Lecture notes on

Mathematical Methods I

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Madrid, September 2023

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Chapter 1

Ordinary differential equations

1.1 Basic concepts and definitions

A **differential equation** can be regarded as a relation between an unknown function u of a variable $\mathbf{x} \in \mathbb{R}^N$, a finite number of partial derivatives of u, and the variable $\mathbf{x}$, which should identically hold at every point of an open set $D \subset \mathbb{R}^N$.

Example 1.1. Poisson's equation

$$\frac{\partial^2 u(\mathbf{x})}{\partial x^2} + \frac{\partial^2 u(\mathbf{x})}{\partial y^2} + \frac{\partial^2 u(\mathbf{x})}{\partial z^2} = \rho(\mathbf{x}) \,, \qquad \mathbf{x} \equiv (x, y, z) \,,$$

where ρ (representing the charge density in electrostatics) is a known function.

- In a differential equation both u and its partial derivatives should be evaluated at the same point. For instance
$$\frac{\partial u(x, y)}{\partial x} + u(x + 3, y) = 0$$
is *not* a differential equation.

If the independent variable $\mathbf{x}$ in a differential equation has multiple components, i.e., if $\mathbf{x} = (x_1, \ldots, x_N)$ with $N > 1$, we say that the equation is a **partial differential equation** (PDE). On the other hand, if $N = 1$ we say that the equation is an **ordinary differential equation** (ODE). In this course we shall be mainly concerned with ordinary differential equations, which we shall formally define next.

Definition 1.2. An n-th order **ordinary differential equation** is an equation of the form

$$\boxed{F\big(x, y, y', \ldots, y^{(n)}\big) = 0} \,, \tag{1.1}$$

where F is defined in an open set $U \subset \mathbb{R}^{n+2}$ and $\dfrac{\partial F}{\partial y^{(n)}} \neq 0$ in U. A **solution** of (1.1) is a function $u : \mathbb{R} \to \mathbb{R}$ which is n-times differentiable in an open interval $D \subset \mathbb{R}$ and

$$\boxed{F\big(x, u(x), u'(x), \ldots, u^{(n)}(x)\big) = 0 \,, \qquad \forall x \in D} \,. \tag{1.2}$$

- The condition $\dfrac{\partial F}{\partial y^{(n)}} \neq 0$ in U is imposed so that the equation is truly of order n. If one can solve Eq. (1.1) explicitly for the highest derivative, that is, if one can rewrite it as
$$\boxed{y^{(n)} = f\big(x, y, y', \ldots, y^{(n-1)}\big)} \,, \tag{1.3}$$
we shall say that the equation is in **normal form**.

- In this part of the course we shall always assume that the independent variable x is real. As we shall see in Chapter 3, in the resolution of certain type of equations it is natural to consider complex-valued solutions. These complex solutions are then combined to yield real-valued solutions, which is usually our purpose.

Example 1.3. As one of the simplest (but important) examples of an ODE, consider the equation

$$y' = -k\,y\,, \qquad k > 0\,, \tag{1.4}$$

which describes the disintegration of a radioactive material, where y represents the mass of the material and x the time.

Resolution: note that $y = 0$ is a solution of (1.4), while if $y \neq 0$ we have

$$\frac{y'(x)}{y(x)} = -k \quad\Longrightarrow\quad \int \frac{y'(x)}{y(x)}\,\mathrm{d}x = -k\int \mathrm{d}x \quad\Longrightarrow\quad \log|y| = -kx + c \quad\Longrightarrow\quad |y| = \mathrm{e}^c\mathrm{e}^{-kx}$$
$$\Longrightarrow\quad y = \pm\mathrm{e}^c\mathrm{e}^{-kx}\,,$$

where c is an arbitrary constant. Thus every solution of the equation (1.4) is of the form

$$y = y_0\mathrm{e}^{-kx}\,, \tag{1.5}$$

where y_0 is an arbitrary constant (in particular, for $y_0 = 0$ we recover the *trivial* solution $y = 0$). We shall say that (1.5) provides the **general solution** of the equation (1.4).

- Note that the general solution (1.5) of the equation (1.4) depends on an arbitrary constant. The general solution of an n-th order equation typically contains n arbitrary constants.

1.2 Elementary integration methods

In this section we shall restrict to the simplest case of first-order equations. We shall assume that the equation can be written in normal form

$$\boxed{y' = f(x, y)}\,. \tag{1.6}$$

We shall next discuss several particular cases of this equation which can be solved by suitable elementary methods.

1.2.1 $y' = f(x)$

Assuming that the function f is continuous in an open interval D, the equation can be easily solved integrating both sides from x_0 to x, where $x_0, x \in D$:

$$\boxed{y = \int_{x_0}^{x} f(t)\,\mathrm{d}t + c}\,, \qquad c = y(x_0)\,. \tag{1.7}$$

We can alternatively express the general solution in terms of an indefinite integral as

$$y = \int^{x} f(t)\,\mathrm{d}t + c\,, \qquad \text{or (with a slight abuse of notation)} \qquad y = \int f(x)\,\mathrm{d}x + c\,.$$

From Eq. (1.7) it follows that the **initial value problem**

$$y' = f(x)\,, \qquad y(x_0) = y_0$$

possesses the *unique solution* $y = \displaystyle\int_{x_0}^{x} f(t)\,\mathrm{d}t + y_0$.

1.2.2 Separable equations

These are equations of the form

$$\boxed{y' = \frac{f(x)}{g(y)}}\,, \tag{1.8}$$

where f (resp. g) is continuous on the open interval U (resp. V), and $g(y) \neq 0$ for all $y \in V$.

Resolution: If $y(x)$ is a solution of the equation (1.8), then

$$g\big(y(x)\big)y'(x) = f(x) \quad \Longrightarrow \quad \int_{x_0}^{x} g\big(y(s)\big)y'(s)\,\mathrm{d}s = \int_{x_0}^{x} f(s)\,\mathrm{d}s$$
$$\underset{t=y(s)}{\Longrightarrow} \quad \int_{y(x_0)}^{y(x)} g(t)\,\mathrm{d}t = \int_{x_0}^{x} f(s)\,\mathrm{d}s$$

Thus any solution of (1.8) satisfies the **implicit equation**

$$\boxed{\int g(y)\,\mathrm{d}y = \int f(x)\,\mathrm{d}x + c}\,, \tag{1.9}$$

where c is an arbitrary constant. Conversely, taking the total derivative of (1.9) with respect to x (regarding y as a function of x) we conclude that any function $y(x)$ satisfying the relation (1.9) is a solution of the equation (1.8). Hence (1.9) is the general solution of (1.8).

The general solution (1.9) of the equation (1.8) is given by the **implicit equation**

$$\phi(x,y) = c\,, \qquad \text{where} \quad \phi(x,y) = \int g(y)\,\mathrm{d}y - \int f(x)\,\mathrm{d}x\,. \tag{1.10}$$

The implicit relation $\phi(x,y) = c$ defines a one-parameter family of curves in the plane, with each curve corresponding to a fixed value of c (even though a curve may possess several branches). These curves are known as the **integral curves** of the equation (1.8). As we have just seen, a function $y(x)$ is a solution of (1.8) if and only if its graph is contained in an integral curve of the equation.

The function ϕ in Eq. (1.10) are of class $C^1(U \times V)$ (since $\frac{\partial \phi}{\partial x} = -f(x)$ and $\frac{\partial \phi}{\partial y} = g(y)$ are assumed to be continuous in U and V, respectively), and $\frac{\partial \phi}{\partial y}$ does not vanish in $U \times V$. Given a point (x_0, y_0) in $U \times V$, the integral curve (1.10) passing through it corresponds to the value $c = \phi(x_0, y_0)$. According to the **implicit function theorem**, there is a neighborhood of (x_0, y_0) on which the relation (1.10) defines a *unique* differentiable function $y(x)$ such that

$$\begin{aligned} &\text{i)} \quad y(x_0) = y_0\,, \\ &\text{ii)} \quad \phi\big(x, y(x)\big) = \phi(x_0, y_0), \qquad \forall x \in \operatorname{dom} y\,. \end{aligned}$$

In the latter neighborhood, the integral curve passing through (x_0, y_0) is thus the graph of a solution $y(x)$. This solution is *locally* (that is, in a certain neighborhood of (x_0, y_0)) the unique solution of the differential equation (1.8) satisfying the initial condition $y(x_0) = y_0$. In other words, the initial value problem associated with the equation (1.8) possesses a *unique local solution* whenever the initial data (x_0, y_0) belong to $U \times V$.

- The fact that the general solution of the separable equation (1.8) is expressed in terms of an implicit relation (cf. eq. (1.10)) is *not* a characteristic feature of this type of equation. In fact, we shall see throughout this section that the general solution of the first-order equation (1.6) is often expressed via an implicit relation. In general, it will not be possible to explicitly solve this relation for y as a function of x, although the implicit function theorem will usually guarantee the local existence of such function.

Example 1.4. Let us consider the separable equation

$$y' = -\frac{x}{y}\,. \tag{1.11}$$

In the previous notation, $f(x) = -x$, $g(y) = y$, $U = \mathbb{R}$, and either $V = \mathbb{R}^+$ or $V = \mathbb{R}^-$, but not $V = \mathbb{R}$ since the function $g(y)$ vanishes at $y = 0$. Proceeding as before (or using directly the formula (1.9)), the general solution of (1.11) is readily found to be

$$x^2 + y^2 = c\,, \qquad c > 0\,. \tag{1.12}$$

Thus in this case the integral curves are circles of radius $\sqrt{c} > 0$ centered at the origin (see Fig. 1.1). In particular, for each point of the plane excepting the origin passes a unique integral curve. Each integral curve contains *two* solutions, given by the functions

$$y = \pm\sqrt{c - x^2}\,, \qquad x \in (-\sqrt{c}, \sqrt{c})\,, \tag{1.13}$$

where the signs $\pm$ corresponds to the choice $V = \mathbb{R}^{\pm}$. The equation (1.11) is not defined for $y = 0$, but the solutions (1.13) have a well-defined limit (equal to zero) as $x \to \pm\sqrt{c}\mp$ (although they are not differentiable at these points, since they have infinite slope). Note that through each point (x_0, y_0) of the plane with $y_0 \neq 0$ passes a unique solution. In contrast, the integral curve passing through a point of the form $(x_0, 0)$ with $x_0 \neq 0$ does not define y as a function of x in a neighborhood of such point. However, this integral curve indeed defines the function $x(y) = \operatorname{sgn} x_0 \sqrt{x_0^2 - y^2}$, $y \in (-|x_0|, |x_0|)$, which is a solution of the equation

$$x' = -\frac{y}{x}\,. \tag{1.14}$$

We shall say that (1.14) is the **associated equation** of (1.11).

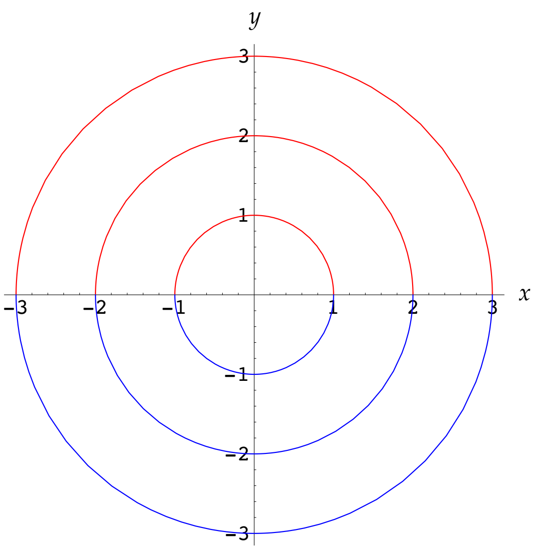

Figure 1.1: Integral curves of the equation $y' = -x/y$. Each integral curve contains two solutions (plotted in red and blue), defined in Eq. (1.13).

- In general, the associated equation of the first-order ODE (1.6) is given by

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{f(x, y)}}\,. \tag{1.15}$$

Equation (1.6) and its associated equation (1.15) possess the same set of integral curves.

Example 1.5. Consider now the equation

$$y' = -\frac{1+y^2}{1+x^2}\,, \tag{1.16}$$

which is also a separable equation. In this case $f(x) = -\frac{1}{1+x^2}$ and $g(y) = \frac{1}{1+y^2}$ are both continuous and g is nonvanishing, so that we can take $U = V = \mathbb{R}$. The equation (1.16) can be readily integrated, with the result

$$\arctan y = -\arctan x + c\,, \tag{1.17}$$

where $|c| < \pi$ for the equation (1.17) to have a solution in y for some value of x. If $c \neq \pm\frac{\pi}{2}$ and we call $C = \tan c$, from Eq. (1.17) it immediately follows that

$$y = \tan(c - \arctan x) = \frac{C-x}{1+Cx}\,. \tag{1.18}$$

Note that the constant C may take any real value. On the other hand, if $c = \pm\frac{\pi}{2}$ we have

$$y = \tan\Big(\pm\frac{\pi}{2} - \arctan x\Big) = \cot(\arctan x) = \frac{1}{x}\,, \qquad x \neq 0\,. \tag{1.19}$$

Notice that this solution is formally obtained from (1.18) in the limit $C \to \pm\infty$.

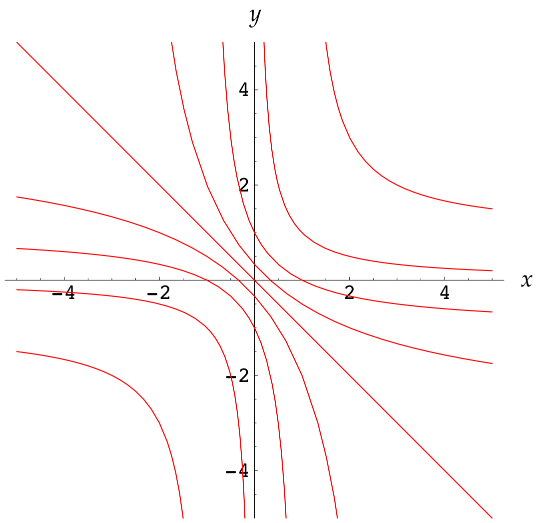

Figure 1.2: Solutions of the equation $y' = -\frac{1+y^2}{1+x^2}$.

The initial value problem associated with equation (1.16) always yields a unique local solution. Indeed, if we impose the initial condition $y(x_0) = y_0$, from (1.18) it follows that the constant C is uniquely given by

$$C = \frac{x_0 + y_0}{1 - x_0 y_0}\,,$$

which is well-defined unless $y_0 = 1/x_0$, whereas in this case the corresponding solution is $y = 1/x$.

- The only solution of equation (1.16) which is defined on the whole real line is $y = -x$, corresponding to $C = 0$. The solution (1.19) and the remaining solutions in (1.18) diverge at a certain finite value of x (more precisely at $x = 0$ and $x = -\frac{1}{C}$, respectively). However, the right-hand side of the equation (1.16) is continuous (in fact, of class C^∞) on the whole plane. In general, the study of the singularities of the function $f(x, y)$ does not provide per se information about the potential singularities of the solutions of the differential equation (1.6).

