differently, different theories at the scale M converge to a same theory with a single renormalizable coupling $\tilde{C}_0(\Lambda_{\mathsf{eff}})$ at a lower scale – the physics at a scale M gets decoupled from the physics at a much lower scale Λ_{eff}. Furthermore, the original coupling $C_1(\Lambda_{\mathsf{eff}}) = \tilde{C}_1(\Lambda_{\mathsf{eff}})/\Lambda_{\mathsf{eff}}^2$ is still power-suppressed in the low energy region, if the cutoff Λ_{eff} is taken much larger than the light particle mass and the characteristic momenta.

1.8 Landau pole in the ϕ^4-theory

In the toy example considered in the previous section, the coupling $\tilde{C}_0$ corresponds to a renormalizable, and the coupling $\tilde{C}_1$ to a non-renormalizable interaction. Let us now consider a purely renormalizable bare interaction, that means that the coupling $\tilde{C}_1$ vanishes at scale M, whereas $\tilde{C}_0$ takes some value $\tilde{C}_0(M)$. One may repeat the argumentation given above, and ensure that $\tilde{C}_1(\Lambda_{\mathsf{eff}})$ is expressed through $\tilde{C}_0(\Lambda_{\mathsf{eff}})$ ($\tilde{C}_1(\Lambda_{\mathsf{eff}})$ is nonzero at a lower scale Λ_{eff}). One now may consider the limit $M \to \infty$, which corresponds to removing the ultraviolet cutoff in the renormalized theory, and ensure that all renormalized couplings stay finite.

The argumentation remains similar in case of the bare non-renormalizable interaction that corresponds to $\tilde{C}_1(M) \neq 0$ – this merely corresponds to a different choice of a starting point on the trajectory in the $(\tilde{C}_0, \tilde{C}_1)$-plane. Only one aspect is different: If one does not start from the bare renormalizable theory, one can not take the UV cutoff M to infinity, since the coefficients in front of the non-renormalizable (irrelevant) operators appear in the Lagrangian with inverse powers of the mass scale M, and in the limit $M \to \infty$ these coefficients necessarily vanish.

In the following, we shall restrict ourselves to renormalizable interaction with a single coupling C_0 and consider the limit $M \to \infty$. It can be proved that this limit is well defined in all orders in perturbation theory (perturbative renormalizability of the ϕ^4-theory in 4 dimensions). A problem, however, arises due to the fact that, beyond perturbation theory, singular points may exist on the trajectories, precluding one from taking the limit $M \to \infty$.

Let us explain what is meant. Consider the situation, when the values of Λ_{eff} and $\tilde{C}_0(\Lambda_{\mathsf{eff}})$ are fixed, whereas M and, correspondingly, $\tilde{C}_0(M)$ are allowed to vary. At lowest order in perturbation theory, the following RG equation holds:

$$M\frac{d\tilde{C}_0(M)}{dM} = \beta_2 \tilde{C}_0(M)^2 + O(\tilde{C}_0^3)\,, \tag{89}$$

where the coefficient β_2 is calculable in perturbation theory and is *positive* (see the respective

exercise). Solving now the RG equation with the boundary condition at $M = \Lambda_{\mathsf{eff}}$, we get

$$\tilde{C}_0(M) = \frac{\tilde{C}_0(\Lambda_{\mathsf{eff}})}{1 - \beta_2 \tilde{C}_0(\Lambda_{\mathsf{eff}}) \ln(M/\Lambda_{\mathsf{eff}})} \,. \tag{90}$$

We see that, at a some *finite* value of M, the quantity $\tilde{C}_0(M)$ becomes infinite, and the limit $M \to \infty$ can not be performed, starting from the low values of the cutoff $M = \Lambda_{\mathsf{eff}}$. This is an example of the so-called Landau-pole [8–10] that leads to *triviality*: it is not possible to remove the UV cutoff of the theory, unless the theory is trivial, i.e., the renormalized coupling vanishes.

It can be, however, argued that, in the vicinity of the Landau pole, the bare constant becomes large and the higher-order terms become important. For this reason, we have to consider the RG equation with the exact β-function:

$$M \frac{d\tilde{C}_0(M)}{dM} = \beta(\tilde{C}_0(M)) \,, \qquad \int_{\tilde{C}_0(\Lambda_{\mathsf{eff}})}^{\tilde{C}_0(M)} \frac{dx}{\beta(x)} = \ln \frac{M}{\Lambda_{\mathsf{eff}}} \,. \tag{91}$$

As we know, the β-function at small values of the coupling constant is small and positive. Below we speculate about the possible form of the β-function on the whole real axis.

a) $\beta(x)$ is strictly positive for all x, and increases as x^α, with $0 < \alpha < 1$, as $x \to \infty$ (modulo logarithms), see Fig. 11a. As $M \to \infty$, the logarithm in the r.h.s. of Eq. (91) diverges. This means that $\tilde{C}_0(M) \to \infty$ as $M \to \infty$, and the integral in the l.h.s. diverges on the upper limit.

b) $\beta(x)$ is strictly positive for all x, and increases as x^α, with $\alpha > 1$, as $x \to \infty$ (modulo logarithms), see Fig. 11b. Now, $\tilde{C}_0(M) \to \infty$ at a finite value of M, and we recover the situation similar to the Landau pole.

