Lectures on Jet Physics: Problems

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Problem 1) Splitting Functions

Infrared enhancements in the quark and gluon branching processes $q \to qg$, $g \to gg$, and $g \to q\bar{q}$ are key ingredient in the formation of jets. The structure of collinear enhancements is described by splitting functions P_{ab}, which to first order in the strong coupling α_s are:

$$\begin{aligned} P_{qq}^{(0)}(x) &= \frac{\alpha_s(\mu)}{2\pi} C_F \Big[\frac{1+x^2}{(1-x)_+} + a_q\, \delta(1-x) \Big] \,, \\ P_{qg}^{(0)}(x) &= \frac{\alpha_s(\mu)}{2\pi} T_R \Big[x^2 + (1-x)^2 \Big] \,, \\ P_{gq}^{(0)}(x) &= \frac{\alpha_s(\mu)}{2\pi} C_F \Big[\frac{1+(1-x)^2}{x} \Big] \,, \\ P_{gg}^{(0)}(x) &= \frac{\alpha_s(\mu)}{2\pi} 2C_A \Big[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) \Big] + a_g\, \delta(1-x) \,. \end{aligned} \tag{1}$$

Here the color factors are $C_F = 4/3$, $T_R = 1/2$, and $C_A = 3$, and you will determine the constants a_q and a_g below. Each $P_{ab}^{(0)}(x)$ should be thought of as the probability of finding a parton of type a inside an initial parton b, with a having a fraction x of the parent b's momentum. These expressions include the familiar Dirac δ-function, and the less familiar $+$-function. The latter is defined by $1/(1-x)_+ = 1/(1-x)$ for any $x < 1$, and by the fact that the singularity at $x = 1$ is regulated such that

$$\int_0^1 dx \frac{1}{(1-x)_+} g(x) = \int_0^1 dx \frac{1}{(1-x)} \Big[g(x) - g(1) \Big] \tag{2}$$

for any function $g(x)$.

a) Derive results for the constants a_q and a_g such that quark number is conserved:

$$\int_0^1 dx\, P_{qq}^{(0)}(x) = 0 \,, \tag{3}$$

and momentum is conserved by the quark and gluon splittings:

$$\int_0^1 dx\, x \left[P_{qq}^{(0)}(x) + P_{gq}^{(0)} \right] = 0 \,, \qquad \int_0^1 dx\, x \left[P_{gg}^{(0)}(x) + 2n_f P_{qg}^{(0)} \right] = 0 \,. \tag{4}$$

Here n_f is the number of light quarks. Show that you can rewrite $P_{qq}^{(0)}$ as $P_{qq}^{(0)}(x) = (\alpha_s(\mu) C_F / 2\pi) \left[(1+x^2)/(1-x) \right]_+$.

Given an initial distribution of quarks $q(\xi, \mu_0)$ and gluons $g(\xi, \mu_0)$ at a momentum scale μ_0, the distribution of quarks at a scale μ_1 is given by

$$q(x, \mu_1) = q(x, \mu_0) + \int_{\mu_0}^{\mu_1} \frac{2\, d\mu}{\mu} \int_x^1 \frac{d\xi}{\xi} \Big[P_{qq}^{(0)}\Big(\frac{x}{\xi}\Big) q(\xi, \mu) + P_{qg}^{(0)}\Big(\frac{x}{\xi}\Big) g(\xi, \mu) \Big] \,, \tag{5}$$

where the terms in the integral account for the possibility that the quark we observe came from a splitting rather than the initital distribution.

b) By iterative use of Eq. (5) derive a series in α_s that writes $q(x,\mu_1)$ in terms of terms only involving q's and g's at $\mu=\mu_0$. Draw Feynman diagrams to describe physically what is happening with the various terms in your infinite series.

The subtraction term from the plus function in $P_{qq}^{(0)}$ in Eq. (5) sets $\xi=x$, and is related to evolution to the scale μ_1 without branching, so strictly speaking Eq. (5) does not yet have a clean separation between branching and no-branching. To better distinguish the two possibilites we will rewrite this equation in a different way. To simplify the formulas below, we'll set $P_{qg}^{(0)}=0$. The probability that a quark does not split when it evolves from μ_0 to μ_1 is then given solely by the quark Sudakov form factor:

$$\Delta_{qq}(\mu_1,\mu_0) \;=\; \exp\Big[-\int_{\mu_0}^{\mu_1}\frac{2\,d\mu}{\mu}\int dx\,\hat{P}_{qq}^{(0)}(x)\Big]\,. \tag{6}$$