1.2.3 Homogeneous equations

These are equations of the form (1.6) with f continuous and homogeneous of degree zero in an open set $U \subset \mathbb{R}^2$, that is,

$$\boxed{f(\lambda x, \lambda y) = f(x, y)\,, \qquad \forall (x, y) \in U, \quad \forall \lambda \neq 0}\,. \tag{1.20}$$

An homogeneous equation becomes separable via the change variable

$$\boxed{y = xu}\,, \qquad x \neq 0\,,$$

where $u(x)$ is the new unknown function. Indeed,

$$y' = u + xu' = f(x, xu) = f(1, u) \quad \Longrightarrow \quad u' = \frac{f(1, u) - u}{x}\,. \tag{1.21}$$

If λ satisfies the condition $f(1, \lambda) = \lambda$, the equation (1.21) has the constant solution $u = \lambda$, which corresponds to $y = \lambda x$. Otherwise, the equation is solved by separating variables and integrating, which leads to the implicit relation

$$\boxed{\int \frac{\mathrm{d}u}{f(1, u) - u} = \log|x| + c\,, \qquad \text{con} \quad u = \frac{y}{x}}\,. \tag{1.22}$$

Example 1.6. Let us consider the equation

$$y' = \frac{y - 2x}{2y + x}\,. \tag{1.23}$$

Since

$$f(x, y) = \frac{y - 2x}{2y + x}$$

is homogeneous of degree zero (and continuous on the whole plane except for the line $y = -x/2$), the equation (1.23) is homogeneous. It is easy to verify that the equation $f(1, \lambda) = \lambda$ possesses no real solutions, so no straight line through the origin can be a solution of (1.23). Since

$$\frac{1}{f(1, u) - u} = -\frac{2u + 1}{2(u^2 + 1)}\,,$$

from Eq. (1.22) we readily obtain

$$\log(1 + u^2) + \arctan u = c - 2\log|x|\,,$$

where c is an arbitrary constant. Substituting u for y/x and simplifying, we obtain the following implicit expression for the integral curves of the equation (1.23):

$$\log(x^2 + y^2) + \arctan\frac{y}{x} = c\,. \tag{1.24}$$

The latter expression cannot be solved explicitly for y as a function of x. However, if we express it in polar coordinates

$$x = r\cos\theta\,, \qquad y = r\sin\theta\,,$$

we immediately obtain

$$2\log r + \theta = c \quad \Longrightarrow \quad r = C\mathrm{e}^{-\theta/2}\,; \quad C = \mathrm{e}^{c/2} > 0\,. \tag{1.25}$$

The integral curves are thus logarithmic spirals such that the radial coordinate grows exponentially as the angular coordinate turns clockwise.

In order to plot these spirals by hand, it is convenient to first determine the isoclines of the equation (1.23). In general, an **isocline** of the first-order equation order equation (1.6) is the locus of the points at which the tangent vectors to the integral curves have all some given direction. The isoclines are hence defined by the implicit equation

$$\boxed{f(x, y) = m}\,, \qquad m \in \mathbb{R} \text{ ó } m = \infty\,,$$

where the value $m = \infty$ is included in order to account for the isoclines with vertical tangent. In the present example, the equation for the isoclines reads

$$\frac{y - 2x}{2y + x} = m. \tag{1.26}$$

Thus in this case the isoclines are straight lines passing through the origin, the one with slope m being given by

$$y = \frac{m + 2}{1 - 2m}\, x\,. \tag{1.27}$$

In particular, the isoclines of slope $0, \infty, 1, -1$ are the straight lines $y = 2x$, $y = -x/2$, $y = -3x$, $y = x/3$, respectively. In Fig.1.3 we plot these isoclines together with the spirals corresponding to $c = 0, \pi/2, \pi, 3\pi/2$. Note that each spiral contains infinitely many solutions of the equation (1.23), each of them defined in an interval of the form (x_0, x_1), where (x_0, y_0) and (x_1, y_1) are two consecutive intersection points between the spiral and the isocline of infinite slope $y = -x/2$.

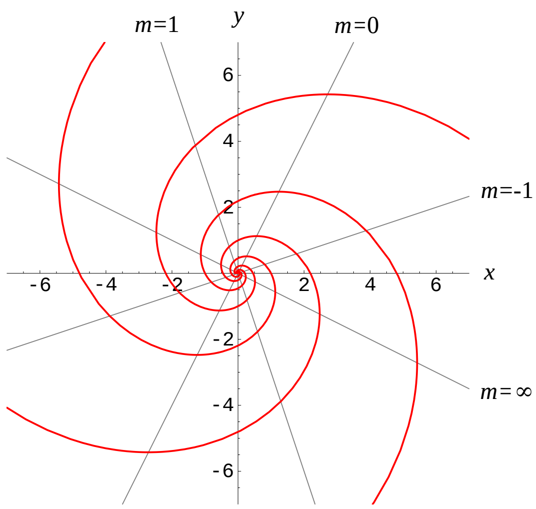

Figure 1.3: Integral curves of the equation $y' = \frac{y-2x}{2y+x}$ (in red) and isoclines with slope $0, \infty, 1, -1$ (in grey).

1.2.4 Exact equations

A differential equation of the form

$$\boxed{P(x, y) + Q(x, y)y' = 0}\,, \tag{1.28}$$

where P, Q are continuous functions on an open set $U \subset \mathbb{R}^2$ and Q does not vanish in U, is said to be **exact** if there is a function $F : U \to \mathbb{R}$ such that

$$\boxed{P(x, y) = F_x(x, y)\,, \qquad Q(x, y) = F_y(x, y)}\,, \qquad \forall (x, y) \in U\,. \tag{1.29}$$

In other words, the equation (1.28) is exact if $(P, Q) = \nabla F$ in U. In this case, if $y(x)$ is a solution of (1.28) we can rewrite this equation as

$$F_x\big(x, y(x)\big) + F_y\big(x, y(x)\big)y'(x) = \frac{\mathrm{d}}{\mathrm{d}x} F\big(x, y(x)\big) = 0\,.$$

Thus the solutions of the exact equation (1.28)-(1.29) satisfy the implicit equation

$$\boxed{F(x, y) = c}\,, \tag{1.30}$$

where c is a constant. Conversely, if (1.30) holds, since $F_y = Q$ does not vanish in U, the implicit function theorem defines y as a function of x in a neighborhood of each point in U, this function being also a solution of the exact equation (1.28) in its domain. Hence the general solution of (1.28)-(1.29) is given by equation (1.30), which defines the *level curves* of F.

When the functions P, Q are of class $C^1(U)$, a *necessary* condition for the equation (1.28) to be exact is

$$\boxed{P_y(x, y) = Q_x(x, y)}\,, \qquad \forall (x, y) \in U\,, \tag{1.31}$$

since $F_{xy} = F_{yx}$ in U by the Schwarz lemma. The condition (1.31) is also *sufficient* if the open set U is **simply connected**[1], as it shall be usually the case in most applications.

Let us see how to determine the function F assuming that the condition (1.31) is satisfied on an open rectangle $U = (a, b) \times (c, d)$. Let (x_0, y_0) be a point in U. Integrating the equation $F_x = P$, we get

$$F(x, y) = \int_{x_0}^{x} P(s, y)\,\mathrm{d}s + g(y)\,, \tag{1.32}$$

where g depends only on y. If $(x, y) \in U$, then all points of the form (s, y) with $s \in [x_0, x]$ or $[x, x_0]$ are also in U, so that the integral in the previous formula is well-defined. Taking the partial derivative of (1.32) with respect to y and using Eq. (1.31) it follows that

$$F_y(x, y) = g'(y) + \int_{x_0}^{x} P_y(s, y)\,\mathrm{d}s = g'(y) + \int_{x_0}^{x} Q_x(s, y)\,\mathrm{d}s = g'(y) + Q(x, y) - Q(x_0, y)\,.$$

From the second equation $F_y = Q$ we obtain

$$g'(y) = Q(x_0, y) \quad \Longrightarrow \quad g(y) = \int_{y_0}^{y} Q(x_0, s)\,\mathrm{d}s\,, \tag{1.33}$$

up to an arbitrary constant. (As before, the integral in (1.33) is well defined since all points of the form (x_0, s) are in U when $s \in [y_0, y]$ or $[y, y_0]$.) Thus in this case the general solution (1.30) of the equation (1.28) is given by

$$\boxed{F(x, y) = \int_{x_0}^{x} P(s, y)\,\mathrm{d}s + \int_{y_0}^{y} Q(x_0, s)\,\mathrm{d}s = c}\,. \tag{1.34}$$

The function F in the latter formula can also be expressed in a more compact way as the line integral

$$F(x, y) = \int_{\gamma_0} (P, Q)\cdot \mathrm{d}\mathbf{r}\,, \tag{1.35}$$

where γ_0 is the broken line path in Fig. 1.4. Since the rectangle U is simply connected and the condition (1.31) is satisfied, the line integral of the vector field (P, Q) along any piecewise C^1 curve contained in U is *path-independent*. We can therefore write

$$\boxed{F(x, y) = \int_{\gamma} (P, Q)\cdot \mathrm{d}\mathbf{r}}\,, \tag{1.36}$$

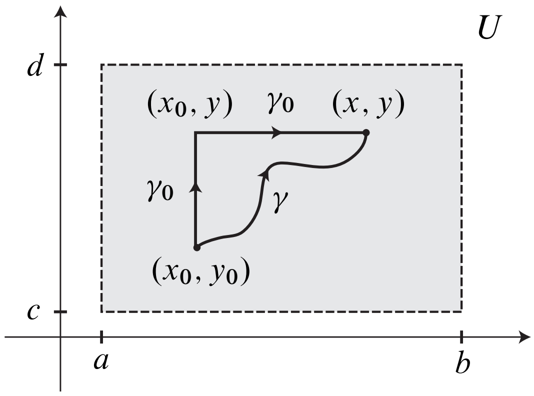

Figure 1.4: Paths joining (x_0, y_0) and (x, y) in U.

where γ is any piecewise C^1 curve joining (x_0, y_0) and (x, y) without leaving U (see Fig. 1.4).

- In fact, it may be shown that the formula (1.36) is valid for any open simply-connected set on which condition (1.31) holds.

Example 1.7. Consider the equation

$$2xy + 1 + (x^2 + y)y' = 0\,, \tag{1.37}$$

which is of the form (1.28) with $P = 2xy+1$, $Q = x^2+y$. Since $P_y = 2x = Q_x$, the equation is exact in any of the simply connected open sets $U_\pm = \{(x, y) \in \mathbb{R}^2 : y \gtrless -x^2\}$ in which Q is nonvanishing. We thus look for a function F such that $\nabla F = (P, Q)$ in $U_\pm$, that is,

$$F_x = 2xy + 1 \quad\Longrightarrow\quad F = x^2 y + x + g(y)$$
$$F_y = x^2 + g'(y) = x^2 + y \quad\Longrightarrow\quad g'(y) = y \quad\Longrightarrow\quad g(y) = \frac{y^2}{2}\,,$$

up to a constant. Hence the integral curves of the equation (1.37) are given by the implicit equation

$$2x^2 y + 2x + y^2 = c\,, \tag{1.38}$$

where c is an arbitrary constant. Solving the previous equation for y yields the expressions

$$y_\pm = -x^2 \pm \sqrt{x^4 - 2x + c} \tag{1.39}$$

for each value of c (see Fig. 1.5), where the $\pm$ sign for the square root corresponds to the choice $U_\pm$.

Sometimes it may be interesting to discuss the behavior of the integral curves in terms of the values of the arbitrary constant appearing in the general solution. For instance, in this example it may be shown that when $c > \frac{3}{2\sqrt[3]{2}}$ each expression $y_\pm$ is a solution of the equation (1.37) defined on the whole real line. Alternatively, if $c < \frac{3}{2\sqrt[3]{2}}$ each expression $y_\pm$ determines *two* solutions of the latter equation, respectively defined in the intervals $(-\infty, x_0)$ and (x_1, ∞), where $x_0 < x_1$ are the two real roots of the polynomial $x^4 - 2x + c$ in the radicand of (1.39). Finally, if $c = \frac{3}{2\sqrt[3]{2}}$, each expression $y_\pm$ also determines two solutions of (1.37) defined in the intervals $\big(-\infty, \frac{1}{\sqrt[3]{2}}\big)$ and $\big(\frac{1}{\sqrt[3]{2}}, \infty\big)$, respectively.

[1] Recall that an open set U is *connected* if every pair of points of U can be joined by a continuous curve entirely contained in U. The open set U is *simply connected* if it is connected and every continuous closed curve contained in U can be shrunk continuously to a point in U without leaving this set. Intuitively, an open set is simply connected if it "consists of only one piece" and "has no holes".

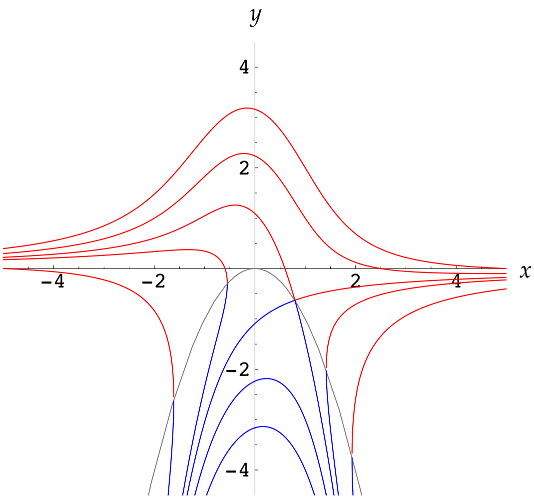

Figure 1.5: Integral curves (1.38) of the equation (1.37). The parabola $y = -x^2$ (in grey) is an isocline of infinite slope dividing the plane in two open simply connected sets $U_\pm$ wherein $Q = x^2 + y$ is nonvanishing and the solutions are respectively given by $y_\pm$.

Consider again the equation (1.28), with P, Q of class $C^1(U)$. Suppose that $P_y \neq Q_x$ in U, so that the equation (1.28) is *not* exact. If $\mu(x, y)$ is a function which does not vanish in U, the equations (1.28) and

$$\mu(x, y)P(x, y) + \mu(x, y)Q(x, y)y' = 0 \tag{1.40}$$

are equivalent in the sense that their sets of solutions are coincident. If the equation (1.40) is exact, we shall say that the function μ is an *integrating factor* of the equation (1.28). In this case we can solve (1.28) by integrating (1.40) using the procedure discussed above.

