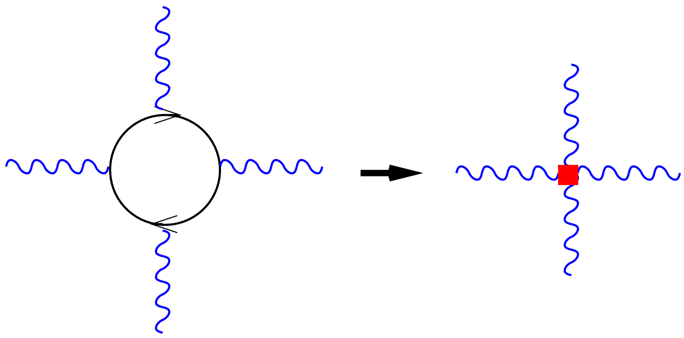

Figure 1: Four-point effective photon vertex emerging from the electron loop. The wiggly and solid lines denote photons and electrons, respectively.

1 Euler-Heisenberg Lagrangian

1.1 The role of symmetry

Consider QED for momenta/energies much smaller than the electron mass. According to the decoupling theorem, the only relevant degrees of freedom in the effective theory will be photons. Consequently, the effective Lagrangian of the theory should be constructed from the photon field A_μ. This is *not* a theory of free photons: the corresponding Lagrangian contains vertices with 4,6,... photons. These vertices describe interactions which in the original theory are mediated by closed electron loops (see Fig. 1).

In order to construct the effective Lagrangian, one could write down all possible terms, which can be built using the field A_μ. At the next step, the couplings in front of these terms should be matched to the underlying theory – QED. Here one arrives at the central question: what is a criterion for *possible* terms? In short, one has to follow the following guidelines:

- Use only those fields that correspond to the relevant degrees of freedom at a given energy.
- Respect all symmetries. For example, Lorentz invariance and the discrete C, P, T symmetries of QED should be maintained. However, in addition to these general symmetries, QED possesses a $U(1)$ gauge symmetry. In this section we shall demonstrate that the requirement of $U(1)$-invariance of the effective theory severely limits the number of the possible terms. This simplifies the procedure of constructing the effective Lagrangian.
- Respect counting rules. At a given order in the low-momentum expansion, only the operators with a pertinent mass dimension should be retained in the Lagrangian.

The rest of the present section is dedicated to the study of the implications of the $U(1)$ gauge symmetry in the construction of the effective Lagrangian. To this end, we find it convenient to use the language of the path integral. In an arbitrary covariant gauge (see, e.g., [21]), the Lagrangian of QED is given by

$$\mathcal{L}_{\mathsf{QED}} = \bar{\psi}(i\gamma^\mu(\partial_\mu + ieA_\mu) - m_e)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\xi}{2}(\partial^\mu A_\mu)^2 \,, \tag{1}$$

where ψ and A_μ are the electron and the photon fields, respectively, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, and ξ denotes the gauge fixing parameter. Observables (e.g., the S-matrix elements) do not depend on ξ, but the Green's functions do.

The generating functional of the Green's functions in QED is given by

$$Z(j, \eta, \bar{\eta}) = \int d\psi d\bar{\psi} dA_\mu \, \exp\left\{ i \int d^4x (\mathcal{L}_{QED} + \bar{\eta}\psi + \bar{\psi}\eta + j_\mu A^\mu) \right\}, \tag{2}$$

where j_μ and η denote external sources for the photon and electron fields, respectively. The Green's functions are obtained in the usual manner, namely by differentiating the generating functional with respect to the sources and, at the end, letting these sources vanish. Since we are interested in the derivation of the effective Lagrangian for the photons only, we may put $\eta = \bar{\eta} = 0$ from the beginning. The generating functional depends then on the argument j_μ only, and we can write