Here $\hat{P}_{qq}^{(0)}(x)=(\alpha_s(\mu)C_F/2\pi)\,(1+x^2)/(1-x)$ and we will assume that the limits on the x integration keep us away from the singularity at $x=1$ (more on this in part d).

c) Taking $\mu_1 d/d\mu_1$ derive differential equations for $q(x,\mu_1)$ and $\Delta_{qq}(\mu_1,\mu_0)$. Next derive an equation for $\mu_1 d/d\mu_1(q/\Delta_{qq})$ and show that its solution yields

$$q(x,\mu_1)=\Delta_{qq}(\mu_1,\mu_0)\,q(x,\mu_0)+\int_{\mu_0}^{\mu_1}\frac{2\,d\mu}{\mu}\frac{\Delta_{qq}(\mu_1,\mu_0)}{\Delta_{qq}(\mu,\mu_0)}\int\frac{d\xi}{\xi}\hat{P}_{qq}^{(0)}\Big(\frac{x}{\xi}\Big)q(\xi,\mu)\,. \tag{7}$$

Since this result does not involve the +-function we can interpret the second term as the probability from splitting, and the first term as the probability of having no splitting. Thus the Sudakov form factor in the first term gives the no-splitting probability when we evolve from μ_0 to μ_1. Can you provide an interpretation for the presence of the ratio of Δ_{qq}'s in the second term? This result with its probabilistic interpretation is used in parton shower Monte Carlo programs that describe parton branching and QCD jets.

Next you will calculate the form of the exponent in $\Delta_{qq}(\mu_1,\mu_0)$. The result can be thought of as an infinite series in $\alpha_s(\mu_0)$, but to keep things simple for this calculation we'll freeze $\alpha_s(\mu)=\alpha_s(\mu_0)$ and approximate $P_{qq}^{(0)}(x)\simeq(\alpha_s(\mu_0)C_F/\pi)/(1-x)$ which will allow us to determine the dominant term for $\mu_1\gg\mu_0$.

d) Lets identify the evolution scale parameter as the parton's virtual mass squared, $\mu^2=p^2\equiv t'$, and hence impose the corresponding kinematic limits on the x-integral: $\mu_0^2/\mu^2<x<1-\mu_0^2/\mu^2$ (obtained for particles with large energy and expanding $\mu_0\ll\mu$). With the approximations above and these limits perform the double integral in Eq. (6), and show that your result involves a $\ln^2(\mu_1/\mu_0)$. This double log is related to the presence in the branching and no-branching probabilities of the soft ($x\to1$) singularity and the collinear singularity described by the splitting function equations.

Problem 2) SCET, Wilson Lines, and Renormalization Group Evolution

In this problem you will get more familiar with the Wilson lines that are present in SCET, and show that solving the anomalous dimension equation of an SCET operator produces a Sudakov exponential.

Consider the two-jet production process through a virtual photon in SCET, namely $e^+e^- \to J_n J_{\bar{n}} X_s$ where J_n is a jet in the $n = (1,0,0,-1)$ direction, $J_{\bar{n}}$ is a jet in the $\bar{n} = (1,0,0,1)$ direction, and any remaining particles in the final state are soft, contained in X_s. We will study the leading operator for this process written down in lecture

$$O = \bar{\xi}_n \, W_n \, \gamma_\perp^\mu \, W_{\bar{n}}^\dagger \, \xi_{\bar{n}} \,. \tag{8}$$

At lowest order in the strong coupling it will produce one collinear quark and one collinear antiquark.

a) In QCD the corresponding current is simply $\bar{\psi}\gamma^\mu\psi$. To derive the presence of W_n consider the Feynman diagram that attaches an $\bar{n}\cdot A_n$ collinear gluon to the $\xi_{\bar{n}}$ collinear antiquark. The resulting propagator is far offshell $\sim Q^2$ and we can expand in small momentum keeping only the leading term. Show that this result corresponds to the momentum space Feynman rule that is linear in the gluon field that can be derived from the position space Wilson line

$$W_n(y^+) = P \exp\left(ig \int_{-\infty}^0 ds \, \bar{n}\cdot A_n(s\bar{n} + y^+) \right).$$

[Here P is path-ordering which is needed in general to tell us how to order color matrices, but which can be ignored for one-gluon. For those looking for more challenge, consider extending your Feynman diagram calculation to two $\bar{n}\cdot A_n$ gluons to pick out the next term.]