c) $\beta(x)$ has a zero at some *fixed point* $x = \tilde{C}_0^*$, see Fig. 11c. In the vicinity of the zero, one may write $\beta(x) = A(x - \tilde{C}_0^*) + \cdots$. The RG equation can be rewritten as:

$$\frac{d\tilde{C}_0}{dt} = A(\tilde{C}_0 - \tilde{C}_0^*) + \cdots \,, \qquad t = \ln \frac{M}{\Lambda_{\mathsf{eff}}} \,, \qquad A < 0 \,. \tag{92}$$

A properly normalized solution to this equation is given by:

$$(\tilde{C}_0(M) - \tilde{C}_0^*) = (\tilde{C}_0(\Lambda_{\mathsf{eff}}) - \tilde{C}_0^*) \left(\frac{M}{\Lambda_{\mathsf{eff}}} \right)^A \,. \tag{93}$$

Since $A < 0$, $C_0(M) \to \tilde{C}_0^*$ as $M \to \infty$. $\tilde{C}_0^*$ is called an *UV fixed point* of the theory.

To summarize, if the β-function is strictly positive and grows faster than a linear function at infinity, fixing the value of $\tilde{C}_0(\Lambda_{\mathsf{eff}})$ and increasing M, it is seen that the bare coupling $\tilde{C}_0(M)$ explodes at a finite value of M. It is not possible to remove the UV cutoff due to the presence of the Landau pole.

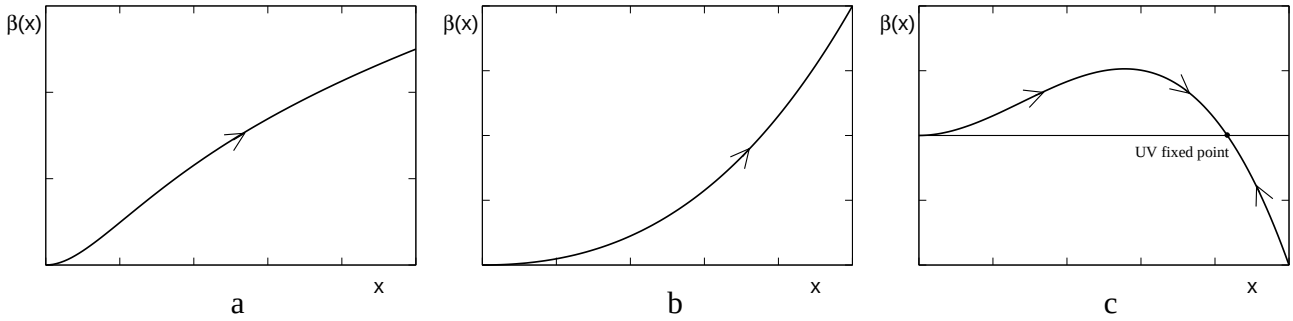

Figure 11: Different behavior of the RG β-functions, see details in the text.

We emphasize that if theory is trivial, it can be defined only with an intrinsic cutoff. However, it will be a meaningful theory well below the cutoff and ceases to be so if the cutoff should taken down to the renormalized mass of a particle. This leads to the constraints on the renormalized mass of the Higgs particle in the Standard Model, known as *triviality bound* [11].

The idea beyond the triviality bound can be simply explained. Consider the Lagrangian of the the Higgs sector of the theory that contains one complex doublet field Φ:

$$\mathcal{L}_H = \frac{1}{2}\,\partial_\mu\Phi^\dagger\partial^\mu\Phi - \frac{m_0^2}{2}\,\Phi^\dagger\Phi - \frac{\tilde{C}_0}{4}\,(\Phi^\dagger\Phi)^2\,. \tag{94}$$

The RG equation for the coupling constant to the lowest order yields:

$$\frac{1}{\tilde{C}_0(\Lambda)} = \frac{1}{\tilde{C}_0(M)} + \frac{3}{2\pi^2}\ln\frac{M}{\Lambda} \geq \frac{3}{2\pi^2}\ln\frac{M}{\Lambda}\,. \tag{95}$$

Here, the scale Λ is of order of the Higgs mass.

In the following, we shall use the following relations which are valid at tree level in the Standard Model:

$$M_H^2 = 2\tilde{C}_0 v^2\,, \qquad M_W^2 = \frac{1}{4}\,g^2v^2\,, \qquad g = \frac{e}{\sin\theta_W}\,. \tag{96}$$

Here, M_H and M_W are the masses of the Higgs and W-bosons, respectively, v is the vacuum expectation value of the Higgs field, e and g are the electromagnetic and the $SU(2)$ gauge couplings, respectively, and $\sin^2\theta_W \simeq 0.23$, where θ_W denotes the Weinberg angle. Using now Eqs. (95) and (96) and replacing the bare coupling with $\tilde{C}_0(\Lambda)$ at tree level, we obtain

$$\left(\frac{M_H}{M_W}\right)^2 = \frac{8\tilde{C}_0(\Lambda)}{g^2}\,. \tag{97}$$

Finally, substituting the value of the coupling g, the following (very rough) estimate can be obtained [11]:

$$\frac{M_H}{M_W} \leq \frac{4\pi}{g\sqrt{3}}\,\frac{1}{(\ln(M/\Lambda))^{1/2}} \simeq \frac{900\ \text{GeV}}{M_W}\,\frac{1}{(\ln(M/\Lambda))^{1/2}}\,. \tag{98}$$

As follows from the discussion above, the logarithm should be of order unity in this expression. Consequently, $M_H \leq 900$ GeV.