If U is an open simply connected set and μ is of class C^1, the necessary and sufficient condition that the function μ must satisfy for the equation (1.40) to be exact is

$$(\mu P)_y = (\mu Q)_x \,.$$

In other words, μ must be a solution of the first-order partial differential equation

$$\boxed{P(x, y)\,\mu_y - Q(x, y)\,\mu_x + \big[P_y(x, y) - Q_x(x, y)\big]\mu = 0}\,. \tag{1.41}$$

Although it can be shown that this PDE has always a solution, the problem is that the usual technique for solving it requires the knowledge of the solution of the ODE (1.28) that we started from. However, we can look for *particular* solutions of (1.41) depending on a single variable, such as $\mu(x)$, $\mu(y)$, $\mu(x+y)$, $\mu(x^2 + y^2)$, etc. In general, these functions will not be solutions of the PDE (1.41) unless P and Q satisfy a suitable condition. For instance, if

$$\boxed{\frac{P_y - Q_x}{Q} \equiv g(x)}\,, \tag{1.42}$$

then (1.41) admits an integrating factor of the form $\mu(x)$. Indeed, if (1.42) holds and $\mu_y = 0$, equation (1.41) yields

$$\mu'(x) = g(x)\mu(x) \quad \Longrightarrow \quad \boxed{\mu(x) = c\,e^{\int g(x)\,\mathrm{d}x}}\,. \tag{1.43}$$

Likewise, if

$$\boxed{\frac{P_y - Q_x}{P} \equiv h(y)}\,, \tag{1.44}$$

then (1.41) admits as a solution the integrating factor depending solely on y

$$\boxed{\mu(y) = c\mathrm{e}^{-\int h(y)\,\mathrm{d}y}}\,, \tag{1.45}$$

Exercise. Show that the equation (1.28) possesses an integrating factor of the form $\mu(r)$ with $r = \sqrt{x^2+y^2}$ if and only if

$$\boxed{\frac{P_y - Q_x}{yP - xQ} = g(r)\,,}$$

which in this case is given by

$$\boxed{\mu(r) = c\,\mathrm{e}^{-\int r\,g(r)\,\mathrm{d}r}\,.}$$

Example 1.8. The equation

$$y(1-x) - \sin y + (x + \cos y)y' = 0\,, \tag{1.46}$$

is not exact, for $P = y(1-x) - \sin y$, $Q = x + \cos y$ do not satisfy the condition (1.31). However, since

$$\frac{P_y - Q_x}{Q} = -1 \equiv g(x)$$

does not depend on y, from equation (1.43) it follows that $\mu(x) = \mathrm{e}^{-x}$ is an integrating factor of equation (1.46) in any of the open simply connected sets

$$U_\pm = \{(x,y) \in \mathbb{R}^2 : x + \cos y \gtrless 0\}.$$

We thus look for a function F such that $\nabla F = \mathrm{e}^{-x}(P,Q)$. In this case it is convenient to start by integrating the second component $F_y = \mathrm{e}^{-x}Q$, which yields

$$\begin{aligned} F_y &= \mathrm{e}^{-x}(x+\cos y) \quad \Longrightarrow \quad F = \mathrm{e}^{-x}(xy + \sin y) + h(x) \\ F_x &= \mathrm{e}^{-x}(y - xy - \sin y) + h'(x) = \mathrm{e}^{-x}\big(y(1-x) - \sin y\big) \quad \Longrightarrow \quad h'(x) = 0\,. \end{aligned}$$

Hence we can choose

$$F(x,y) = \mathrm{e}^{-x}(xy + \sin y)\,,$$

so that the integral curves of the equation (1.37) satisfy the transcendental equation

$$\mathrm{e}^{-x}(xy + \sin y) = c\,, \tag{1.47}$$

where c is an arbitrary constant. The previous equation cannot be solved explicitly for y as a function of x (although for $c = 0$ it can be solved for x as a function of y). However, the implicit function theorem guarantees that if $F_y(x_0,y_0) = \mathrm{e}^{-x_0}(x_0 + \cos y_0) \neq 0$, i.e. if $(x_0,y_0) \in U_\pm$, the equation (1.47) defines y as a function of x in a neighborhood of (x_0,y_0).

1.2.5 Linear equations

These are equations of the form

$$\boxed{y' = a(x)y + b(x)}\,, \tag{1.48}$$

where a and b are continuous functions on an open interval U. The equation (1.48) is said to be **homogeneous** if $b \equiv 0$, or **inhomogeneous** or **complete** otherwise. We shall see that the general solution of a linear equation can always be expressed via quadratures. Indeed, in the homogeneous case

$$\boxed{y' = a(x)y} \tag{1.49}$$

it admits the *trivial solution* $y = 0$, whereas if $y \neq 0$ it can be tackled as a separable equation:

$$\frac{y'}{y} = a(x) \quad \Longrightarrow \quad \log|y| = \int a(x)\,\mathrm{d}x + c_0 \quad \Longrightarrow \quad |y| = \mathrm{e}^{c_0}\mathrm{e}^{\int a(x)\,\mathrm{d}x}\,.$$

The general solution of (1.49) is thus

$$\boxed{y = c\,\mathrm{e}^{\int a(x)\,\mathrm{d}x}}\,, \tag{1.50}$$

where c is an arbitrary constant (since either $c = \pm\mathrm{e}^{c_0}$, or $c = 0$ for the trivial solution).

- The set of solutions (1.50) of the homogeneous equation (1.49) is a one-dimensional vector space.

The inhomogeneous equation (1.48) can be solved by the **method of variation of constants**, due to Lagrange. The method consists in making the following ansatz for the solution of the inhomogeneous equation (1.48):

$$y = c(x)\mathrm{e}^{A(x)}\,, \tag{1.51}$$

where

$$A(x) = \int a(x)\,\mathrm{d}x$$

is any *fixed* primitive of the function $a(x)$. In other words, one assumes that the solution of the inhomogeneous equation is given by the general solution of the homogeneous equation with the arbitrary constant c replaced by an unknown function $c(x)$. Inserting (1.51) into the equation (1.48) we readily obtain

$$c'(x)\mathrm{e}^{A(x)} + c(x)\mathrm{e}^{A(x)}a(x) = a(x)c(x)\mathrm{e}^{A(x)} + b(x)\,,$$

so that

$$c'(x) = b(x)\mathrm{e}^{-A(x)} \quad \Longrightarrow \quad c(x) = c + \int b(x)\mathrm{e}^{-A(x)}\,\mathrm{d}x\,,$$

where c is an arbitrary constant. Hence the general solution of the complete equation (1.48) is given by

$$\boxed{y = c\,\mathrm{e}^{A(x)} + \mathrm{e}^{A(x)}\int b(x)\mathrm{e}^{-A(x)}\,\mathrm{d}x}\,. \tag{1.52}$$

- The previous expression shows that the general solution of the equation (1.48) is of the form

  $$y = y_{\mathrm{h}}(x) + y_{\mathrm{p}}(x)\,,$$

  where y_{h} is the general solution of the homogeneous equation and y_{p} is a particular solution of the complete equation.

Consider now the initial-value problem

$$\begin{cases} y' = a(x)y + b(x)\,, \\ y(x_0) = y_0\,, \end{cases} \tag{1.53}$$

where $x_0 \in U$. Choosing as the primitive of $a(x)$ the function $A(x) = \int_{x_0}^{x} a(s)\,\mathrm{d}s$, it follows immediately that the unique solution of Eq. (1.48) satisfying the initial condition $y(x_0) = y_0$ is given by

$$\boxed{y = y_0\mathrm{e}^{\int_{x_0}^{x} a(s)\,\mathrm{d}s} + \mathrm{e}^{\int_{x_0}^{x} a(s)\,\mathrm{d}s}\int_{x_0}^{x} b(s)\mathrm{e}^{-\int_{x_0}^{s} a(t)\,\mathrm{d}t}\,\mathrm{d}s}\,.$$

Example 1.9. Solve the linear inhomogeneous equation

$$y' = \frac{y}{x} + x^2\,, \tag{1.54}$$

defined for $x \neq 0$.

The general solution of the corresponding homogeneous equation reads

$$\frac{y'}{y} = \frac{1}{x} \quad \Longrightarrow \quad \log|y| = \log|x| + c_0 \quad \Longrightarrow \quad y = cx\,,$$

where $c \in \mathbb{R}$ (the value $c = 0$ comes from the trivial solution $y \equiv 0$). For the complete equation we try a particular solution of the form $y_{\mathrm{p}} = c(x)x$, which yields

$$y_{\mathrm{p}}' = c'x + c = c + x^2 \quad \Longrightarrow \quad c = \frac{x^2}{2} \quad \Longrightarrow \quad y_{\mathrm{p}} = \frac{x^3}{2}\,.$$

The general solution of the equation (1.54) is therefore given by

$$y = cx + \frac{x^3}{2}\,, \qquad c \in \mathbb{R}\,.$$

Note that even though the differential equation is not defined if $x = 0$, its solutions are analytic on the whole real line.

1.2.6 Bernoulli equation

This is an equation of the form

$$\boxed{y' = a(x)y + b(x)y^r}\,, \qquad r \neq 0, 1\,, \tag{1.55}$$

where a and b are continuous functions on an open interval U. The equation (1.55) is not defined for $y < 0$ unless $r = p/q$ is an irreducible rational number with odd q, nor for $y = 0$ when $r < 0$. The Bernoulli equation can be transformed into a linear equation (and is thus solvable via quadratures) by the change of variable

$$\boxed{u = y^{1-r}}\,.$$

Indeed, taking the derivative of u with respect to x and using (1.55) we get

$$u' = (1-r)y^{-r}y' = (1-r)a(x)y^{1-r} + (1-r)b(x) = (1-r)a(x)u + (1-r)b(x)\,,$$

which is a linear equation for the new unknown variable u.

Example 1.10. The equation

$$y' = \frac{y - y^2}{x}\,, \tag{1.56}$$

is a Bernoulli equation with $r = 2$. A possible solution is $y \equiv 0$. If $y \neq 0$, the suitable change of variable $u = 1/y$ yields

$$u' = -\frac{y'}{y^2} = -\frac{1}{xy} + \frac{1}{x} = -\frac{u}{x} + \frac{1}{x}\,, \tag{1.57}$$

which is linear in u. The general solution of the homogeneous equation reads

$$u_{\mathrm{h}} = \frac{c}{x}\,.$$

As to the complete equation, we try a particular solution of the form $u_{\mathrm{p}} = \frac{c(x)}{x}$, which leads to

$$\frac{c'}{x} = \frac{1}{x} \quad \Longrightarrow \quad c = x \quad \Longrightarrow \quad u_{\mathrm{p}} = 1\,.$$

Thus the general solution of (1.57) is

$$u = u_{\mathrm{h}} + u_{\mathrm{p}} = \frac{x+c}{x}\,,$$

so that

$$y = \frac{x}{x+c}$$

is the general solution of (1.56). (The solution $y \equiv 0$ is formally obtained in the limit $c \to \infty$.)

Exercise. Solve the equation (1.56) treating it as a separable equation.

1.2.7 Riccati equation

The Riccati equation

$$\boxed{y' = a(x) + b(x)y + c(x)y^2}\,, \qquad a, c \not\equiv 0\,, \tag{1.58}$$

where the functions a, b, c are continuous on an open interval U is of great importance in Mathematical Physics due to its close relation with second order linear equations (such as the Schrödinger equation). In general it is not possible to solve a Riccati equation by quadratures. However, if a particular solution $y_0(x)$ is known it can be transformed into a linear equation via the change of variable

$$\boxed{u = \frac{1}{y - y_0(x)}}\,.$$

Indeed,

$$\begin{aligned} u' &= -\frac{y' - y_0'(x)}{\big(y - y_0(x)\big)^2} = -\frac{b(x)\big(y - y_0(x)\big) + c(x)\big(y^2 - y_0^2(x)\big)}{\big(y - y_0(x)\big)^2} = -b(x)u - c(x)\,\frac{y + y_0(x)}{y - y_0(x)} \\ &= -\big[b(x) + 2c(x)y_0(x)\big]u - c(x)\,, \end{aligned}$$

which is a linear equation for u.

Example 1.11. Consider the Riccati equation

$$y' = y^2 - \frac{2}{x^2}\,. \tag{1.59}$$

If we try a solution of the form $y = \lambda/x$, we readily obtain

$$-\frac{\lambda}{x^2} = \frac{\lambda^2}{x^2} - \frac{2}{x^2} \quad \Longrightarrow \quad \lambda^2 + \lambda - 2 = 0 \quad \Longrightarrow \quad \lambda = -2, 1\,.$$

Let us take the particular solution $y_0 = 1/x$. The change of variable

$$u = \frac{1}{y - 1/x} \tag{1.60}$$

leads to the linear equation

$$u' = -\frac{y' + 1/x^2}{(y - 1/x)^2} = -\frac{y^2 - 1/x^2}{(y - 1/x)^2} = -\frac{y + 1/x}{y - 1/x} = -\frac{2u}{x} - 1\,. \tag{1.61}$$

The general solution of the homogeneous equation is given by

$$u_{\mathrm{h}} = \frac{C}{x^2}\,.$$

In order to determine a particular solution of the inhomogeneous equation, we can use the method of variation of constants, or more directly, try a solution of the form $u_{\mathrm{p}} = kx$. Substituting into Eq. (1.61) we obtain

$$k = -2k - 1 \quad\Longrightarrow\quad k = -\frac{1}{3}\,.$$

Thus

$$u = \frac{C}{x^2} - \frac{x}{3} = -\frac{x^3 + c}{3x^2}\,, \qquad c = -3C\,,$$

is the general solution of (1.61). From (1.60) it immediately follows that

$$y = \frac{1}{x} - \frac{3x^2}{x^3 + c}\,.$$

is the general solution of (1.59).

Remark. As we have mentioned above, the Riccati (1.58) equation is closely related to second order linear equations. More precisely, it is possible to transform (1.58) into a second-order linear equation through the change of variable

$$y = -\frac{1}{c(x)}\,\frac{u'}{u}\,.$$

Indeed,

$$y' = -\frac{1}{c(x)}\,\frac{u''}{u} + \frac{c'(x)}{c(x)^2}\,\frac{u'}{u} + \frac{1}{c(x)}\,\frac{u'^2}{u^2} = a(x) - \frac{b(x)}{c(x)}\,\frac{u'}{u} + \frac{1}{c(x)}\,\frac{u'^2}{u^2}\,,$$

so that u satisfies the equation

$$\boxed{u'' - \left[b(x) + \frac{c'(x)}{c(x)}\right]u' + a(x)c(x)u = 0}\,.$$

In the next chapter we shall see that the general solution of the latter equation is of the form

$$u = k_1\,u_1(x) + k_2\,u_2(x)\,,$$

where k_1, k_2 are real constants and u_1, u_2 are two linearly independent solutions, which in general will not be possible to compute in closed form. When this is the case, the general solution of the initial Riccati equation (1.58) can be expressed in terms of u_1 y u_2 as

$$\boxed{y = -\frac{1}{c(x)}\,\frac{k_1u_1'(x) + k_2u_2'(x)}{k_1u_1(x) + k_2u_2(x)}}\,.$$

(Note that this solution depends on a single arbitrary constant, namely k_2/k_1 or k_1/k_2.)