Next we will consider the coupling of soft gluons to our collinear quarks. The interactions simplify to such a large extent that this is often referred to as soft-collinear decoupling. In particular, at leading order they only involve a Wilson line built from soft gluons

$$Y_n(x) = P \exp\left(ig \int_{-\infty}^0 ds \, n\cdot A_{us}(sn^\mu + x^\mu) \right), \tag{9}$$

and the analog $Y_{\bar{n}}$ where we swap $n \leftrightarrow \bar{n}$. The result in Eq. (9) satisfies $Y_n^\dagger Y_n = 1$ and has the equation of motion $n\cdot D_{us}\, Y_n = 0$.

b) Start with the leading order Lagrangian, $\mathcal{L}^{(0)}$ for a collinear quark in SCET given in the lecture notes. This action gives eikonal couplings to ultrasoft gluons. Make the field redefinitions $\xi_n = Y_n \xi_n^{(0)}$ and $A_n = Y_n A_n^{(0)} Y_n^\dagger$ to obtain a Lagrangian $\mathcal{L}^{(0)}(\xi_n^{(0)}, A_n^{(0)})$. Show explicitly that this new Lagrangian has no coupling to $n\cdot A_{us}$ gluons. Demonstrate that after this field redefinition the new form of the operator we are studying is

$$O = (\bar{\xi}_n \, W_n \,)\gamma_\perp^\mu (Y_n^\dagger Y_{\bar{n}})(W_{\bar{n}}^\dagger \, \xi_{\bar{n}}) \,. \tag{10}$$

What is the Feynman rule that this operator generates for one-soft gluon? Besides a connection through color indices this result involves the product of three sets of fields that do not interact through their Lagrangians. This property leads directly to a factorization theorem for the cross-section for this process of the type discussed in lecture.

c) This part of the problem is more challenging, and you can solve d) below without first solving this part. Draw the five one-loop Feynman diagrams that are non-zero for $e^+e^- \to q_n \bar{q}_{\bar{n}}$ (use Feynman gauge for all gluons when determining which graphs are zero). Here q_n has n-collinear momentum p, and $\bar{q}_{\bar{n}}$ has $\bar{n}$-collinear momentum $\bar{p}$ and you should work in the CM frame. All graphs but one can be directly read off using the loop computations given in the handout notes, as long as you use the same IR regulator. That is, you should keep both collinear quarks offshell, $p^2 \neq 0$ and $\bar{p}^2 \neq 0$. Compute the divergent terms in the one remaining ultrasoft graph using dimensional regularization in the UV. Add up the $1/\epsilon$ terms from the graphs in c) and determine the lowest order anomalous dimension equation for C the Wilson coefficient of O.

d) Keeping only the leading logarithmic term in the anomalous dimension equation for C we have

$$\frac{d}{d\ln\mu} C(Q,\mu) \;=\; -\frac{2C_F\alpha_s(\mu)}{\pi} \ln\Big(\frac{\mu}{Q}\Big) C(Q,\mu) + \ldots\,, \tag{11}$$

which must be solved simultaneously with the equation for the running coupling

$$\frac{d}{d\ln\mu}\alpha_s(\mu) \;=\; -\frac{\beta_0}{2\pi}\alpha_s^2(\mu) + \ldots\,. \tag{12}$$

Here the constant $\beta_0 = 11C_A/3 - 2n_f/3$ where we have n_f light quarks, and the ellipses are higher order terms we are dropping. (After you see what the strategy here is, keeping higher order terms is no more difficult.) Solve Eq. (11) to find $C(Q,\mu_1)$ given boundary conditions $C(Q,\mu_0)$ and $\alpha_s(\mu_0)$. You can start by considering the case with a frozen coupling $\alpha_s(\mu) = \alpha_s(\mu_0)$ where you should find that the result for $C(Q,\mu_1)$ involves a Sudakov double logarithm. Next solve including the full μ dependence from Eq. (12). Voilá, a formula for Sudakov double logs that are summed up into an exponential with a running coupling.

[Hint: To derive the result for the Sudakov exponent with a running coupling $\alpha_s(\mu)$ you can either i) first solve Eq. (12) and then substitute into Eq. (11), or ii) use Eq. (12) to make a change of variable $\mu \to \alpha_s(\mu)$ in the differential equation for C, and then solve for $C(Q,\mu)$ as a function of the running coupling $\alpha_s(\mu)$ and the boundary conditions. The second approach is easier.]