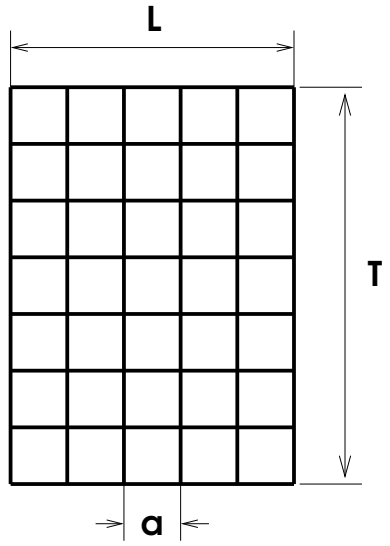

Figure 12: A schematic representation of an asymmetric lattice. The size of the box in space and time directions are $L = N_s a$ and $T = N_t a$, respectively. The lattice spacing a is taken to be universal.

1.9 Continuum limit on the lattice and the triviality of the ϕ^4-theory

Instead of a space-time continuum, a field theory can be formulated on a *lattice* (see, e.g., [6, 12, 13]). At present, the lattice represents the most popular (if not the only) ab initio non-perturbative approach in field theory.

In this section, we construct a ϕ^4 field theory on the lattice. We shall consider here the simplest case of the cubic Euclidean lattice, with the same number of sites N both in space and time direction. The lattice spacing is denoted by a, so the size of a cubic box is $L = Na$, and the total number of the lattice sites is equal to N^4. The role of an UV cutoff M is played by the inverse lattice spacing $a^{-1} \sim M$. In general, the number of sites in space and time directions can be taken different: N_s and N_t, respectively (see Fig. 12), as well as the lattice spacing, a_s and a_t. In this section, however, we shall not explore the general case.

The fields $\phi(x)$ live on the sites. The field derivatives in the continuum field theory are replaced by

$$\partial_\mu \phi(x) = \frac{1}{a} \left(\phi(x + a\hat{\mu}) - \phi(x) \right), \tag{99}$$

where $\hat{\mu}$ denotes a unit vector in the direction $\mu = 0, 1, 2, 3$.

The lattice action takes the form

$$S = a^4 \sum_{x,\mu} \frac{1}{2} \partial_\mu \phi(x) \partial_\mu \phi(x) + a^4 \sum_x \left(\frac{m_0^2}{2} \phi^2(x) + \frac{\tilde{C}_0}{4} \phi^4(x) \right), \tag{100}$$

where the sum over x runs over all lattice sites (periodic boundary conditions $\phi(x+Na\hat{\mu}) = \phi(x)$ are assumed), and $m_0, \tilde{C}_0$ are the bare mass and coupling, respectively. Using this discretized

action functional, one may transform the path integral for the Euclidean Green's functions into conventional multiple integrals that can be evaluated on (super)computers by using Monte-Carlo techniques. For example, the two-point function is given by

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \frac{1}{Z}\int \prod_z d\phi(z)\phi(x)\phi(y)e^{-S}\,, \tag{101}$$

where Z denotes a normalization factor.

At large Euclidean times, the two-point function decreases exponentially, defining the *correlation length*:

$$\sum_{\mathbf{x}} \langle 0|T\phi(\mathbf{x},t)\phi(\mathbf{0},0)|0\rangle \to \text{const}\cdot e^{-t/\xi}\,, \tag{102}$$

where the summation is carried out over the three-vector components $\mathbf{x}$.

In order to understand this behavior, let us consider the same Euclidean two-point function in the continuum and restrict ourselves to the free-field case

$$\int d^3\mathbf{x}\,\langle 0|T\phi(\mathbf{x},t)\phi(\mathbf{0},0)|0\rangle = \int d^3\mathbf{x}\int \frac{d^4k}{(2\pi)^4}\,\frac{e^{ik_4t+i\mathbf{k}\mathbf{x}}}{m_P^2+k^2}\,, \tag{103}$$

where m_P denotes the *physical mass.* Performing the Cauchy integration over k_4, we obtain that, at a large positive values of t,

$$\int d^3\mathbf{x}\,\langle 0|T\phi(\mathbf{x},t)\phi(\mathbf{0},0)|0\rangle \to \frac{e^{-m_Pt}}{2m_P}\,. \tag{104}$$

Comparing Eqs. (102) and (104), one may identify the correlation length with the inverse physical mass. The statement stays valid in the interacting theory as well. This can be easily checked by using Källèn-Lehmann representation for the two-point function. Finally, since we are considering the lattices which are invariant under the interchange of the space and time axes, the correlation langths in all directions are the same.