1.3 Existence and uniqueness of solutions

In this section we shall study the *existence* and *uniqueness* of solution of the initial value problem

$$\begin{cases} y' = f(x, y)\,, \\ y(x_0) = y_0\,. \end{cases} \tag{1.62}$$

In several examples of the previous section in which the function f was sufficiently regular we have seen that this problem has a unique local solution. In this section we will state without proof a fundamental result guaranteeing the existence of a unique (in general, *local*) solution of the initial value problem. We shall assume that the dependent variable y and the function f are vector-valued[2]. In other words, we shall consider the initial value problem for a system of first-order equations:

Definition 1.12. A **system of n first-order ordinary differential equations in normal form** for an unknown function $y = (y_1, \ldots, y_n)$ is a vector-valued equation

$$\boxed{y' = f(x, y)}\,, \tag{1.63}$$

where $f = (f_1, \ldots, f_n)$ is defined on an open set $U \subset \mathbb{R}^{n+1}$ and takes values in $\mathbb{R}^n$. Given $(x_0, y_0) \in U$, the **initial value problem** associated with the system (1.63) consists in finding a solution $y(x)$ defined on an interval I containing x_0 such that

$$\boxed{y(x_0) = y_0}\,. \tag{1.64}$$

- The system (1.63) is equivalent to the n scalar equations

  $$\begin{cases} y_1' = f_1(x, y_1, \ldots, y_n)\,, \\ \quad\vdots \\ y_n' = f_n(x, y_1, \ldots, y_n)\,, \end{cases}$$

  while the initial data (1.64) corresponds to the n conditions

  $$y_1(x_0) = y_{01}\,,\ \ldots\ , y_n(x_0) = y_{0n}\,.$$

- The initial value problem (1.63)–(1.64) includes as a particular case the initial value problem associated with a *scalar* n-th order equation in normal form

  $$\begin{cases} u^{(n)} = F\big(x, u, u', \ldots, u^{(n-1)}\big)\,, \\ u(x_0) = u_0\,,\ u'(x_0) = u_1\,,\ \ldots\ , u^{(n-1)}(x_0) = u_{n-1}\,. \end{cases} \tag{1.65}$$

  Indeed, if we introduce the n dependent variables

  $$y_1 = u\,,\ y_2 = u'\,,\ \ldots\ , y_n = u^{(n-1)}\,,$$

  the initial value problem (1.65) may be rewritten as the system of first-order equations given by

  $$\begin{cases} y_1' = y_2\,, \\ \quad\vdots \\ y_{n-1}' = y_n\,, \\ y_n' = F(x, y_1, \ldots, y_n)\,, \end{cases}$$

  with the initial conditions

  $$y_1(x_0) = u_0\,,\ y_2(x_0) = u_1\,, \ldots\ , y_n(x_0) = u_{n-1}\,.$$

  (More in general, a system of m ordinary differential equations of order n may be written as a system of mn first-order equations.)

The following theorem shows that the continuity of the function f on its domain U is sufficient to guarantee the local existence of solutions of the initial value problem (1.63)–(1.64):

[2]Hereafter we shall drop the vector notation, e.g., we shall write y instead of $\mathbf{y}$

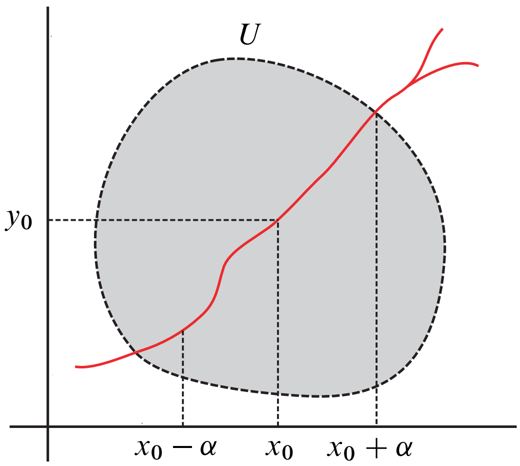

Figure 1.6: If the function $f(x, y)$ satisfies the hypothesis of the existence and uniqueness theorem on the open set U, the initial value problem (1.63)–(1.64) has a unique solution defined in the interval $(x_0 - \alpha, x_0 + \alpha)$. The solution might not exist or be unique outside the open set U.

Peano's theorem. *Let $f : U \to \mathbb{R}^n$ be continuous on an open set U, and let $(x_0, y_0) \in U$. Then the initial value problem (1.63)–(1.64) has (at least) one solution $y(x)$ defined on an interval of the form $(x_0 - \alpha, x_0 + \alpha)$, with $\alpha > 0$ sufficiently small.*

- The number α can be estimated explicitly, and it depends both on the point (x_0, y_0) and a bound of the values of $\|f(x, y)\|$ on U.

The continuity of the function f on the open set U does not guarantee (even locally) the uniqueness of the solution of the initial value problem (1.63)–(1.64) with initial data in U, as illustrated by the following example:

Example 1.13. Consider the initial value problem

$$\begin{cases} y' = 3y^{2/3}\,, \\ y(x_0) = y_0\,. \end{cases} \tag{1.66}$$

Since $f(x, y) = 3y^{2/3}$ is continuous on $U = \mathbb{R}^2$, according to Peano's theorem the problem (1.66) possesses at least a local solution for any initial data (x_0, y_0). Let us show that when $y_0 = 0$ this solution is not unique. Indeed, $y \equiv 0$ is a possible solution of the problem (1.66) for $y_0 = 0$. On the other hand, since the equation $y' = 3y^{2/3}$ is separable, it can be immediately integrated, with the result

$$y = (x + c)^3\,, \qquad c \in \mathbb{R}\,. \tag{1.67}$$

In particular, $y = (x - x_0)^3$ is another solution of (1.66) with $y_0 = 0$, differing from $y \equiv 0$ in any open interval centered at x_0.

Let us now present the key result that we shall apply to establish the existence and uniqueness of solution of the initial value problem (1.63)–(1.64).

Existence and uniqueness theorem. *If the function $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ and its partial derivatives $\frac{\partial f_i}{\partial y_j}$ $(1 \leqslant i, j \leqslant n)$ are continuous on the open set U, then for all $(x_0, y_0) \in U$ the initial value problem (1.63)–(1.64) has a unique solution on an interval of the form $(x_0 - \alpha, x_0 + \alpha)$, with $\alpha > 0$ depending on (x_0, y_0).*

The previous theorem is a consequence of a more general result, known as the *Picard–Lindelöf theorem*, whose statement and proof can be found for instance in F. Finkel y A. González-López, *Manual de Ecuaciones Diferenciales I*, UCM, 2009[3]. Sometimes we shall make use of the following direct corollary of the existence and uniqueness theorem:

Corollary 1.14. *If $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ is of class C^1 on an open set U, the initial value problem* (1.63)–(1.64) *has a unique local solution for any initial data $(x_0, y_0) \in U$.*

- If the hypothesis of the existence and uniqueness theorem (or its Corollary 1.14) hold, then no two solutions can intersect each other inside U, since otherwise there would be two solutions in U with the same initial data (see Fig. 1.6).

- The hypothesis of the existence and uniqueness theorem are definitely *not necessary* for the initial value problem (1.63)–(1.64) to have unique solution. For instance, the function

$$f(x, y) = \begin{cases} -\dfrac{2y}{x} + 4x\,, & x \neq 0\,, \\ 0\,, & x = 0\,, \end{cases}$$

  is discontinuous on the vertical axis $x = 0$. Since the equation $y' = f(x, y)$ is linear it can be easily solved, with the result

$$y = \frac{c}{x^2} + x^2\,.$$

  Thus, the differential equation $y' = f(x, y)$ with the initial condition $y(0) = 0$ has the unique solution $y = x^2$, corresponding to $c = 0$. On the other hand, for the initial condition $y(0) = y_0 \neq 0$ the initial value problem has no solution, for the only solution of the differential equation defined at $x = 0$ is $y = x^2$.

The existence and uniqueness theorem (or its Corollary 1.14) can also be employed to determine if there is a unique integral curve passing through a given point.

Example 1.15. Let us study for which points of the plane passes a unique integral curve of the differential equation

$$y' = \frac{y^2}{2x(y - x)}\,. \tag{1.68}$$

Notice in the first place that the function

$$f(x, y) = \frac{y^2}{2x(y - x)}$$

is of class C^1 on the whole real plane excepting the lines $x = 0$, $y = x$. According to Corollary 1.14, through any point $(x_0, y_0) \in \mathbb{R}^2$ not belonging to these lines passes a unique local solution, and thus a unique integral curve. In order to determine if this is also the case when the initial data (x_0, y_0) belong to the lines $x = 0$ or $y = x$, we consider the associated equation

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{2x(y - x)}{y^2}\,,$$

whose integral curves (but not its solutions!) coincide with those of the original equation. Since the right-hand side of the previous equation is of class C^1 on the whole plane excepting the line $y = 0$, from Corollary 1.14 it follows that through any point on the lines $x = 0$ or $y = x$ but the origin passes a

[3] From now on we shall use the abbreviation [EDI2009] to cite this reference.

unique solution of the associated equation, which is also an integral curve (with vertical tangent) of the equation (1.68)[4]

The only point in the plane where the existence and uniqueness theorem cannot be applied to either the starting equation or its associated one is the origin. In order to determine how many integral curves pass through this point there is no choice other than solving the differential equation, which in this case can be done since it is a homogeneous equation. Performing the change of variable $y = xu$ in equation (1.68) we obtain

$$xu' + u = \frac{u^2}{2(u-1)} \quad \Longrightarrow \quad xu' = \frac{u(2-u)}{2(u-1)}\,.$$

The latter equation admits the particular solutions $u = 0$, $u = 2$, corresponding to the linear solutions $y = 0$ and $y = 2x$. If $u \neq 0, 2$, solving the equation for u as a separable equation we obtain

$$-\int \frac{2u-2}{u^2-2u}\,\mathrm{d}u = -\log|u^2-2u| = \log|x| + c_0 \quad \Longrightarrow \quad u(u-2) = \frac{c}{x}\,, \quad c \in \mathbb{R}\,.$$

Expressing u in terms of y in the previous expression yields

$$y(y-2x) = cx\,,$$

which includes the previous solutions $y = 0$ and $y = 2x$ when $c = 0$. Since the above equation is satisfied identically for $x = y = 0$ and arbitrary c, all integral curves have a branch passing through the origin. In summary, through every point in the plane except the origin passes a single integral curve, while an infinite number of integral curves pass through the origin (cf. Fig. 1.7).

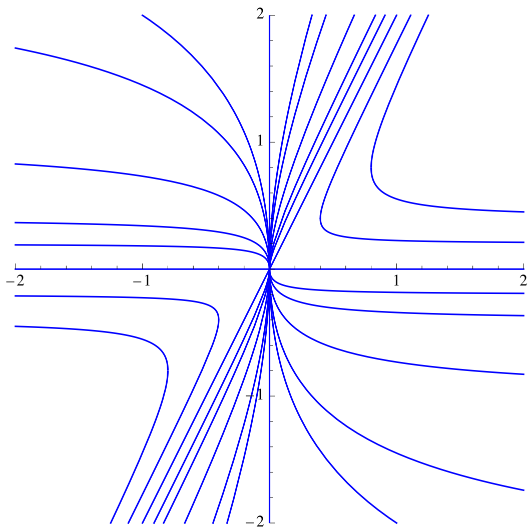

Figure 1.7: Integral curves of the equation (1.68).

[4]Note that the line $x = 0$ is clearly a solution of the associated equation, and thus an integral curve of the equation (1.68). However, the line $y = x$ is *not* an integral curve.

Chapter 2

Linear equations and systems

2.1 Space of solutions of a linear system

Definition 2.1. A **first order linear system** is a system of n equations of the form

$$\boxed{y' = A(x)\, y + b(x)}\,, \tag{2.1}$$

where $A : \mathbb{R} \to M_n(\mathbb{R})$ (resp. $b : \mathbb{R} \to \mathbb{R}^n$) is a matrix-valued (resp. vector-valued) function, that is,

$$A(x) = \begin{pmatrix} a_{11}(x) & \dots & a_{1n}(x) \\ \vdots & \ddots & \vdots \\ a_{n1}(x) & \dots & a_{nn}(x) \end{pmatrix}, \qquad b(x) = \begin{pmatrix} b_1(x) \\ \vdots \\ b_n(x) \end{pmatrix}. \tag{2.2}$$

The system (2.1) is said to be **homogeneous** if $b \equiv 0$, or **inhomogeneous** otherwise.

- The set $M_n(\mathbb{R})$ of square matrices of order n with real entries is a vector space of dimension n^2. The canonical basis of this space consists of the matrices E_{ij} whose only nonzero element is a 1 at the i-th row and the j-th column. The coordinates of a matrix A in this basis are its matrix elements a_{ij}.

- Recall that a vector function $b : \mathbb{R} \to \mathbb{R}^n$ is continuous at x if and only if its components $b_i : \mathbb{R} \to \mathbb{R}$ are continuous. Similarly, a matrix function $A : \mathbb{R} \to M_n(\mathbb{R})$ is continuous at x if and only if its n^2 matrix elements $a_{ij} : \mathbb{R} \to \mathbb{R}$ are continuous functions at x.

If the matrix function A and the vector function b are continuous on an open interval I, the existence and uniqueness theorem discussed in the previous chapter implies that the initial value problem

$$\begin{cases} y' = A(x)\, y + b(x)\,, \\ y(x_0) = y_0 \end{cases} \tag{2.3}$$

associated with the linear system (2.1) has a unique local solution for any initial data $(x_0, y_0) \in I \times \mathbb{R}^n$. In fact, for linear systems it can proved the following stronger result, whose proof can be found in [EDI2009]:

Theorem 2.2. *If $A : I \to M_n(\mathbb{R})$ and $b : I \to \mathbb{R}^n$ are continuous on an interval $I \subset \mathbb{R}$, then the initial value problem (2.3) has a unique solution defined on the* entire *interval I for any initial data $(x_0, y_0) \in I \times \mathbb{R}^n$.*

In what follows we shall assume that the functions A y b in the linear system (2.1)-(2.2) are continuous on an interval $I \subset \mathbb{R}$, and the hypothesis of Theorem 2.2 hold. We shall denote by $\mathcal{S}$ the set of solutions of the system (2.1), that is,

$$\mathcal{S} = \left\{ y : I \to \mathbb{R}^n \mid y'(x) = A(x)y(x) + b(x)\,,\ \forall x \in I \right\} \subset C^1(I, \mathbb{R}^n)\,.$$

Likewise, we shall denote by $\mathcal{S}_0$ the set of solutions of the corresponding homogeneous system

$$\boxed{y' = A(x)\, y}\,. \tag{2.4}$$

- If φ^1, φ^2 are two solutions of the homogeneous system (2.4), then any linear combination $\lambda\varphi^1 + \mu\varphi^2$ with coefficients $\lambda, \mu \in \mathbb{R}$ is also a solution. Indeed,

  $$(\lambda\varphi^1+\mu\varphi^2)'(x) = \lambda\varphi^{1\prime}(x)+\mu\varphi^{2\prime}(x) = \lambda A(x)\varphi^1(x)+\mu A(x)\varphi^2(x) = A(x)\big(\lambda\varphi^1(x)+\mu\varphi^2(x)\big).$$

  In other words, *the set $\mathcal{S}_0$ of solutions of the homogeneous system* (2.4) *is a real vector space*. This important property of homogeneous linear systems is known as the **linear superposition principle**.

- Since $A(x)$ is a real matrix, if φ is a *complex* solution of the homogeneous system (2.4), then $\bar{\varphi}$ is also a solution of this system. Similarly, if φ is a solution of (2.4), then $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi$ are both solutions of this system. Is this also the case for the inhomogeneous system (2.1)?