$$\begin{aligned} Z(j) &= Z(j, \eta, \bar{\eta})\Big|_{\eta=\bar{\eta}=0} \\ &= \int dA_\mu \exp\left\{ i \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{\xi}{2}(\partial^\mu A_\mu)^2 + \mathcal{L}_{\mathsf{eff}}(A_\mu) + j_\mu A^\mu \right) \right\}, \end{aligned} \tag{3}$$

where

$$\exp\left\{ i \int d^4x \mathcal{L}_{\mathsf{eff}}(A_\mu) \right\} = \int d\psi d\bar{\psi} \exp\left\{ i \int d^4x \bar{\psi}(i\gamma^\mu(\partial_\mu + ieA_\mu) - m_e)\psi \right\}. \tag{4}$$

Now, let us focus on the role of gauge invariance. It is straightforward to see that the integrand in Eq. (4) is invariant under the gauge transformations

$$\psi(x) \mapsto e^{-i\alpha(x)}\psi(x), \quad \bar{\psi}(x) \mapsto \bar{\psi}(x)e^{i\alpha(x)}, \quad A_\mu \mapsto A_\mu + \frac{1}{e}\,\partial_\mu\alpha(x). \tag{5}$$

Here, $\alpha(x)$ denotes the parameter of the gauge transformation.

Consequently, assuming that the path integral measure is also invariant with respect to the gauge transformations[1], and performing these transformations in Eq. (4), we easily obtain

$$\exp\left\{ i \int d^4x \mathcal{L}_{\mathsf{eff}}(A_\mu) \right\} = \exp\left\{ i \int d^4x \mathcal{L}_{\mathsf{eff}}(A_\mu + \partial_\mu\alpha) \right\}. \tag{6}$$

This means the effective Lagrangian $\mathcal{L}_{\mathsf{eff}}(A_\mu)$ is gauge-invariant, i.e., depends only on the gauge-invariant field tensor

$$\mathcal{L}_{\mathsf{eff}}(A_\mu) = \mathcal{L}_{\mathsf{eff}}(F_{\mu\nu}). \tag{7}$$

[1]At the first glance, this seems self-evident, since $d\psi d\bar{\psi} \doteq \prod_x d\psi(x)d\bar{\psi}(x) = \prod_x (e^{-i\alpha(x)}d\psi(x))(e^{i\alpha(x)}d\bar{\psi}(x))$. However, a certain care is needed performing the continuum limit, where the number of integration variables tends to infinity. One, in particular, needs to regularize the ultraviolet divergence emerging in this limit, and remove the regularization at the end of the calculations. In a given particular case, this can be done without a problem, justifying the assumption about the gauge-invariance of the fermionic measure. However, if the gauge transformation contains γ_5, the fermionic measure is, in general, no more gauge-invariant, giving rise to the so-called *anomalies*. In the following, we shall consider this issue in detail.

Note that the gauge invariance naturally leads to the modification of the counting rules: since $F_{\mu\nu}$ contains field derivatives, insertions of $F_{\mu\nu}$ into the loop diagrams result in the suppression of the loop corrections at low energies.

The non-linear contributions to the Lagrangian arise first at $O(m_e^{-4})$. To this order, there are only two such terms, consistent with all symmetries:

$$\mathcal{L}_{\mathsf{eff}} = -\frac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + \frac{\alpha^2}{m_e^4}\left\{c_1(F_{\mu\nu}F^{\mu\nu})^2 + c_2(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right\} + O(m_e^{-6})\,, \tag{8}$$

where $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. The overall factor m_e^{-4} appears on dimensional grounds, and the factor α^2, where $\alpha = e^2/(4\pi)$ is the fine-structure constant, appears because this term couples with four photons, each carrying a factor e. So, to this order, only two constants c_1, c_2 have to be determined from matching to QED.