The central question is, how the continuum limit $a \to 0$ can be explicitly performed at the end in the lattice simulations. Physically, the continuum limit means $a \ll \xi$. We remind the reader that the physical mass m_P equals to the inverse of the correlation length. Then, in the continuum limit the physical mass measured in lattice units vanishes, $am_P \to 0$. What does this mean physically? Recall that the tree-level potential in our theory in Minkowski space is equal to $V(\phi) = \frac{1}{2}\,m_P^2\phi^2 + \frac{1}{4}\,\tilde{C}_0\phi^4$. If $m_P^2 > 0$, the vacuum of a system $\phi = 0$ is symmetric under $\phi \to -\phi$. This symmetry is however spontaneously broken for $m_P^2 < 0$, when the field ϕ acquires a nonzero vacuum expectation value. Hence, one may argue that in the continuum limit the system undergoes a phase transition from the unbroken phase with $m_P^2 > 0$ and $\langle 0|\phi|0\rangle = 0$ to the broken phase with $m_P^2 < 0$ and $\langle 0|\phi|0\rangle \neq 0$.

In the lattice simulations, one can always use units $a = 1$ (lattice units). In these units, one has only two dimensionless parameters am_0 and $\tilde{C}_0$. The continuum limit is achieved at the *critical surface*, i.e., at such values of the bare parameters $am_0, \tilde{C}_0$, for which $am_P \to 0$.

In order to investigate the continuum limit, let us specify the expression of the renormalized parameters of the theory in terms of the bare ones. The renormalized one-particle irreducible Green's functions in the theory are related to the unrenormalized ones, according to

$$\Gamma_R^{(n)}(p_1 \cdots p_n; \tilde{C}_R, m_R, m_R a) = Z_R^{n/2}(\tilde{C}_0, am_0)\Gamma_0^{(n)}(p_1 \cdots p_n; \tilde{C}_0, m_0, am_0)\,, \tag{105}$$

where $\tilde{C}_R$ denotes the renormalized coupling constant and m_R is the renormalized mass. $m_R = m_P$ in the vicinity of the critical surface, so the continuum limit implies $am_R \to 0$ as well. Moreover, renormalizability of the theory gives

$$\Gamma_R^{(n)}(p_1 \cdots p_n; \tilde{C}_R, m_R, m_R a) = \Gamma_R^{(n)}(p_1 \cdots p_n; \tilde{C}_R, m_R, 0) + O(a^2(\ln a)^k)\,. \tag{106}$$

The wave function renormalization constant Z_R ensures the normalization of the two-point function:

$$\frac{\partial}{\partial p^2}\,\Gamma_R^{(2)}(p, -p; \tilde{C}_R, m_R, m_R a)\bigg|_{p^2=0} = -1\,. \tag{107}$$

Finally, the renormalized mass and coupling constant are defined via

$$m_R^2 = -\Gamma_R^{(2)}(0, 0; \tilde{C}_R, m_R, m_R a)\,, \qquad \tilde{C}_R = -\Gamma_R^{(4)}(0, 0, 0, 0; \tilde{C}_R, m_R, m_R a)\,. \tag{108}$$

With the help of the above definitions, it is possible to express the renormalized couplings in terms of the bare ones. Using dimensional arguments, this relation will have the following form:

$$am_R = am_R(\tilde{C}_0, am_0)\,, \qquad \tilde{C}_R = \tilde{C}_R(\tilde{C}_0, am_0)\,. \tag{109}$$

Using the first equation, one may express am_0 in terms of am_R and substitute into the second equation. As a result, one gets:

$$\tilde{C}_R = \tilde{C}_R(\tilde{C}_0, am_R)\,. \tag{110}$$

Let us approach the continuum limit, fixing $\tilde{C}_0$ and approaching the critical surface by varying m_0. This is equivalent to fixing $\tilde{C}_0$ and considering the limit $am_R \to 0$. The renormalized coupling constant obeys the RG equation

$$(am_R)\frac{d}{d(am_R)}\,\tilde{C}_R\bigg|_{\tilde{C}_0 \text{ fixed}} = \beta_R(\tilde{C}_R, am_R)\bigg|_{\tilde{C}_0 \text{ fixed}}\,. \tag{111}$$

In the scaling region $am_R \to 0$ the dependence of the beta-function on the second argument disappears, and the RG equation takes the form:

$$(am_R)\frac{d}{d(am_R)}\,\tilde{C}_R\bigg|_{\tilde{C}_0 \text{ fixed}} = \beta_R(\tilde{C}_R)\,. \tag{112}$$

Note that the above β_R-function does not coincide with the β-function for the *unrenormalized* coupling, see Eq. (91), albeit these two are related. The solution of Eq. (111) is given by:

$$\ln am_R = \int^{\tilde{C}_R} \frac{dC'}{\beta_R(C')} + \text{const}\,. \tag{113}$$

If $\beta_R(C')$ does not have fixed points, except the trivial fixed point at $C' = 0$, then in the continuum limit $am_R \to 0$ the renormalized constant $\tilde{C}_R$ vanishes, i.e., the theory is trivial. To date, the positiveness of the β_R-function is a well established fact beyond perturbation theory (see, e.g. [14] and references therein).

Finally, we would like to mention that in this section, as well as in the previous section, we have used the word "triviality," referring, in fact, to the different phenomena. Namely, it should be stressed that the trivial theory in the continuum limit does not necessarily imply the emergence on the Landau pole, see, e.g., [15]. For more information on the triviality issue, see, e.g., Refs. [16, 17].