- The general solution of the inhomogeneous system (2.1) is of the form $y = y_{\mathrm{p}} + y_{\mathrm{h}}$, where y_{p} is a fixed particular solution of that system and y_{h} is the general solution of the corresponding homogeneous system (2.4). Indeed, if y is of this form it is clearly a solution of (2.1). Conversely, if y is any solution of (2.1), then $y - y_{\mathrm{p}}$ is obviously a solution of the homogeneous system (2.4). In mathematical language, we say that *the set of solutions of the inhomogeneous system is the affine space $\mathcal{S} = y_{\mathrm{p}} + \mathcal{S}_0$, where y_{p} is a fixed element of $\mathcal{S}$.*

Using the existence and uniqueness Theorem 2.2, we shall prove next that the dimension of the space $\mathcal{S}_0$ of solutions of the homogeneous system (2.4) is precisely n:

Theorem 2.3. *The solution set of the homogeneous system $y' = A(x)y$, with $y \in \mathbb{R}^n$, is a real vector space of dimension n.*

Proof. Let $x_0 \in I$ be a fixed but arbitrary point of I, and let e_i be the i-th vector of the canonical basis of $\mathbb{R}^n$. If $Y^i(x)$ denotes the solution of the initial value problem

$$\begin{cases} y' = A(x)y\,, \\ y(x_0) = e_i\,, \end{cases} \tag{2.5}$$

let us first show that $\{Y^1(x), \dots, Y^n(x)\}$ span the vector space $\mathcal{S}_0$. Indeed, let $y(x)$ be any solution of the homogeneous system (2.4), and let

$$y_0 = y(x_0) \equiv (y_{01}, \dots, y_{0n}) = \sum_{i=1}^n y_{0i}\, e_i\,.$$

Then the function

$$\tilde{y}(x) = \sum_{i=1}^n y_{0i}\, Y^i(x)$$

is a solution of the homogeneous system (2.4) (being a linear combination of solutions), and satisfies the initial condition

$$\tilde{y}(x_0) = \sum_{i=1}^n y_{0i}\, e_i = y_0 = y(x_0)\,.$$

From the existence and uniqueness Theorem 2.2 it follows that $\tilde{y} = y$ in I. Then *any solution is a linear combination of the n solutions Y^i*, which are in turn a generator set of $\mathcal{S}_0$. Let us prove that the

n solutions Y^i are also linearly independent, so that they form a basis of $\mathcal{S}_0$. Indeed, consider the linear combination

$$\sum_{i=1}^{n} \lambda_i Y^i = 0\,,$$

with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. The previous equality is equivalent to

$$\sum_{i=1}^{n} \lambda_i Y^i(x) = 0\,, \qquad \forall x \in I\,,$$

whence

$$\sum_{i=1}^{n} \lambda_i Y^i(x_0) = \sum_{i=1}^{n} \lambda_i e_i = 0\,,$$

which implies that $\lambda_1 = \cdots = \lambda_n = 0$ since $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$. This shows that $\{Y^1, \ldots, Y^n\}$ is a basis of $\mathcal{S}_0$, and thus $\dim \mathcal{S}_0 = n$. $\square$

2.2 Homogeneous systems

Definition 2.4. A **fundamental system of solutions** of the homogeneous system (2.4) is a basis $\{y^1, \ldots, y^n\}$ of its solution space $\mathcal{S}_0$.

In other words, a fundamental system of solutions of (2.4) is a set of n linearly independent solutions. For instance, the n solutions Y^i of the initial value problem (2.5) are a fundamental system of solutions of the homogeneous system $y' = A(x)y$. Note that, by construction, these solutions satisfy $Y^i(x_0) = e_i$.

By definition, any solution $y(x)$ of the homogeneous system (2.4) is a linear combination of the elements of a fundamental system of solutions $\{y^1, \ldots, y^n\}$, that is,

$$y(x) = \sum_{i=1}^{n} c_i y^i(x)\,, \qquad x \in I\,, \tag{2.6}$$

for certain real constants $c_1, \ldots, c_n$. The vector equality (2.6) is equivalent to the n scalar equalities

$$y_k(x) = \sum_{i=1}^{n} c_i y^i_k(x)\,, \qquad k = 1, \ldots, n\,,$$

for each of the components of the solution $y(x)$. In turn, we can write the equality (2.6) in matrix form as

$$\boxed{y(x) = Y(x)\, c}\,, \tag{2.7}$$

where

$$Y(x) = \begin{pmatrix} y^1(x) & \cdots & y^n(x) \end{pmatrix} \equiv \begin{pmatrix} y^1_1(x) & \ldots & y^n_1(x) \\ \vdots & \ddots & \vdots \\ y^1_n(x) & \ldots & y^n_n(x) \end{pmatrix}, \qquad c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Definition 2.5. A **fundamental matrix** of the homogeneous system (2.4) is any matrix-valued function $Y : I \to M_n(\mathbb{R})$ whose *columns* form a fundamental system of solutions.

- From what we have just seen, if $Y(x)$ is a fundamental matrix of the homogeneous system (2.4), the system's general solution is given by Eq. (2.7).

- A matrix-valued function $Y : I \to M_n(\mathbb{R})$ is a fundamental matrix of the homogeneous system (2.4) if and only if its columns are linearly independent, and it satisfies

$$Y'(x) = A(x)Y(x)\,, \qquad \forall x \in I\,.$$

  Indeed, the latter matrix equality is equivalent to the n vector equalities

$$y^{i\prime}(x) = A(x)y^i(x)\,, \qquad \forall x \in I\,, \quad \forall i = 1\,, \ldots\,, n\,,$$

  where $y^i(x)$ is the i-th column of $Y(x)$.

- A homogeneous linear system obviously has infinitely many fundamental matrices. If $Y_1(x)$ and $Y_2(x)$ are two fundamental matrices of (2.4) such that $Y_1(x_0) = Y_2(x_0)$, then $Y_1 = Y_2$ on the whole interval I. Indeed, each column of Y_1 and the corresponding column of Y_2 are solutions of the system taking the same value at x_0, so they must coincide in all I by virtue of the existence and uniqueness Theorem 2.2.

2.2.1 Wronskian

Given n solutions $\varphi^1, \ldots, \varphi^n$ (not necessarily independent) of the homogeneous system (2.4), let us consider the **matrix of solutions**

$$\Phi(x) = \begin{pmatrix}\varphi^1(x) & \cdots & \varphi^n(x)\end{pmatrix}$$

whose i-th column is given by the solution φ^i. Since by hypothesis $\varphi^{i\prime}(x) = A(x)\varphi^i(x)$ for $i = 1, \ldots, n$, the matrix $\Phi(x)$ satisfies the matrix equation

$$\Phi'(x) = A(x)\Phi(x)\,. \tag{2.8}$$

Definition 2.6. Given n solutions $\varphi^1, \ldots, \varphi^n$ of the homogeneous system (2.4), their **Wronskian** is the determinant of the corresponding matrix of solutions $\Phi(x)$, that is,

$$W[\varphi^1, \ldots, \varphi^n](x) \equiv \det \Phi(x) = \begin{vmatrix} \varphi^1_1(x) & \ldots & \varphi^n_1(x) \\ \vdots & \ddots & \vdots \\ \varphi^1_n(x) & \ldots & \varphi^n_n(x) \end{vmatrix}\,. \tag{2.9}$$

Notation. When it is clear from the context to which solutions $\varphi^1, \ldots, \varphi^n$ we are referring to we will denote their Wronskian simply as $W(x)$.

The key property of the Wronskian is that its vanishing at *any* point of the interval I implies the linear dependence of the solutions $\varphi^1, \ldots, \varphi^n$ in that interval, according to the following

Proposition 2.7. *Let $\varphi^1, \ldots, \varphi^n$ be solutions of the homogeneous system* (2.4) *on the interval I. Then $\{\varphi^1, \ldots, \varphi^n\}$ are linearly independent $\iff$ $W[\varphi^1, \ldots, \varphi^n](x) \neq 0$, $\forall x \in I$.*

Proof. Consider first the implication ($\Leftarrow$). If the solutions $\{\varphi^1, \ldots, \varphi^n\}$ were linearly dependent, the *vectors* $\{\varphi^1(x), \ldots, \varphi^n(x)\}$ would be linearly dependent at any point $x \in I$. But then $W[\varphi^1, \ldots, \varphi^n](x) = 0$ *for all* $x \in I$.

Regarding the implication ($\Rightarrow$), if there is a point $x_0 \in I$ such that $W[\varphi^1, \ldots, \varphi^n](x_0) = 0$, then the vectors $\{\varphi^1(x_0), \ldots, \varphi^n(x_0)\}$ would be linearly dependent, that is, there would be n real constants λ_k not all equal to zero such that

$$\sum_{k=1}^{n} \lambda_k \varphi^k(x_0) = 0\,.$$

But then

$$y(x) = \sum_{k=1}^{n} \lambda_k \varphi^k(x)$$

would be a solution of the system (2.4) satisfying the initial condition $y(x_0) = 0$. From the existence and uniqueness Theorem 2.2 it would follow that $y \equiv 0$ on I, and hence $\{\varphi^1, \ldots, \varphi^n\}$ would be linearly dependent. $\square$

- If $\varphi^1, \ldots, \varphi^n$ are solutions of the system $y' = A(x)y$, from the latter proof it follows that either $W[\varphi^1, \ldots, \varphi^n](x) \neq 0$ for all $x \in I$, or $W[\varphi^1, \ldots, \varphi^n](x) = 0$ for all $x \in I$.

- Note that a matrix-valued function $\Phi : I \to M_n(\mathbb{R})$ is a fundamental matrix of the system (2.4) if and only if

$$\text{i) } \Phi'(x) = A(x)\Phi(x)\,, \quad \forall x \in I, \tag{2.10a}$$

$$\text{ii) } \det \Phi(x) \neq 0\,, \quad \forall x \in I. \tag{2.10b}$$

  Indeed, the second condition is equivalent to the linear independence of the columns of $\Phi(x)$ by virtue of the previous proposition.

- If $\Phi(x)$ is a fundamental matrix and P is any *invertible* constant matrix, it is immediate to check that $\Psi(x) = \Phi(x)P$ satisfies i) and ii), and is therefore a fundamental matrix. Conversely, let $\Phi(x)$ and $\Psi(x)$ be two fundamental matrices of the system. Since the matrices $\Phi(x_0)$ and $\Psi(x_0)$ are invertible on account of the Proposition 2.7, the matrix $P = \Phi(x_0)^{-1}\Psi(x_0)$ is well-defined and invertible, and satisfy $\Psi(x_0) = \Phi(x_0)P$ by construction. Then $\Psi(x)$ and $\Phi(x)P$ are both fundamental matrices of the system (2.4) and coincide at x_0, so they must be equal on the whole interval I by virtue of the remark on page 24. In summary, any fundamental matrix of the homogeneous system (2.4) can be obtained from a fixed fundamental matrix by right-multiplying it by an appropriate invertible matrix.

Exercise. If $\Phi(x)$ is a fundamental matrix of the system (2.4) and P is a constant invertible matrix, what can be said about the matrix $\Psi(x) = P\Phi(x)$?

Definition 2.8. The **canonical fundamental matrix** of the homogeneous system (2.4) at the point x_0 is the unique fundamental matrix $Y(x)$ of this system satisfying the condition $Y(x_0) = \mathbb{1}$.

- Given any fundamental matrix $Y(x)$ of the system (2.4), it is immediate to check that $Y(x)Y(x_0)^{-1}$ is its canonical fundamental matrix at the point x_0.

- If $Y(x)$ is the canonical fundamental matrix of the system (2.4) at x_0, the solution of the initial value problem

$$\begin{cases} y' = A(x)y \\ y(x_0) = y_0 \end{cases}$$

  associated with this system is given by

$$y(x) = Y(x)y_0\,.$$

Remark. Given n arbitrary differentiable functions $\varphi^1, \ldots, \varphi^n$ (not necessarily solutions of a homogeneous linear system of the form (2.4) with A continuous on I), the vanishing of their Wronskian (even identically) does not imply their linear dependence. For instance, the functions

$$\varphi^1(x) = \begin{pmatrix} \sin x \\ x \end{pmatrix}, \qquad \varphi^2(x) = \begin{pmatrix} \mathrm{e}^x \sin x \\ \mathrm{e}^x x \end{pmatrix}$$

are linearly independent in spite of the fact that their Wronskian vanish identically on $\mathbb{R}$.

2.2.2 The Abel–Liouville formula

Let φ^k, $k = 1, \ldots, n$, be solutions of the homogeneous system (2.4), and let $W(x)$ be their Wronskian. Then

$$W'(x) = \sum_{i=1}^{n} \begin{vmatrix} \varphi_1^1(x) & \ldots & \varphi_1^n(x) \\ \vdots & & \vdots \\ \varphi_i^{1\prime}(x) & \ldots & \varphi_i^{n\prime}(x) \\ \vdots & & \vdots \\ \varphi_n^1(x) & \ldots & \varphi_n^n(x) \end{vmatrix} . \tag{2.11}$$

Since φ^k is a solution of (2.4) it follows that

$$\varphi_i^{k\prime}(x) = \sum_{j=1}^{n} a_{ij}(x)\varphi_j^k(x) , \qquad k = 1, \ldots, n .$$

Then

$$W'(x) = \sum_{i=1}^{n} \begin{vmatrix} \varphi_1^1(x) & \ldots & \varphi_1^n(x) \\ \vdots & & \vdots \\ \sum_{j=1}^{n} a_{ij}(x)\varphi_j^1(x) & \ldots & \sum_{j=1}^{n} a_{ij}(x)\varphi_j^n(x) \\ \vdots & & \vdots \\ \varphi_n^1(x) & \ldots & \varphi_n^n(x) \end{vmatrix} = \sum_{i=1}^{n} \begin{vmatrix} \varphi_1^1(x) & \ldots & \varphi_1^n(x) \\ \vdots & & \vdots \\ a_{ii}(x)\varphi_i^1(x) & \ldots & a_{ii}(x)\varphi_i^n(x) \\ \vdots & & \vdots \\ \varphi_n^1(x) & \ldots & \varphi_n^n(x) \end{vmatrix} ,$$

since a determinant does not change if one adds to a row a linear combination of the remaining ones. Thus

$$W'(x) = \sum_{i=1}^{n} a_{ii} W(x) = \operatorname{tr} A(x) \cdot W(x) .$$

Integrating the latter first-order linear equation starting from a certain point $x_0 \in I$, we obtain

$$\boxed{W(x) = W(x_0)\, \mathrm{e}^{\int_{x_0}^{x} \operatorname{tr} A(t)\, \mathrm{d}t}} , \qquad \forall x \in I . \tag{2.12}$$

This identity is known as the **Abel–Liouville formula**. From this formula we can also deduce that either $W(x)$ does not vanish on I, or it vanishes identically on I.