Irrespective of the actual values of these constants, one may investigate, e.g., the dependence of the photon-photon scattering cross section on photon energy E at $E \ll m_e$. From the explicit form of the Euler-Heisenberg Lagrangian given in Eq. (8), it is straightforward to conclude that the scattering amplitude behaves like

$$A_{2\gamma} \sim \frac{\alpha^2 E^4}{m_e^4}\,, \tag{9}$$

where the factor E^4 stems from the four derivatives. The cross section behaves as

$$\sigma_{2\gamma} \sim \left(\frac{\alpha^2 E^4}{m_e^4}\right)^2 \frac{1}{E^2} = \frac{\alpha^4 E^6}{m_e^8}\,. \tag{10}$$

Note that, in the above expression, the phase-space factor E^{-2} is established on purely dimensional grounds: in the absence of a photon mass, the photon energy E is the only dimensionful parameter, on which the phase space factor can depend.

In the subsequent section we shall discuss the matching of the coefficients c_1, c_2 to the underlying theory. The direct method, based on the matching of Feynman integrals, turns out to be very cumbersome. We shall see that using path integral methods allows one to achieve the goal with a substantially smaller effort.

Historical note: While Euler, Heisenberg and Kockel analyzed light-by-light scattering using effective field theory as described (this might in fact be the first use of an EFT) in the mid 1930ties, the full calculation of this process based on the finite sum of box diagrams in full QED was only be performed by Karplus and Neumann in 1951 [23]. In fact, the low energy result of Karplus and Neumann limit exactly recovers the Euler-Heisenberg result. This is a beautiful example that in case of scale separation the EFT approach is much more *effective* than the calculation in the full theory. For a nice discussion on the history of the Euler-Heisenberg approach, sere Ref. [24].

1.2 Matching of the couplings in the effective Lagrangian

Below, we generally follow the discussion of the question given in the textbook of Itzykson and Zuber [21], which is based on the Fock-Schwinger proper time method. In the path integral by

Eq. (4) one may carry out Grassman integration over the variables $\psi, \bar{\psi}$. The answer is given by

$$\exp\left\{ i \int d^4x \mathcal{L}_{\mathsf{eff}}(A_\mu) \right\} = \det(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m_e)\,, \tag{11}$$

so that

$$\mathcal{L}_{\mathsf{eff}}(A_\mu) = -i \ln\det(D)\,, \qquad D = i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m_e\,. \tag{12}$$

In other words, calculating the determinant and expanding in powers of A_μ, one will reproduce all terms of the effective Lagrangian.

The key observation that simplifies the calculations dramatically, consists in the following: in order to establish the coefficients c_1, c_2, it suffices to consider the determinant for the *constant* electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. Defining the quantities a and b so that

$$a^2 - b^2 = \mathbf{E}^2 - \mathbf{B}^2 = \frac{1}{2}\, F_{\mu\nu}F^{\mu\nu}\,, \qquad ab = \mathbf{EB} = \frac{1}{4}\, F_{\mu\nu}\tilde{F}^{\mu\nu}\,, \tag{13}$$

It can be shown that (see below)

$$-i \ln\det(D) = \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s}\, e^{-is(m_e^2 - i0)} \left(e^2 ab \frac{\cosh(eas)\cos(ebs)}{\sinh(eas)\sin(ebs)} - \frac{1}{s^2} \right). \tag{14}$$

Expanding in powers of a, b and using Eq. (13), we obtain

$$\begin{aligned} -i \ln\det(D) &= \frac{e^2}{24\pi^2}\,(\mathbf{E}^2 - \mathbf{B}^2) \int_0^\infty \frac{ds}{s}\, e^{-is(m_e^2 - i0)} \\ &- \left(\frac{e^4}{360\pi^2}\,(\mathbf{E}^2 - \mathbf{B}^2)^2 + \frac{7e^4}{360\pi^2} (\mathbf{EB})^2 \right) \int_0^\infty ds s\, e^{-is(m_e^2 - i0)} + \cdots\,. \end{aligned} \tag{15}$$