1.10 Symanzik effective action

In the previous section, we have considered the continuum limit $a \to 0$ in lattice field theories. Here we concentrate on the minimization of the *lattice artifacts*, which emerge at a finite a (for simplicity, we do not consider another type of the artifacts, which stem from the lattice box size L being finite, assuming $L \to \infty$). The example of such artifacts is given in Eq. (106), where the terms of order $a^2(\ln a)^k$ are responsible for the *scaling violation*. The question we ask is the following: is it possible to modify the action in Eq. (100), adding higher-order terms in a, so that the finite-spacing artifacts in the physical observables are canceled up to a given order in a? Stated differently, we are looking for the expansion

$$S = a^4 \sum_x \mathcal{L}(\phi, \partial_\mu \phi)\,, \qquad \mathcal{L}(\phi, \partial_\mu \phi) = \mathcal{L}_0 + a^2 \mathcal{L}_2 + a^4 \mathcal{L}_4 + \cdots\,, \tag{114}$$

where $\mathcal{L}_0$ can be read off from Eq. (100). Note that *formally* the action functional in Eq. (114) reduces to the one in Eq. (100) in the continuum limit $a \to 0$, so one may expect that they yield the same physics in this limit.

Let us illustrate the idea first for a free field. To this end, let us perform the Fourier transform

$$\phi(x) = \sum_p e^{-ipx} \tilde{\phi}(p)\,, \quad \tilde{\phi}^*(p) = \tilde{\phi}(-p)\,, \qquad p_\mu = \frac{2\pi}{L} n_\mu\,, \quad n_\mu = -N/2+1, \cdots, N/2\,, \tag{115}$$

in the quadratic part of the effective action, which now reads

$$S_0 = \frac{1}{2}(Na)^4 \sum_{p\mu} \tilde{\phi}^*(p) \frac{2 - e^{ip_\mu a} - e^{-ip_\mu a}}{a^2} \tilde{\phi}(p) + \frac{m_0^2}{2}(Na)^4 \sum_p \tilde{\phi}^*(p)\tilde{\phi}(p)\,. \tag{116}$$

The lattice dispersion law is given by

$$\sum_\mu (2 - 2\cos p_\mu a) + (am_0)^2 = 0\,. \tag{117}$$

Expanding, in powers of a, at the lowest order we get:

$$m_0^2 + \sum_\mu p_\mu^2 - \frac{a^2}{12} \sum_\mu p_\mu^4 + O(a^4) = 0\,. \tag{118}$$

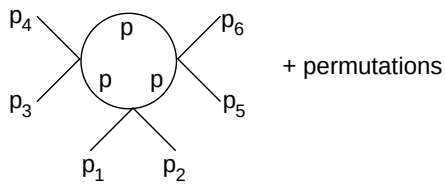

Figure 13: 6-point function at one loop in the ϕ^4-theory.

Consider now a modification to the original Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \frac{a^2}{24}\,\phi(x)\partial_\mu^4\phi(x)\,. \tag{119}$$

It is clear that the dispersion law for the modified Lagrangian coincides with the continuum result up to $O(a^4)$. The leading $O(a^2)$ corrections are exactly canceled. Thus, the modified Lagrangian will yield more accurate numerical results at a finite a than the original one.

The general solution beyond the tree level has been provided by Symanzik [18]. Here we illustrate Symanzik's idea on a simple example. Consider the 6-point function at one loop, see Fig. 13. This diagram is ultraviolet-finite (i.e., stays finite in the limit $a \to 0$). The scalar propagator on the lattice is given by (cf. Eq. (117)):

$$D(p) = \frac{1}{m_0^2 + \hat{p}^2}\,, \qquad \hat{p}_\mu = \frac{2}{a}\,\sin\!\left(\frac{ap_\mu}{2}\right), \qquad \hat{p}^2 = \sum_\mu \hat{p}_\mu\hat{p}_\mu\,. \tag{120}$$

For vanishing external momenta $p_i = 0,\ i = 1, \cdots, 6$, the one-loop proper diagram for the 6-point function is given by

$$\Gamma_6(0,0,0,0,0,0) \propto \tilde{C}_0^3 I_6\,, \qquad I_6 = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4}\,\frac{1}{(m_0^2+\hat{p}^2)^3}\,. \tag{121}$$

In the above integral, the integration is performed in the first Brillouin zone $-\pi/a < p_\mu < \pi/a$. Thus, as mentioned before, the inverse lattice spacing a^{-1} plays a role of an ultraviolet cutoff.