2.3 Space of solutions of an n-th order linear differential equation

Definition 2.9. An **n-th order linear equation** is a differential equation of the form

$$\boxed{u^{(n)} + a_{n-1}(x)\, u^{(n-1)} + \cdots + a_1(x)\, u' + a_0(x)\, u = b(x)} , \tag{2.13}$$

where the functions $a_i : \mathbb{R} \to \mathbb{R}$ ($i = 0, \ldots, n-1$) and $b : \mathbb{R} \to \mathbb{R}$ are continuous on an interval I. We shall say that the equation (2.13) is **homogeneous** if $b \equiv 0$ on I, and **inhomogeneous** or **complete** otherwise.

As we have seen in Chapter 1 (page 16), any differential equation of order n can be written as a system of n first order equations. The first-order system (1.65) associated with the equation (2.13) is the linear system

$$y' = A(x)\, y + b(x)\, e_n , \tag{2.14}$$

where

$$A(x) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0(x) & -a_1(x) & -a_2(x) & \dots & -a_{n-1}(x) \end{pmatrix} \tag{2.15}$$

is called the **companion matrix** of the equation (2.13). Note, in particular, that

$$\boxed{\operatorname{tr} A(x) = -a_{n-1}(x)}\,, \tag{2.16}$$

which shall be used in the sequel. Since the entries of the companion matrix $A(x)$ and the function $b(x)$ are continuous on the interval I, from Theorem 2.2 it follows that the initial value problem given by the equation (2.13) with the initial conditions

$$u(x_0) = u_0\,,\; u'(x_0) = u_1\,,\; \dots\,,\; u^{(n-1)}(x_0) = u_{n-1} \qquad (x_0 \in I) \tag{2.17}$$

has a unique solution defined on the entire interval I:

Theorem 2.10. *If the functions $a_i : I \to \mathbb{R}$ $(i = 0, \dots, n-1)$ and $b : I \to \mathbb{R}$ are continuous on the interval I, the initial value problem (2.13)-(2.17) possesses a unique solution defined on the entire interval I for any initial data $(x_0, u_0, \dots, u_{n-1}) \in I \times \mathbb{R}^n$.*

We will use a notation analogous to that for first-order linear systems, denoting by $\mathcal{S}$ the set of solutions of the equation (2.13), and by $\mathcal{S}_0$ that of the corresponding homogeneous equation

$$\boxed{u^{(n)} + a_{n-1}(x)\,u^{(n-1)} + \dots + a_1(x)\,u' + a_0(x)\,u = 0}\,. \tag{2.18}$$

Note that both sets are contained in $C^n(I)$. Reasoning as in the case of the first-order systems one can easily prove the following properties:

- If φ_1, φ_2 are two solutions of the homogeneous equation (2.18), then any linear combination $\lambda\varphi_1 + \mu\varphi_2$ with coefficients $\lambda, \mu \in \mathbb{R}$ is also a solution. In other words, *the set $\mathcal{S}_0$ of solutions of the homogeneous equation homogeneous equation* (2.18) *is a real vector space* (linear superposition principle).

- If φ is a complex solution of the homogeneous equation (2.18), then $\bar{\varphi}$, $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi$ are also solutions of this equation.

- The general solution of the inhomogeneous equation (2.13) is of the form $u = u_{\mathrm{p}} + u_{\mathrm{h}}$, where u_{p} is a fixed particular solution of this equation and u_{h} is the general solution of the corresponding homogeneous equation (2.18). Equivalently, *the solution set of the inhomogeneous equation* (2.13) *is the affine space $\mathcal{S} = u_{\mathrm{p}} + \mathcal{S}_0$, where u_{p} is a fixed element of $\mathcal{S}$.*

In order to determine the dimension of the space $\mathcal{S}_0$ we shall use the following property:

- If $\varphi_1, \dots, \varphi_k$ are solutions of the homogeneous linear equation (2.18), and $y^i = (\varphi_i, \varphi_i', \dots, \varphi_i^{(n-1)})$, $i = 1, \dots, k$, denote the corresponding solutions of the associated first-order linear system, then

  $$\{\varphi_1, \dots, \varphi_k\} \text{ are linearly independent} \quad \Longleftrightarrow \quad \{y^1, \dots, y^k\} \text{ are linearly independent}\,.$$

  The implication $(\Rightarrow)$ is trivial. Regarding the converse, assume that

  $$\lambda_1\varphi_1 + \dots + \lambda_k\varphi_k = 0\,, \qquad \lambda_i \in \mathbb{R}\,.$$

  Differentiating $n-1$ times this equality it follows that

  $$\lambda_1 y^1 + \dots + \lambda_k y^k = 0\,,$$

  and thus $\lambda_1 = \dots = \lambda_k = 0$, since $\{y^1, \dots, y^k\}$ are linearly independent by hypothesis.

Theorem 2.3 and the previous property yield the following result:

Theorem 2.11. *The space $\mathcal{S}_0$ of solutions of the homogeneous equation (2.18) is of dimension n.*

Definition 2.12. A **fundamental system of solutions** of the homogeneous equation (2.18) is a basis $\{\varphi_1, \ldots, \varphi_n\}$ of its solution space $\mathcal{S}_0$.

- If $\{\varphi_1, \ldots, \varphi_n\}$ is a fundamental system of solutions of (2.18), any solution u of this equation may be expressed as

$$\boxed{u(x) = \sum_{i=1}^{n} c_i \varphi_i(x)\,, \qquad c_i \in \mathbb{R}}\,.$$

Definition 2.13. Given n solutions $\varphi_1, \ldots, \varphi_n$ (not necessarily independent) of the homogeneous equation (2.18), its **Wronski matrix** is defined as the matrix of the corresponding solutions of the associated linear system, i.e.,

$$\Phi(x) = \begin{pmatrix} \varphi_1(x) & \ldots & \varphi_n(x) \\ \varphi_1'(x) & \ldots & \varphi_n'(x) \\ \vdots & & \vdots \\ \varphi_1^{(n-1)}(x) & \ldots & \varphi_n^{(n-1)}(x) \end{pmatrix}. \tag{2.19}$$

Definition 2.14. Given n solutions $\varphi_1, \ldots, \varphi_n$ of the homogeneous equation (2.18), its **Wronskian** is the determinant of the corresponding Wronski matrix, that is,

$$W[\varphi_1, \ldots, \varphi_n](x) = \det \Phi(x) = \begin{vmatrix} \varphi_1(x) & \ldots & \varphi_n(x) \\ \varphi_1'(x) & \ldots & \varphi_n'(x) \\ \vdots & & \vdots \\ \varphi_1^{(n-1)}(x) & \ldots & \varphi_n^{(n-1)}(x) \end{vmatrix}. \tag{2.20}$$

We shall also use the abbreviated notation $W(x)$ instead of $W[\varphi_1, \ldots, \varphi_n](x)$ when it is clear from the context to which solutions $\varphi_1, \ldots, \varphi_n$ we are referring to. As in the case of a first-order linear system, the Wronskian can be used to easily determine the linear independence of a set of n solutions of the homogeneous linear equation (2.18):

Proposition 2.15. *Let $\varphi_1, \ldots, \varphi_n$ be solutions of the homogeneous equation (2.18) over an interval I. Then $\{\varphi_1, \ldots, \varphi_n\}$ are linearly independent $\iff W[\varphi_1, \ldots, \varphi_n](x) \neq 0$, $\forall x \in I$.*

Proof. If $y^1, \ldots, y^n$ are the solutions of the associated first-order system corresponding to $\varphi_1, \ldots, \varphi_n$, from the remark just before Theorem 2.11 and Proposition 2.7 it follows that

$$\{\varphi_1, \ldots, \varphi_n\} \text{ l.i.} \iff \{y^1, \ldots, y^n\} \text{ l.i.} \iff W[y^1, \ldots, y^n](x) \neq 0, \ \forall x \in I\,.$$

But $W[y^1, \ldots, y^n](x) = W[\varphi_1, \ldots, \varphi_n](x)$ by definition. □

- If $\varphi_1, \ldots, \varphi_n$ are solutions of the equation (2.18), by the remark following Proposition 2.7 either $W[\varphi_1, \ldots, \varphi_n](x) \neq 0$ for all $x \in I$, or $W[\varphi_1, \ldots, \varphi_n](x) = 0$ for all $x \in I$. Thus it suffices to check that $W[\varphi_1, \ldots, \varphi_n](x_0) \neq 0$ at any point $x_0 \in I$ to ensure the linear independence of these solutions on I.

- Let $\varphi_1, \dots, \varphi_n$ be solutions of the equation (2.18), and let $W(x)$ be their Wronskian. Since the companion matrix (2.15) satisfies $\operatorname{tr} A(x) = -a_{n-1}(x)$, the Abel–Liouville formula (2.12) becomes

$$\boxed{W(x) = W(x_0)\, \mathrm{e}^{-\int_{x_0}^{x} a_{n-1}(t)\,\mathrm{d}t}}\,, \qquad x \in I\,. \tag{2.21}$$

  Note that the statement in the previous remark follows directly from this formula, and that the Wronskian is constant if the coefficient $a_{n-1}(x)$ vanishes identically over I.

2.3.1 Reduction of order

In general, it is not possible to compute a fundamental system of solutions of the homogeneous equation (2.18) in closed form, namely in terms of the coefficients $a_i(x)$ and their primitives. However, in the case of a second-order equation

$$u'' + a_1(x)u' + a_0(x)u = 0\,, \tag{2.22}$$

if a nontrivial solution is known, one can determine the general solution in terms of quadratures. Indeed, if $\varphi(x)$ is a nontrivial particular solution of the equation (2.22) and u is any solution of the latter equation, from the Abel–Liouville formula (2.21) it follows that

$$\varphi(x)u' - \varphi'(x)\,u = k\,\mathrm{e}^{-\int_{x_0}^{x} a_1(s)\,\mathrm{d}s}\,,$$

where $k = W[\varphi, u](x_0)$. Integrating this first-order linear equation for u we easily obtain the following expression for the general solution of (2.22):

$$u(x) = c\,\varphi(x) + k\,\varphi(x) \int_{x_0}^{x} \frac{\mathrm{e}^{-\int_{x_0}^{t} a_1(s)\,\mathrm{d}s}}{\varphi^2(t)}\,\mathrm{d}t\,.$$

Thus $\varphi(x)$ and the new solution

$$\boxed{\psi(x) = \varphi(x) \int_{x_0}^{x} \frac{\mathrm{e}^{-\int_{x_0}^{t} a_1(s)\,\mathrm{d}s}}{\varphi^2(t)}\,\mathrm{d}t} \tag{2.23}$$

constitute a fundamental system of solutions of the homogeneous equation (2.22), since (by construction) $W[\varphi, \psi](x_0) = 1 \neq 0$.

Remark. In the case of the homogeneous equation (2.18) of order $n > 2$, the explicit knowledge of a nontrivial particular solution $\varphi(x)$ makes it possible to transform the equation into a homogeneous linear equation of order $n - 1$ via the change of variable

$$z = \left(\frac{u}{\varphi(x)}\right)'. \tag{2.24}$$

For instance, assume that $\varphi(x) \not\equiv 0$ is a particular solution of the third-order equation

$$u''' + a_2(x)u'' + a_1(x)u' + a_0(x)u = 0\,, \tag{2.25}$$

Writing the change of variable (2.24) as $u = \varphi(x) \int^{x} z$, we obtain

$$\begin{aligned}
u' &= \varphi'(x) \textstyle\int^{x} z + \varphi(x)z\,, \\
u'' &= \varphi''(x) \textstyle\int^{x} z + 2\varphi'(x)z + \varphi(x)z'\,, \\
u''' &= \varphi'''(x) \textstyle\int^{x} z + 3\varphi''(x)z + 3\varphi'(x)z' + \varphi(x)z''\,.
\end{aligned}$$

Substituting these expressions into (2.25), and taking into account that $\varphi(x)$ is a solution of this equation, we obtain the following second-order equation for z:

$$\varphi(x)z'' + \big[3\varphi'(x) + a_2(x)\varphi(x)\big]z' + \big[3\varphi''(x) + 2a_2(x)\varphi'(x) + a_1(x)\varphi(x)\big]z = 0\,.$$

In general, it may be proved (see, e.g., L. Elsgolts, *Differential Equations and the Calculus of Variations*, University Press of the Pacific, 2003) that if k linearly independent solutions of a homogeneous linear equation of order n are known, it is possible to transform this equation into a homogeneous linear equation of order $n-k$ by successive changes of variable of the form form (2.24). In particular, if $n-1$ solutions of the equation (2.18) are known, it is possible to express its general solution in terms of quadratures after transforming it by this procedure into a first-order linear equation.

2.4 Method of variation of constants

In general, it is not possible to determine explicitly a fundamental system of solutions of the homogeneous system (2.4) or of the homogeneous equation (2.18). However, if such a fundamental system is known it will be possible to determine the general solution of the corresponding system or inhomogeneous equation by using the **method of variation of constants**, which we shall explain in what follows.

2.4.1 Method of variation of constants for an inhomogeneous system

Similarly to the scalar case (see p. 12), in this method one considers as a trial solution of the system (2.1) the function obtained by substituting the constant vector c in the general solution (2.7) of the homogeneous system by a vector-valued function $c(x)$, that is,

$$y(x) = Y(x)c(x)\,, \qquad c(x) = \begin{pmatrix} c_1(x) \\ \vdots \\ c_n(x) \end{pmatrix}.$$

Inserting this expression into (2.1), we obtain

$$y'(x) = Y'(x)c(x) + Y(x)c'(x) = A(x)Y(x)c(x) + b(x)\,,$$

which yields, taking into account that the fundamental matrix $Y(x)$ satisfies the conditions (2.10),

$$c'(x) = Y^{-1}(x)b(x)\,, \qquad \forall x \in I\,.$$

Thus

$$c(x) = c + \int^x Y^{-1}(s)b(s)\,\mathrm{d}s\,, \qquad c \in \mathbb{R}^n\,.$$

Hence the general solution of the inhomogeneous system (2.1) is given by

$$\boxed{y(x) = Y(x)\,c + Y(x)\int^x Y^{-1}(s)b(s)\,\mathrm{d}s}\,, \qquad \forall x \in I\,. \tag{2.26}$$

In accordance with Proposition 2.1, the solution (2.26) is of the form

$$y(x) = y_{\mathrm{h}}(x) + y_{\mathrm{p}}(x)\,,$$

where $y_{\mathrm{h}}(x)$ is the general solution of the homogeneous system and $y_{\mathrm{p}}(x)$ is a particular of the inhomogeneous one. Finally, it can be easily checked that the solution of the initial value problem (2.3) is

$$\boxed{y(x) = Y(x)Y^{-1}(x_0)\,y_0 + \int_{x_0}^x Y(x)Y^{-1}(s)b(s)\,\mathrm{d}s}\,, \qquad \forall x \in I\,. \tag{2.27}$$

2.4.2 Method of variation of constants for an inhomogeneous equation

Just as for first-order linear systems, if a fundamental system of solutions of the homogeneous equation (2.18) is known, it is possible to express the general solution of the corresponding inhomogeneous equation (2.13) by means of quadratures. Indeed, let $\{\varphi_1, \dots, \varphi_n\}$ be a fundamental system of solutions of the equation (2.18), and let

$$\Phi(x) = \begin{pmatrix} \varphi_1(x) & \dots & \varphi_n(x) \\ \varphi_1'(x) & \dots & \varphi_n'(x) \\ \vdots & & \vdots \\ \varphi_1^{(n-1)}(x) & \dots & \varphi_n^{(n-1)}(x) \end{pmatrix}$$

be their corresponding Wronski matrix. The general solution of the first-order linear system (2.14) associated with the inhomogeneous equation (2.13) is (cf. eq. (2.26))

$$y(x) = \Phi(x)c + \int^x b(t)\,\Phi(x)\,\Phi(t)^{-1} e_n\,\mathrm{d}t\,, \qquad c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n\,. \tag{2.28}$$