The ultraviolet divergence at $s = 0$ in the first integral can be removed by the renormalization of the free-photon term $\sim F_{\mu\nu}F^{\mu\nu}$ in the Lagrangian. The second term is finite. Performing the integration over s in this term, we finally get

$$-i \ln\det(D) = C F_{\mu\nu}F^{\mu\nu} + \frac{\alpha^2}{90 m_e^4}\,(F_{\mu\nu}F^{\mu\nu})^2 + \frac{7\alpha^2}{720 m_e^4}\,(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \cdots\,, \tag{16}$$

where C denotes an ultraviolet-divergent constant. From this equation, one may directly read off the values of c_1, c_2:

$$c_1 = \frac{1}{90}\,, \qquad c_2 = \frac{7}{360}\,. \tag{17}$$

1.3 The fermion determinant in a constant field*

Below we give an explicit calculation of the fermion determinant in a constant field. Namely, our final goal will be to derive Eq. (14), which was already used to match the coefficients c_1, c_2 in the Euler-Heisenberg Lagrangian.

Subtracting a constant that does not depend on the field A_μ, we may define

$$\ln\det(\bar{D}) = \ln\det(D) - \ln\det(i\gamma^\mu\partial_\mu - m_e) = \mathrm{Tr}\,\ln\Big((i\,\not{\partial} - e\,\not{A} - m_e)(i\,\not{\partial} - m_e)^{-1}\Big)\,, \quad (18)$$

where "Tr" denotes the trace both in the x-space and in the space of the Dirac indices.

Using $C\gamma_\mu C^{-1} = -\gamma_\mu^T$, where $C = i\gamma^2\gamma^0$, Eq. (18) can be rewritten as

$$2\ln\det(\bar{D}) = \mathrm{Tr}\,\ln\Big(((i\,\not{\partial} - e\,\not{A})^2 - m_e^2)((i\,\not{\partial})^2 - m_e^2)^{-1}\Big)\,. \quad (19)$$

Further, using the relation

$$\ln\frac{\alpha}{\beta} = \int_0^\infty \frac{ds}{s}\,(e^{is(\beta+i0)} - e^{is(\alpha+i0)})\,, \quad (20)$$

the equation (19) can be rewritten as

$$2\ln\det(\bar{D}) = -\int_0^\infty \frac{ds}{s}\,e^{-is(m_e^2-i0)}\int d^4x\,\mathrm{tr}\Big(\langle x|e^{is(i\not{\partial}-e\not{A})^2}|x\rangle - \langle x|e^{is(i\not{\partial})^2}|x\rangle\Big)\,, \quad (21)$$

where "tr" denotes the trace over the Dirac indices only.

In order to further simplify this expression, we use the identity

$$(i\,\not{\partial} - e\,\not{A})^2 = (i\partial_\mu - eA_\mu)^2 - \frac{e}{2}\,\sigma_{\mu\nu}F^{\mu\nu}\,, \qquad \sigma_{\mu\nu} = \frac{i}{2}\,[\gamma_\mu, \gamma_\nu]\,. \quad (22)$$

As mentioned above, we restrict ourselves to constant electric and magnetic fields

$$E_3 = F^{30} = -F^{03} = a\,, \qquad B^3 = F^{12} = -F^{21} = b\,. \quad (23)$$

All other entries in the tensor $F^{\mu\nu}$ are equal to 0. Therefore,

$$-\frac{e}{2}\,\sigma_{\mu\nu}F^{\mu\nu} = e\sigma_3 \otimes \begin{pmatrix} b & -ia \\ -ia & b \end{pmatrix}\,, \quad (24)$$

where σ_3 stands for the 2×2 Pauli matrix, and

$$\mathrm{tr}\,\exp\Big(-\frac{ise}{2}\,\sigma_{\mu\nu}F^{\mu\nu}\Big) = 4\cos(ebs)\cosh(eas)\,. \quad (25)$$