In order to calculate the above integral, one may split the integral into two parts $|p| < \delta/a$ and $|p| > \delta/a$, with $\delta \ll 1$ and $|p| = \left(\sum_\mu p_\mu p_\mu\right)^{1/2}$ (see e.g., Ref. [6]). In the first integral one may use the continuum form of a propagator

$$\begin{aligned} I_6^< &= \int_{|p|<\delta/a} \frac{d^4p}{(2\pi)^4}\,\frac{1}{(m_0^2+p^2)^3} = \frac{1}{16\pi^2}\int_0^{(\delta/a)^2} \frac{p^2dp^2}{(m_0^2+p^2)^3} \\ &= \frac{1}{32\pi^2 m_0^2} - \frac{a^2}{16\pi^2\delta^2} + O(a^4)\,. \end{aligned} \tag{122}$$

On the other hand,

$$I_6^> = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4}\,\frac{\theta(|p|-\delta/a)}{(m_0^2+\hat{p}^2)^3} = a^2\int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4}\,\frac{\theta(|p|-\delta)}{((am_0)^2 + 4\sum_\mu \sin^2(p_\mu/2))^3}\,. \tag{123}$$

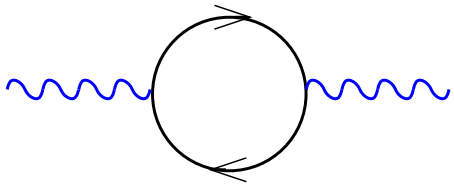

Figure 14: Photon self-energy in QED at one loop. The wiggly and solid lines denote photons and electrons, respectively.

Adding these two expressions, one gets:

$$
\begin{aligned}
I_6 &= I_6^< + I_6^> = \frac{1}{32\pi^2 m_0^2} + a^2 D + O(a^4)\,, \\
D &= \lim_{\delta\to 0} \int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4} \frac{\theta(|p|-\delta)}{(\sum_\mu 4\sin^2(p_\mu/2))^3} - \frac{1}{16\pi^2\delta^2}\,.
\end{aligned} \tag{124}
$$

The quantity D is finite and can be found by numerical integration.

Now, adding a counterterm

$$
\Delta\mathcal{L}_6 \propto a^2 \tilde{C}_0^3 D \phi^6(x) \tag{125}
$$

one may cancel the terms $\propto a^2$ in the 6-point function at a finite a. For such choice of the Lagrangian, the scaling violation starts at order a^4.

It is clear that the approach can be straightforwardly generalized to systematically remove lattice artifacts in higher orders in the lattice spacing a.

1.11 Decoupling in the different renormalization schemes

Finally, we wish to discuss the role of the choice of the regularization/renormalization schemes. As we have seen in the previous sections, the decoupling of a heavy scale in the theory proceeds differently, if different regularizations and renormalization schemes are used (for example, $\overline{\text{MS}}$ scheme in dimensional regularization vs cutoff regularization). In this section, we wish to elaborate on this issue.

Consider a well-known example: charge renormalization in QED (see, e.g., [19]). In dimensional regularization, the expression of the unrenormalized self-energy operator (see Fig. 14) takes the form:

$$
\begin{aligned}
\Sigma_{\mu\nu}(p) &= i(p_\mu p_\nu - p^2 g_{\mu\nu})\Sigma(p^2) \\
&= \frac{ie^2}{2\pi^2}(p_\mu p_\nu - p^2 g_{\mu\nu})\left\{-\frac{16\pi^2}{3}L - \int_0^1 dx\, x(1-x)\ln\frac{m_e^2 - p^2 x(1-x)}{\mu^2}\right\},
\end{aligned} \tag{126}
$$

where m_e denotes the electron mass. The renormalized propagator in the $\overline{\text{MS}}$ scheme takes the form

$$\Sigma^{\overline{\text{MS}}}(p^2;\mu^2) = -\frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\frac{m_e^2 - p^2x(1-x)}{\mu^2}\,. \tag{127}$$

Along with the $\overline{\text{MS}}$ scheme, we consider the mass-dependent scheme – a version of the BPHZ scheme[5], where the subtraction in the self-energy is made at $p^2 = \Lambda^2$:

$$\Sigma^{\text{MD}}(p^2;\Lambda^2) = \Sigma(p^2) - \Sigma(\Lambda^2) = -\frac{e^2}{2\pi^2}\int_0^1 dx x(1-x)\ln\frac{m_e^2 - p^2x(1-x)}{m_e^2 - \Lambda^2x(1-x)}\,. \tag{128}$$

The β-function in $\overline{\text{MS}}$ scheme is defined by

$$\mu\frac{de(\mu)}{d\mu} = \beta^{\overline{\text{MS}}}(e(\mu))\,, \qquad \beta^{\overline{\text{MS}}}(e) = \frac{e}{2}\,\mu\frac{d}{d\mu}\,\Sigma^{\overline{\text{MS}}}(p^2;\mu^2) = \frac{e^3}{12\pi^2}\,. \tag{129}$$

whereas in the mass-dependent scheme,

$$\Lambda\frac{de(\Lambda)}{d\Lambda} = \beta^{\text{MD}}(e(\Lambda))\,, \qquad \beta^{\text{MD}}(e) = \frac{e}{2}\,\Lambda\frac{d}{d\Lambda}\,\Sigma^{\text{MD}}(p^2;\Lambda^2) = \frac{e^3}{4\pi^2}\int_0^1 \frac{dx\, x^2(1-x)^2\Lambda^2}{m_e^2 - \Lambda^2x(1-x)}\,. \tag{130}$$