The general solution of (2.13) is the first component of the right-hand side of the last equation. In order to write this solution more explicitly, note that the first component of $\Phi(x)\,\Phi(t)^{-1} e_n$ is given by

$$\sum_{i=1}^n \Phi_{1i}(x)\big[\Phi(t)^{-1}\big]_{in} = \sum_{i=1}^n \varphi_i(x)(-1)^{i+n}\frac{M_{ni}(t)}{W(t)} = \frac{1}{W(t)}\begin{vmatrix} \varphi_1(t) & \dots & \varphi_n(t) \\ \varphi_1'(t) & \dots & \varphi_n'(t) \\ \vdots & & \vdots \\ \varphi_1^{(n-2)}(t) & \dots & \varphi_n^{(n-2)}(t) \\ \varphi_1(x) & \dots & \varphi_n(x) \end{vmatrix},$$

where $M_{ni}(t)$ denotes the minor of the matrix $\Phi(t)$ associated with the matrix element ni, and the last equality is obtained by expanding the determinant by the last row. Substituting the latter expression into (2.28) we finally obtain the following expression for the general solution of the equation (2.13):

$$\boxed{u(x) = \sum_{i=1}^n c_i\,\varphi_i(x) + \int^x \begin{vmatrix} \varphi_1(t) & \dots & \varphi_n(t) \\ \varphi_1'(t) & \dots & \varphi_n'(t) \\ \vdots & & \vdots \\ \varphi_1^{(n-2)}(t) & \dots & \varphi_n^{(n-2)}(t) \\ \varphi_1(x) & \dots & \varphi_n(x) \end{vmatrix} \frac{b(t)}{W(t)}\,\mathrm{d}t}\,. \tag{2.29}$$

In particular, for the second-order equation

$$u'' + a_1(x)\,u' + a_0(x)\,u = b(x) \tag{2.30}$$

we have

$$\boxed{u(x) = c_1\,\varphi_1(x) + c_2\,\varphi_2(x) + \int^x \frac{b(t)}{W(t)}\big[\varphi_1(t)\varphi_2(x) - \varphi_2(t)\varphi_1(x)\big]\,\mathrm{d}t\,.} \tag{2.31}$$

Note that the function

$$u_{\mathrm{p}}(x) = \int_{x_0}^x \frac{b(t)}{W(t)}\big[\varphi_1(t)\varphi_2(x) - \varphi_2(t)\varphi_1(x)\big]\,\mathrm{d}t \tag{2.32}$$

is a particular solution of the inhomogeneous equation (2.30) satisfying the initial conditions

$$u_{\mathrm{p}}(x_0) = u_{\mathrm{p}}'(x_0) = 0\,.$$

Example 2.16. Consider the second-order equation

$$u'' + u = \tan x\,. \tag{2.33}$$

Although we shall postpone to next chapter the problem of how to construct a fundamental system of solutions of the homogeneous equation

$$u'' + u = 0\,, \tag{2.34}$$

it is immediate to *verify* that the functions

$$\varphi_1(x) = \cos x\,, \quad \varphi_2(x) = \sin x$$

form such a system, as they are obviously solutions of the equation (2.34) and their Wronskian is

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0\,.$$

(Notice that $W(x)$ is a constant; this is a consequence of the Abel–Liouville formula (2.21), since in this case $a_{n-1} \equiv a_1 = 0$.) From equation (2.31) it follows that a particular solution of the equation (2.33) is given by

$$\begin{aligned} u_{\mathrm{p}} &= \int^x \tan t\,[\cos t \sin x - \sin t \cos x]\,\mathrm{d}t \\ &= \sin x \int^x \sin t\,\mathrm{d}t - \cos x \int^x \frac{1-\cos^2 t}{\cos t}\,\mathrm{d}t = -\cos x \int^x \sec t\,\mathrm{d}t\,. \end{aligned} \tag{2.35}$$

In order to evaluate the latter integral we perform the change of variable $s = \tan\frac{t}{2}$, so that

$$\mathrm{d}s = \tfrac{1}{2}\,(1 + \tan^2\tfrac{t}{2})\,\mathrm{d}t = \tfrac{1}{2}(1+s^2)\,\mathrm{d}t \quad\Longrightarrow\quad \mathrm{d}t = \frac{2}{1+s^2}\,\mathrm{d}s\,.$$

On the other hand,

$$\begin{cases} \cos t = 2\cos^2\frac{t}{2} - 1 = \dfrac{2}{\sec^2\frac{t}{2}} - 1 = \dfrac{2}{1+\tan^2\frac{t}{2}} - 1 = \dfrac{2}{1+s^2} - 1 = \dfrac{1-s^2}{1+s^2}\,, \\[2ex] \sin t = 2\sin\frac{t}{2}\cos\frac{t}{2} = \dfrac{2\tan\frac{t}{2}}{\sec^2\frac{t}{2}} = \dfrac{2s}{1+s^2}\,, \end{cases}$$

and thus

$$\begin{aligned} \int \sec t\,\mathrm{d}t &= \int \frac{1+s^2}{1-s^2}\cdot\frac{2}{1+s^2}\,\mathrm{d}s = \int \frac{2}{1-s^2}\,\mathrm{d}s = \int \frac{\mathrm{d}s}{1-s} + \int \frac{\mathrm{d}s}{1+s} = \log\left|\frac{1+s}{1-s}\right| \\ &= \log\left|\frac{1+s^2}{1-s^2} + \frac{2s}{1-s^2}\right| = \log|\sec t + \tan t|\,. \end{aligned}$$

Substituting this expression into (2.35) and adding the general solution of the homogeneous equation we finally conclude that

$$u = c_1 \cos x + c_2 \sin x - \cos x\,\log|\sec x + \tan x|\,, \qquad c_1, c_2 \in \mathbb{R}$$

is the general solution of the equation (2.33).

Chapter 3

Linear equations and systems with constant coefficients

3.1 Equations with constant coefficients. Method of undetermined coefficients

As we have seen in the previous chapter, the difficulty in solving the inhomogeneous equation (2.13) lies in determining a fundamental system of solutions of the corresponding homogeneous equation. Indeed, the method of variation of constants makes it possible to express the general solution of the inhomogeneous equation via quadratures provided that such system is known (cf. eq. (2.29)). A particular case of great practical interest for which it is possible to find the general solution of the linear equation (2.13) occurs when the coefficients $a_i(x)$ are constant, that is,

$$\boxed{u^{(n)} + a_{n-1}\, u^{(n-1)} + \cdots + a_1\, u' + a_0\, u = b(x)}\,, \qquad a_0, \ldots, a_{n-1} \in \mathbb{R}\,. \tag{3.1}$$

In this section we shall see that one can construct a fundamental system of solutions of the corresponding homogeneous equation

$$\boxed{u^{(n)} + a_{n-1}\, u^{(n-1)} + \cdots + a_1\, u' + a_0\, u = 0} \tag{3.2}$$

provided that one can determine all the zeros of the **characteristic polynomial** of the equation (3.2), defined as

$$\boxed{p(\lambda) \equiv \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0}\,. \tag{3.3}$$

- The companion matrix of the equation (3.2) is the constant matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{pmatrix}. \tag{3.4}$$

  Note that the characteristic polynomial (3.3) coincides with the characteristic polynomial of the matrix A, defined by

$$p_A(\lambda) \equiv \det(\lambda \mathbb{1} - A)\,,$$

  as it can be easily verified by expanding the latter determinant by the last row,

Let us begin by rewriting equation (3.2) in the form

$$\big(D^n + a_{n-1}\, D^{n-1} + \cdots + a_1\, D + a_0\big)u \equiv p(D)u = 0\,, \qquad D \equiv \frac{\mathrm{d}}{\mathrm{d}x}\,.$$

From the equality

$$(D-\lambda)\big(f(x)\mathrm{e}^{\lambda x}\big) = f'(x)\mathrm{e}^{\lambda x}\,,$$

it immediately follows that

$$\boxed{(D-\lambda)^k\big(f(x)\mathrm{e}^{\lambda x}\big) = f^{(k)}(x)\mathrm{e}^{\lambda x}}\,, \tag{3.5}$$

which is a very useful identity, as we shall see later in this section. Assume that the characteristic polynomial can be factorized as

$$p(\lambda) = (\lambda-\lambda_1)^{r_1}\cdots(\lambda-\lambda_m)^{r_m}\,, \qquad r_1+\cdots+r_m = n\,,$$

where the (possibly complex) roots $\lambda_1,\dots,\lambda_m$ are all distinct. If λ_i is one of these roots we can thus write

$$p(\lambda) = q(\lambda)(\lambda-\lambda_i)^{r_i}\,,$$

where $q(\lambda)$ is a polynomial of degree $n-r_i$ with $q(\lambda_i)\neq 0$. Then $p(D) = q(D)(D-\lambda_i)^{r_i}$, and from equation (3.5) it follows that

$$p(D)\big(x^k\mathrm{e}^{\lambda_i x}\big) = q(D)\left[\mathrm{e}^{\lambda_i x}\frac{\mathrm{d}^{r_i}}{\mathrm{d}x^{r_i}}\,x^k\right] = 0\,, \qquad k = 0,1,\dots,r_i-1\,.$$

This shows that the functions

$$\boxed{x^k\mathrm{e}^{\lambda_i x}\,, \qquad i = 1,\dots,m\,, \quad k = 0,1,\dots,r_i-1}\,, \tag{3.6}$$

are all solutions of the equation (3.2). Since there are precisely $r_1+\cdots+r_m = n$ solutions of this type, in order to show that they form a fundamental system of solutions of the latter equation it suffices to check that they are linearly independent.

Lemma 3.1. *The functions* (3.6) *are linearly independent,*

Proof. Consider the linear combination

$$\sum_{i=1}^{m}\sum_{k=0}^{r_i-1} c_{ik}x^k\mathrm{e}^{\lambda_i x} = 0\,, \qquad \forall x\in\mathbb{R}\,, \tag{3.7}$$

where the coefficients c_{ik} are complex constants. We can rewrite the latter equality as

$$P_1(x)\mathrm{e}^{\lambda_1 x}+\cdots+P_m(x)\mathrm{e}^{\lambda_m x} = 0\,, \qquad \forall x\in\mathbb{R}\,, \tag{3.8}$$

where $P_i(x) = \sum\limits_{k=0}^{r_i-1} c_{ik}x^k$ is a polynomial of degree at most r_i-1. Note that proving that all the coefficients c_{ik} in equation (3.7) are zero is equivalent to showing that the polynomials P_i vanish identically. In order to establish this result, we first multiply the equation (3.8) by $\mathrm{e}^{-\lambda_1 x}$, obtaining

$$P_1(x)+P_2\mathrm{e}^{\mu_2 x}+\cdots+P_m(x)\mathrm{e}^{\mu_m x} = 0\,, \qquad \forall x\in\mathbb{R}\,, \tag{3.9}$$

where the exponents $\mu_i \equiv \lambda_i-\lambda_1 \neq 0$ are all distinct. Differentiating this equation r_1 times we get

$$Q_2(x)\mathrm{e}^{\mu_2 x}+\cdots+Q_m(x)\mathrm{e}^{\mu_m x} = 0\,, \qquad \forall x\in\mathbb{R}\,, \tag{3.10}$$

where Q_i is a polynomial of the same degree as P_i. Repeating this procedure $m-1$ times one finally arrives at an equation of the form

$$R_m(x)\mathrm{e}^{\nu_m x} = 0\,, \qquad \forall x\in\mathbb{R}\,, \tag{3.11}$$

where R_m is a polynomial of the same degree as P_m. From the latter condition it immediately follows that $R_m \equiv 0$, which implies that $P_m \equiv 0$ on account that $\deg P_m = \deg R_m$. Since the ordering of the roots λ_i is irrelevant for the previous argument, we conclude that all the polynomials P_i are identically zero. ☐

The previous discussion and Lemma 3.1 lead to the following theorem:

Theorem 3.2. *The functions* (3.6)*, where r_i is the multiplicity of the root λ_i of the characteristic polynomial* (3.3)*, form a fundamental system of solutions of the homogeneous linear equation with constant coefficients* (3.2)*.*

- If the root $\lambda_j = a_j + \mathrm{i}b_j$ is complex, from the $2r_j$ complex solutions

$$x^k \mathrm{e}^{a_j x} \mathrm{e}^{\pm \mathrm{i} b_j x}, \qquad k = 0, 1, \ldots, r_j - 1,$$

associated with the roots λ_j and $\overline{\lambda_j}$ of the characteristic polynomial, we obtain the $2r_j$ real solutions

$$\boxed{x^k \mathrm{e}^{a_j x} \cos(b_j x), \quad x^k \mathrm{e}^{a_j x} \sin(b_j x) \qquad k = 0, 1, \ldots, r_j - 1}$$

taking the real and imaginary parts of the solutions corresponding to the root λ_j (or $\overline{\lambda_j}$).

Example 3.3. Let us find the general solution of the fourth-order equation with constant coefficients

$$u^{(4)} + u''' + u' + u = 0. \tag{3.12}$$

The characteristic polynomial associated with this equation reads

$$p(\lambda) = \lambda^4 + \lambda^3 + \lambda + 1 = (\lambda + 1)^2(\lambda^2 - \lambda + 1). \tag{3.13}$$

The zeros are thus $\lambda_1 = -1$ (with multiplicity $r_1 = 2$) and the roots of the equation

$$\lambda^2 - \lambda + 1 = 0,$$

given by

$$\lambda_{2,3} = \frac{1}{2}\big(1 \pm \mathrm{i}\sqrt{3}\big),$$

with multiplicity $r_2 = r_3 = 1$. The general solution of the equation (3.12) is then

$$u(x) = \mathrm{e}^{-x}(c_1 + c_2 x) + \mathrm{e}^{\frac{x}{2}} \left[c_3 \cos\left(\tfrac{\sqrt{3}}{2}\, x\right) + c_4 \sin\left(\tfrac{\sqrt{3}}{2}\, x\right) \right], \tag{3.14}$$

with $c_1, \ldots, c_4$ arbitrary real constants.

3.1.1 Method of undetermined coefficients

Once a fundamental system of solutions of the homogeneous equation (3.2) has been found, the general solution of the inhomogeneous equation (3.1) can be computed for any function $b(x)$ using the method of variation of constants (see equation (2.29)). However, for certain simple forms of the function $b(x)$ that appear frequently in the applications, the **method of undetermined coefficients** that we shall discuss in what follows enables one to compute a particular solution of (3.1) in a simpler way. The general solution of (3.1) is then found by adding to this particular solution the general solution of the corresponding homogeneous equation.