Consequently,

$$\begin{aligned}\ln\det(\bar{D}) &= -2\int_0^\infty \frac{ds}{s}\,e^{-is(m_e^2-i0)}\int d^4x\,\Big(\cos(ebs)\cosh(eas)\langle x|e^{is(i\partial_\mu-eA_\mu)^2}|x\rangle \\ &- \langle x|e^{is(i\partial_\mu)^2}|x\rangle\Big)\,. \end{aligned} \quad (26)$$

In order to calculate matrix elements in Eq. (26), define

$$p_\mu = i\partial_\mu \,, \qquad [x_\mu, p_\nu] = -ig_{\mu\nu} \,. \tag{27}$$

Further, the electromagnetic field in our case is

$$A_0 = A_2 = 0 \,, \qquad A_1 = -bx_2 \,, \qquad A_3 = ax_0 \tag{28}$$

Then,

$$\begin{aligned} H &\doteq (p_\mu - eA_\mu)^2 = p_0^2 - (p_1 + ebx_2)^2 - p_2^2 - (p_3 - eax_0)^2 \\ &= e^{ip_0p_3/ea} e^{ip_1p_2/eb} (p_0^2 - e^2a^2x_0^2 - p_2^2 - e^2b^2x_2^2) e^{-ip_1p_2/eb} e^{-ip_0p_3/ea} \doteq H_{03} + H_{12} \,, \end{aligned} \tag{29}$$

and

$$\langle x|e^{isH}|x\rangle = \langle x_0x_3|e^{isH_{03}}|x_0x_3\rangle\langle x_1x_2|e^{isH_{12}}|x_1x_2\rangle \,. \tag{30}$$

In the equation above, we use the notation $|x\rangle \doteq |x_0x_1x_2x_3\rangle = |x_0\rangle \otimes |x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle$. Calculating the matrix elements in Eq. (30) separately, for the first one we get

$$\begin{aligned} \langle x_0x_3|e^{isH_{03}}|x_0x_3\rangle &= \int \frac{dp_0dp_3dp_0'dp_3'dq_0dq_3dq_0'dq_3'}{(2\pi)^8}\, e^{ix_0(p_0-p_0')+ix_3(p_3-p_3')} \\ &\times \langle p_0p_3|e^{ip_0p_3/ea}|q_0q_3\rangle\langle q_0q_3|e^{is(p_0^2-e^2a^2q_0^2)}|q_0'q_3'\rangle\langle q_0'q_3'|e^{-ip_0p_3/ea}|p_0'p_3'\rangle \,. \end{aligned} \tag{31}$$

Using the relations

$$\begin{aligned} \langle p_0p_3|e^{\pm ip_0p_3/ea}|q_0q_3\rangle &= e^{\pm ip_0p_3/ea}(2\pi)^2\delta(p_0-q_0)\delta(p_3-q_3) \,, \\ \langle q_0q_3|e^{is(p_0^2-e^2a^2q_0^2)}|q_0'q_3'\rangle &= (2\pi)\delta(q_3-q_3')\langle q_0|e^{is(p_0^2-e^2a^2q_0^2)}|q_0'\rangle \,, \end{aligned} \tag{32}$$

we easily get

$$\langle x_0x_3|e^{isH_{03}}|x_0x_3\rangle = \frac{ea}{4\pi^2}\int dp_0\, \langle p_0|e^{is(p_0^2-e^2a^2x_0^2)}|p_0\rangle \,. \tag{33}$$

In order to calculate the matrix element in Eq. (33), we consider the quantum-mechanical problem of a harmonic oscillator with a Hamiltonian

$$h_{\mathsf{osc}} = \frac{1}{2}\, p_0^2 + \frac{\omega_0^2}{2}\, x_0^2 \,. \tag{34}$$

The eigenfunctions of this Hamiltonian are labeled by an index $n = 0, 1, \cdots$

$$h_{\mathsf{osc}}|n\rangle = \omega_0\left(n + \frac{1}{2}\right)|n\rangle \,. \tag{35}$$