This gives:

$$\begin{aligned} \beta^{\text{MD}}(e) &= \frac{e^3}{12\pi^2}\,, && \text{if } m_e \ll \Lambda\,, \quad \text{the same result as in } \overline{\text{MS}}\,, \\ \beta^{\text{MD}}(e) &= \frac{e^3\Lambda^2}{60\pi^2 m_e^2}\,, && \text{if } m_e \gg \Lambda\,, \quad \text{decoupling}\ . \end{aligned} \tag{131}$$

Does this mean that the decoupling of the heavy scale occurs only within the cutoff regularization? Of course not, as can be seen from the discussion in section 1.4. The lesson to be learnt here is different. The decoupling is explicit if everything is expressed in terms of the *low-energy quantities*, like the the electric charge defined at a scale $\Lambda \ll m_e$ in the cutoff regularization. However, in the dimensional regularization defines the charge at a scale μ, and one is forced to take $\mu \simeq m_e$, in order to avoid large logarithms in the perturbation theory. Hence, the decoupling in not explicit. Additional renormalization is necessary to remove all high-energy contributions from the observables, see section 1.4. Stated differently, if one expresses $e(\mu)$ via the electromagnetic coupling at a some low-energy scale, and then re-expresses the physical observables in terms of this low-energy coupling, this would automatically ensure the decoupling of a heavy scale in all physical observables also in case of the dimensional regularization [4].

Below, we give an explicit example at one loop, which serves as an illustration to the above discussion. Let us sum up all self-energy insertions in the renormalized photon propagator, see Fig. 15. The resummed propagator then obeys the Dyson-Schwinger equation

$$G_{\mu\nu}(p) = G^0_{\mu\nu}(p) + iG^0_{\mu\lambda}(p)\Sigma_{\lambda\rho}(p)G_{\rho\nu}(p)\,, \tag{132}$$

[5]See, e.g., [20], chapter 5.

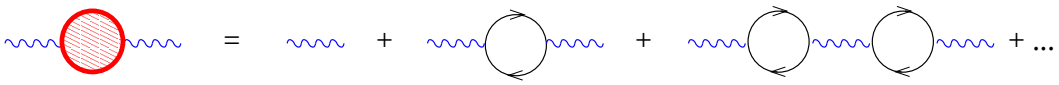

Figure 15: The renormalized photon propagator, where the self-energy insertions are summed up.

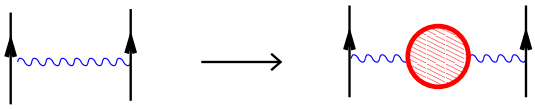

Figure 16: The vacuum polarization correction to the one-photon exchange diagram in the electron-electron scattering.

where $G^0_{\mu\nu}(p) = -g_{\mu\nu}/p^2$ is the free photon propagator (in the Feynman gauge). The solution of the above equation is given by

$$G_{\mu\nu}(p) = -\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2(1+\Sigma(p^2))} - \frac{p_\mu p_\nu}{p^4}\,. \tag{133}$$

Here $\Sigma(p^2)$ stands for the *renormalized* self-energy operator $\Sigma^{\overline{\mathsf{MS}}}(p^2;\mu^2)$ or $\Sigma^{\mathsf{MD}}(p^2;\Lambda^2)$, respectively.

Let us now insert the full photon propagator in the diagram for the electron-electron scattering, see Fig. 16. This corresponds to

$$\begin{aligned} T_{ee\to ee} &= \bar{u}(p_1's_1')\gamma^\mu u(p_1s_1)\,\frac{e^2 g_{\mu\nu}}{-q^2}\,\bar{u}(p_2's_2')\gamma^\nu u(p_2s_2) \\ &\to \bar{u}(p_1's_1')\gamma^\mu u(p_1s_1)\,e^2(\mu)G^{\overline{\mathsf{MS}}}_{\mu\nu}(p;\mu^2)\,\bar{u}(p_2's_2')\gamma^\nu u(p_2s_2) \\ &= \bar{u}(p_1's_1')\gamma^\mu u(p_1s_1)\,\frac{e^2(\mu)}{-q^2(1+\Sigma^{\overline{\mathsf{MS}}}(p^2;\mu^2))}\,\bar{u}(p_2's_2')\gamma_\mu u(p_2s_2)\,, \end{aligned} \tag{134}$$

where, p_i, s_i and p_i', s_i' for $i = 1,2$ denote the momenta and the spins of the electrons in the initial and in the final states, and $q = p_1' - p_1 = p_2' - p_2$. To ease notations, we do not display explicitly the second diagram, which is obtained by a permutation of two electrons in the initial or in the final state. Further, for definiteness, we have chosen the $\overline{\mathsf{MS}}$ scheme. The expression in the mass-dependent scheme is similar.