Now, consider the matrix element

$$\begin{aligned}\frac{ea}{4\pi^2}\int dp_0\,\langle p_0|e^{2ish_{\mathrm{osc}}}|p_0\rangle &= \frac{ea}{4\pi^2}\sum_{n=0}^{\infty}\int dp_0|\langle p_0|n\rangle|^2\exp\biggl\{2is\omega_0\biggl(n+\frac{1}{2}\biggr)\biggr\} \\ &= \frac{ea}{2\pi}\sum_{n=0}^{\infty}\exp\biggl\{2is\omega_0\biggl(n+\frac{1}{2}\biggr)\biggr\}. \qquad (36)\end{aligned}$$

In order to recover the original matrix element in Eq. (33), one has to substitute $\omega_0 \to iea$. Carrying out the summation over n, we finally arrive at the following result

$$\langle x_0x_3|e^{isH_{03}}|x_0x_3\rangle = \frac{ea}{4\pi\sinh(eas)}\,. \qquad (37)$$

Evaluating the second matrix element in Eq. (30) with the same method, we obtain

$$\langle x_1x_2|e^{isH_{12}}|x_1x_2\rangle = \frac{eb}{4\pi i\sin(ebs)}\,. \qquad (38)$$

Finally, substituting Eqs. (37) and (38) into Eqs. (30) and (26), we arrive at Eq. (14), which was used for matching the couplings of the Euler-Heisenberg Lagrangian.

1.4 Blue sky

In this section, we shall consider another example with very long-wavelength photons. In particular, we wish to address the scattering of long-wavelength photons by atoms. This process goes under the name of the Rayleigh scattering. Below, we mainly follow the reasoning outlined in Ref. [19].

Since the momentum transfer in the process is very small, the atoms can be described non-relativistically. The pertinent free Lagrangian is given by

$$\mathcal{L}_{\mathsf{atom}} = \Psi^\dagger\biggl(i\partial_t - M_{\mathsf{atom}} + \frac{\nabla^2}{2M_{\mathsf{atom}}}\biggr)\Psi\,, \qquad (39)$$

where Ψ denotes the non-relativistic field

$$\Psi(\mathbf{x},t) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,e^{-ik_0t+i\mathbf{kx}}a(\mathbf{k})\,. \qquad (40)$$

Here, $a(\mathbf{k})$ denotes an annihilation operator of the atom.

The interaction Lagrangian of an atom with photons must be gauge-invariant. Since atoms are electrically neutral, the only gauge-invariant objects from which this Lagrangian can be constructed, are the atom fields, along with the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. To the lowest order, the interaction Lagrangian can be written as

$$\mathcal{L}_{\mathsf{int}} = a_0^3\Psi^\dagger(d_1\mathbf{E}^2 + d_2\mathbf{B}^2)\Psi\,, \qquad (41)$$

where, from dimensional counting, a_0 has the dimension of length, and d_1, d_2 are dimensionless. a_0 is the typical size of the atom which is the only scale in the problem. We further assume that d_1, d_2 are of natural size, $d_1, d_2 \sim \mathcal{O}(1)$, so that the operator in Eq. (41) is indeed leading.

According to Eq. (41), the scattering amplitude $A_{\gamma-\mathsf{atom}} \sim a_0^3 E^2$, where E is the energy of the photon. The scattering cross section is given by

$$\sigma_{\gamma-\mathsf{atom}} = \text{phase space} \times |A|^2 \sim \text{phase space} \times a_0^6 E^4 \,. \tag{42}$$

From the above formula it follows that the phase space factor should be dimensionless and thus energy-independent. Consequently, the cross section of the Rayleigh scattering is proportional to the fourth power of the photon energy. This in particular, explains, why the sky is blue – because the blue light is scattered more intensively by the atoms in the atmosphere than the red light.

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