Consider these expressions at very low momenta $\mathbf{p}_i^2 \ll m_e^2$, $\mathbf{p}_i'^2 \ll m_e^2$. It is easily seen that

$$p_i^0 = \sqrt{m_e^2 + \mathbf{p}_i^2} = m_e + \frac{\mathbf{p}_i^2}{2m_e} + \cdots = m_e + O(m_e^{-1})\,. \tag{135}$$

Similar relations hold for $p_i'^0$. Further, $q^2 = (p_i'^0 - p_i^0)^2 - (\mathbf{p}_i' - \mathbf{p}_i)^2 = -(\mathbf{p}_i' - \mathbf{p}_i)^2 + O(m_e^{-1})$. The non-relativistic reduction of the Dirac spinors takes the form

$$\bar{u}(p's')\gamma^\mu u(ps) = \bar{u}(0,s')\,\frac{\not{p}' + m_e}{\sqrt{p'^0 + m_e}}\,\gamma^\mu\,\frac{\not{p} + m_e}{\sqrt{p^0 + m_e}}\,u(0,s) = (2m_e)g^{\mu 0}\delta_{s's}(1 + O(m_e^{-1}))\,. \tag{136}$$

According to this, the tree-level amplitude at low momenta is given by

$$\bar{u}(p_1' s_1')\gamma^\mu u(p_1 s_1)\,\frac{e^2 g_{\mu\nu}}{-q^2}\,\bar{u}(p_2' s_2')\gamma^\nu u(p_2 s_2) = (2m_e)^2 \delta_{s_1' s_1}\delta_{s_2' s_2}\frac{e^2}{\mathbf{q}^2}(1+O(m_e^{-1}))\,. \tag{137}$$

The Fourier-transform of Eq. (137) gives the Coulomb law

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3}\,e^{i\mathbf{q}\mathbf{r}}\,\frac{e^2}{\mathbf{q}^2} = \frac{e^2}{4\pi r} \doteq V_{\mathsf{Coul}}(r)\,. \tag{138}$$

It is immediately seen that taking into account the vacuum polarization leads to the modification of the Coulomb potential

$$V_{\mathsf{Coul}}(r) \to \int \frac{d^3\mathbf{q}}{(2\pi)^3}\,e^{i\mathbf{q}\mathbf{r}}\,\frac{e^2(\mu)}{\mathbf{q}^2(1+\Sigma^{\overline{\mathsf{MS}}}(-\mathbf{q}^2;\mu^2))} \doteq V(r)\,. \tag{139}$$

The expression for the mass-dependent scheme looks similarly.

We fix the parameters of theory (the electric charge) at large distances, i.e., by measuring the force acting on small charged oil droplets (Millikan-type experiment). The elementary charge measured in this manner corresponds to $\alpha = e^2/(4\pi) \simeq 1/137$. Since the distances in such an experiment are much larger than the Compton wavelength of an electron, in the momentum space we are looking the region $\mathbf{q}^2 \to 0$. In this region, the modified potential is well approximated by

$$V(\mathbf{q}) = \frac{e^2(\mu)}{\mathbf{q}^2(1+\Sigma^{\overline{\mathsf{MS}}}(0;\mu^2))} + \cdots\,. \tag{140}$$

From this expression, one can easily read off the relation between e and $e(\mu)$

$$e^2 = \frac{e^2(\mu)}{1+\Sigma^{\overline{\mathsf{MS}}}(0;\mu^2)} = \frac{e^2(\mu)}{1-\dfrac{e^2(\mu)}{12\pi^2}\,\ln\dfrac{m_e^2}{\mu^2}}\,. \tag{141}$$

There is a similar expression for the mass-dependent scheme

$$e^2 = \frac{e^2(\Lambda)}{1+\Sigma^{\mathsf{MD}}(0;\Lambda^2)} = \frac{e^2(\Lambda)}{1+\dfrac{e^2(\Lambda)}{2\pi^2}\displaystyle\int_0^1 dx\,x(1-x)\ln\left(1-\frac{\Lambda^2}{m_e^2}\,x(1-x)\right)}\,. \tag{142}$$

Differentiating $e(\mu)$ and $e(\Lambda)$ by μ and Λ, respectively, and taking into account that the quantity $e = \sqrt{4\pi\alpha}$ is a physical observable which is scale-independent, we again arrive at the RG equations (129) and (130). Finally, expressing everything in terms of the physical charge e, the modified Coulomb potential becomes

$$V(\mathbf{q}) = \frac{e^2}{\mathbf{q}^2(1+F(\mathbf{q}^2))}\,, \qquad F(\mathbf{q}^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\,x(1-x)\ln\left(1+\frac{\mathbf{q}^2}{m_e^2}\,x(1-x)\right). \tag{143}$$

Note that the quantity $F(\mathbf{q}^2)$ is scale-independent and is the same in both regularizations. The decoupling is explicit: $F(\mathbf{q}^2) \to 0$ as $m_e^2 \to \infty$. Thus, the whole difference between the two regularizations is hidden in Eqs. (141) and (142), which describe, how the renormalized charge $e(\mu)$ and $e(\Lambda)$ behave at $m_e \to \infty$, when e is fixed. This behavior is different. Namely, $e(\Lambda) \to e$, meaning that $e(\Lambda)$ stays a perfectly low-energy quantity in this limit. On the contrary, the limit $m_e \to \infty$ can not be performed at a fixed e and μ in the quantity $e(\mu)$, because of the large logarithms $\ln \dfrac{m_e^2}{\mu^2}$ in the perturbation theory. In order to suppress these logarithms, one has to take $\mu \sim m_e$, meaning that one is fixing the charge at a scale of order m_e. Thus, $e(\mu)$ is not a quantity defined at low energy, and the decoupling is not explicit if the expressions are written in terms of $e(\mu)$ instead of e.

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