Suppose first that

$$b(x) = q(x)\mathrm{e}^{\mu x}, \tag{3.15}$$

where $q(x)$ is a *polynomial*. If r is the multiplicity of μ as a root of the characteristic polynomial (3.3) of the homogeneous equation (3.2), then

$$p(\lambda) = p_1(\lambda - \mu)^r + p_2(\lambda - \mu)^{r+1} + \cdots + p_{n-r}(\lambda - \mu)^{n-1} + (\lambda - \mu)^n,$$

with $p_1, \dots, p_{n-r} \in \mathbb{R}$ (or $\mathbb{C}$, if μ is complex) and $p_1 \neq 0$. Note that this expression is also valid when μ is not a root of the characteristic polynomial, with $r = 0$ in this case. From the previous expression and equation (3.5) it follows that

$$p(D)\big(f(x)e^{\mu x}\big) = \Big[p_1 f^{(r)}(x) + p_2 f^{(r+1)}(x) + \cdots + p_{n-r} f^{(n-1)}(x) + f^{(n)}(x)\Big]\, e^{\mu x} .$$

This suggest trying a particular solution of the form

$$\boxed{u_p(x) = x^r Q(x) e^{\mu x}}\,, \tag{3.16}$$

where

$$\boxed{Q(x) = Q_0 + Q_1 x + \cdots + Q_d x^d\,, \qquad d \equiv \deg q} \tag{3.17}$$

is a polynomial to be determined by the condition

$$p_1 (x^r Q)^{(r)} + p_2 (x^r Q)^{(r+1)} + \cdots + p_{n-r} (x^r Q)^{(n-1)} + (x^r Q)^{(n)} = q\,. \tag{3.18}$$

It may be proved that the last equation yields a linear system with $d + 1$ equations for the $d + 1$ coefficients of Q, which is always compatible. Therefore, the constant coefficients equation (3.1) with the inhomogeneous term (3.15) always has a particular solution of the form (3.16)-(3.17), where r is the multiplicity of μ as a root of the characteristic polynomial and the polynomial Q is determined via equation (3.18) (or by direct substitution of (3.16)-(3.17) into (3.1)).

Example 3.4. Let us find a particular solution of the equation

$$u^{(4)} + u''' + u' + u = x e^{-x}\,. \tag{3.19}$$

Here $\mu = -1$ is a double root of the characteristic polynomial of the homogeneous equation (see the equation (3.13) in Example 3.3), and $q(x) = x$ is a first-degree polynomial. We thus look for a particular solution of the form

$$u_p(x) = x^2(a + bx)e^{-x}\,.$$

In order to compute $p(D)\, u_p$ it is convenient to expand $p(D)$ in powers of $D+1$, since $(D+1)^k\big(f(x)e^{-x}\big) = f^{(k)}(x)e^{-x}$ on account of equation (3.5). Using Taylor's formula to expand $\lambda^2 - \lambda + 1$ in powers of $\lambda + 1$ we obtain

$$p(\lambda) = (\lambda + 1)^2\big[3 - 3(\lambda + 1) + (\lambda + 1)^2\big].$$

Thus

$$p(D)u_p = \big[3(2a + 6bx) - 3 \cdot 6b\big]e^{-x} = x e^{-x} \quad \Longleftrightarrow \quad 3(2a + 6bx) - 18b = x\,,$$

so that

$$6a - 18b = 0\,, \quad 18b = 1\,.$$

Hence, the sought-for particular solution reads

$$u_p(x) = \frac{1}{18}(x^3 + 3x^2)e^{-x}\,. \tag{3.20}$$

The general solution of the equation (3.19) is the sum of this particular solution and the general solution (3.14) of the homogeneous equation.

Example 3.5. Consider next the equation

$$u^{(4)} + 4u = x\big(1 + e^x \cos x\big)\,. \tag{3.21}$$

The characteristic polynomial of the homogeneous equation is

$$p(\lambda) = \lambda^4 + 4\,,$$

whose zeros λ_k are the fourth roots of -4:

$$\lambda_k = \sqrt{2}\,\mathrm{e}^{\mathrm{i}(\frac{\pi}{4}+k\frac{\pi}{2})}\,, \qquad k = 0,1,2,3\,,$$

i.e.,

$$\lambda_0 = 1+\mathrm{i}\,, \quad \lambda_1 = -1+\mathrm{i}\,, \quad \lambda_2 = -1-\mathrm{i}\,, \quad \lambda_3 = 1-\mathrm{i}\,.$$

Thus the general solution of the homogeneous equation is given by

$$u_{\mathrm{h}}(x) = \mathrm{e}^{x}(c_1\cos x + c_2\sin x) + \mathrm{e}^{-x}(c_3\cos x + c_4\sin x)\,, \qquad c_i \in \mathbb{R}\,. \tag{3.22}$$

It seems that the method of undetermined coefficients cannot be applied to the equation (3.21), since the inhomogeneous term is not of the form (3.15). However, since the right-hand side of (3.21) is the sum of the terms

$$b_1(x) = x\,, \qquad b_2(x) = x\mathrm{e}^{x}\cos x\,,$$

the sum of the corresponding particular solutions $u_i(x)$ of the equations

$$u^{(4)} + 4u = b_i(x)\,, \qquad i = 1,2\,, \tag{3.23}$$

is (by linearity) a particular solution of (3.21). The inhomogeneous term of the first of these equations is clearly of the form (3.15), with $q(x) = x$ and $\mu = 0$. Since 0 is not is not a root of the characteristic polynomial, we look for a particular solution of the form $u_1(x) = a + bx$. Substituting it into the corresponding complete equation (3.23), we readily obtain

$$u_1(x) = \frac{x}{4}\,.$$

Turning to the second equation in (3.23), since $b_2(x) = \operatorname{Re}\left(x\mathrm{e}^{(1+\mathrm{i})x}\right)$ we can look for a particular solution of the form $u_2(x) = \operatorname{Re} u(x)$, where $u(x)$ is any solution of the equation

$$u^{(4)} + 4u = x\mathrm{e}^{(1+\mathrm{i})x}\,. \tag{3.24}$$

The right-hand side of the latter equation is again of the form (3.15), with $q(x) = x$ and $\mu = 1+\mathrm{i}$ a simple root of the characteristic polynomial, so we try a particular solution of the form

$$u(x) = x(a+bx)\mathrm{e}^{(1+\mathrm{i})x} \equiv f(x)\mathrm{e}^{\mu x}\,.$$

Substituting this expression into (3.24) and using the general Leibniz rule

$$(fg)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(k)}g^{(n-k)}$$

we obtain

$$\mu^4\mathrm{e}^{\mu x}f + 4\mu^3\mathrm{e}^{\mu x}f' + 6\mu^2\mathrm{e}^{\mu x}f'' + 4\mathrm{e}^{\mu x}f = x\mathrm{e}^{\mu x} \quad\Longleftrightarrow\quad 4\mu^3(a+2bx) + 6\mu^2\cdot 2b = x\,,$$

where we have taken into account that $\mu^4 = -4$. Thus

$$\begin{cases} 8\mu^3 b = 1 \\ \mu a + 3b = 0 \end{cases} \quad\Longrightarrow\quad \begin{cases} b = \dfrac{1}{8\mu^3} = \dfrac{\mu}{-32} = -\dfrac{1}{32}\,(1+\mathrm{i}) \\ a = -\dfrac{3b}{\mu} = \dfrac{3}{32}\,, \end{cases}$$

so that

$$u(x) = \frac{x}{32}\left[3-(1+\mathrm{i})x\right]\mathrm{e}^{(1+\mathrm{i})x}\,.$$

Taking the real part of this function we get

$$u_2(x) = \frac{x}{32}\,\mathrm{e}^x\Big[(3-x)\cos x + x\sin x\Big].$$

Therefore, the inhomogeneous equation (3.21) admits the particular solution

$$u_\mathrm{p}(x) = u_1(x) + u_2(x) = \frac{x}{4} + \frac{x}{32}\,\mathrm{e}^x\Big[(3-x)\cos x + x\sin x\Big].$$

The general solution of the equation (3.21) is the sum of the latter particular solution and the general solution (3.22) of the homogeneous equation.

- In general, the method of undetermined coefficients can be applied to the equation (3.1) when the inhomogeneous term is of the form

$$b(x) = \sum_{i=1}^{l} b_i(x)\,, \tag{3.25}$$

where

$$b_i(x) = \mathrm{e}^{\alpha_i x}\big[q_i(x)\cos(\beta_i x) + \tilde{q}_i(x)\sin(\beta_i x)\big]\,, \tag{3.26}$$

with $\alpha_i, \beta_i \in \mathbb{R}$ ($\beta_i \geqslant 0$), and $q_i, \tilde{q}_i$ are polynomials. Note that if $\beta_i = 0$ the function $b_i(x)$ is of the form (3.15) with $\mu = \alpha_i$. It may be verified that the equation (3.1) with the inhomogeneous term (3.25)-(3.26) possesses a particular solution of the form

$$u_\mathrm{p}(x) = \sum_{i=1}^{l} u_i(x)\,, \tag{3.27}$$

where

$$u_i(x) = x^{r_i}\mathrm{e}^{\alpha_i x}\big[Q_i(x)\cos(\beta_i x) + \widetilde{Q}_i(x)\sin(\beta_i x)\big]\,. \tag{3.28}$$

Here r_i is the multiplicity of $\mu_i = \alpha_i + \mathrm{i}\beta_i$ as a root of the characteristic polynomial of the homogeneous equation, and $Q_i, \widetilde{Q}_i$ are polynomials such that $\deg Q_i, \deg \widetilde{Q}_i \leqslant \max(\deg q_i, \deg \tilde{q}_i)$, with coefficients to be determined upon substitution of (3.27)-(3.28) into (3.1).

3.2 Systems with constant coefficients. Exponential of a matrix

In Section 2.4 we proved that if a fundamental matrix of the homogeneous system (2.4) is known, then it is possible to express the general solution of the inhomogeneous system (2.1) by quadratures (cf. eq. (2.26)). The problem is that in general it is not possible to determine such a fundamental matrix explicitly. In this section we will show that when the matrix $A(x)$ of the system (2.1) is *constant* it is possible in principle to construct a fundamental matrix of the corresponding homogeneous system

$$\boxed{y' = A\,y\,, \quad A \in M_n(\mathbb{R})\,.} \tag{3.29}$$

More explicitly, we shall prove that the canonical fundamental matrix at $x_0 = 0$ of the system (3.29) is given by the matrix exponential e^{xA}, whose definition shall be discussed next.

Let $E(x)$ denote the canonical fundamental matrix at $x_0 = 0$, which satisfies the matrix initial value problem

$$\begin{cases} E'(x) = AE(x)\,, \\ E(0) = \mathbb{1}\,. \end{cases} \tag{3.30}$$

Differentiating repeatedly the above equation it follows that

$$E^{(k)}(x) = A^k E(x)\,, \qquad k = 0, 1, \dots\,, \tag{3.31}$$

where by definition $B^0 \equiv \mathbb{1}$ for any matrix B, so that

$$E^{(k)}(0) = A^k\,, \qquad k = 0, 1, \dots\,. \tag{3.32}$$

It may be shown (see, e.g., [EDI2009]) that for any matrix A the Taylor series of $E(x)$ centered at the origin converges for all $x \in \mathbb{R}$, that is,

$$E(x) = \sum_{k=0}^{\infty} \frac{A^k}{k!} x^k\,, \qquad \forall x \in \mathbb{R}\,. \tag{3.33}$$

By analogy with the scalar case, the **exponential** of a matrix $B \in M_n(\mathbb{C})$ is defined as

$$\boxed{\mathrm{e}^B = \sum_{k=0}^{\infty} \frac{B^k}{k!}}\,. \tag{3.34}$$

Again, it can be shown (cf. [EDI2009]) that the previous series converges for any matrix B. From the latter definition and (3.33) it follows that

$$E(x) = \mathrm{e}^{xA}\,,$$

Hence, the solution of the initial value problem

$$\begin{cases} y' = Ay \\ y(0) = y_0 \end{cases}$$

is given by

$$\boxed{y(x) = \mathrm{e}^{xA} y_0}\,. \tag{3.35}$$

In this course we shall make use of the following properties of the matrix exponential:

$$\text{i)} \quad \mathrm{e}^{(x+t)A} = \mathrm{e}^{xA}\mathrm{e}^{tA} = \mathrm{e}^{tA}\mathrm{e}^{xA}\,, \qquad \forall x, t \in \mathbb{R}\,, \tag{3.36}$$

$$\text{ii)} \quad \left(\mathrm{e}^{xA}\right)^{-1} = \mathrm{e}^{-xA}\,, \qquad \forall x \in \mathbb{R}\,, \tag{3.37}$$

$$\text{iii)} \quad \mathrm{e}^{PBP^{-1}} = P\mathrm{e}^B P^{-1}\,, \tag{3.38}$$

where $A \in M_n(\mathbb{R})$ and $B, P \in M_n(\mathbb{C})$, with P invertible[1].

Proof. For any fixed $t \in \mathbb{R}$, the matrix $E(x+t)$ (regarded as a function of x) is a fundamental matrix of the system (3.29), since it is invertible everywhere (recall that $E(x)$ is invertible for all $x \in \mathbb{R}$, for it is a fundamental matrix) and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x} E(x+t) = E'(x+t) = AE(x+t)\,, \qquad \forall x \in \mathbb{R}\,.$$

On the other hand, $E(x)E(t)$ is also a fundamental matrix of (3.29), since $E(t)$ is constant invertible matrix. At $x = 0$, both $E(x+t)$ and $E(x)E(t)$ take the same value $E(t)$ (since $E(0) = \mathbb{1}$), so that $E(x+t) = E(x)E(t)$ for all $x, t \in \mathbb{R}$. This establishes the first property. Property ii) follows from property i) taking $t = -x$ and noting that $\mathrm{e}^{0\cdot A} = E(0) = \mathbb{1}$. Regarding the third property,

$$P\mathrm{e}^B P^{-1} = P\left(\sum_{k=0}^{\infty} \frac{B^k}{k!}\right)P^{-1} = \sum_{k=0}^{\infty} \frac{PB^kP^{-1}}{k!} = \sum_{k=0}^{\infty} \frac{(PBP^{-1})^k}{k!} = \mathrm{e}^{PBP^{-1}}\,.$$

□

[1] In fact, property i) is a consequence of the following more general property [EDI2009]: if any two matrices A y B commute (i.e., $[A, B] \equiv AB - BA = 0$), then $\mathrm{e}^{A+B} = \mathrm{e}^A\mathrm{e}^B = \mathrm{e}^B\mathrm{e}^A$.

From the identities (3.36)-(3.37) we deduce that $\mathrm{e}^{xA}(\mathrm{e}^{x_0A})^{-1} = \mathrm{e}^{(x-x_0)A}$ is the canonical fundamental matrix of (3.29) at x_0. Thus, the solution of the initial value problem

$$\begin{cases} y'(x) = Ay(x) \\ y(x_0) = y_0 \end{cases}$$

is given by

$$\boxed{y(x) = \mathrm{e}^{A(x-x_0)}y_0}\,.$$

On the other hand, from the identities (3.36)-(3.37) and the variation of constants formula (2.26) it follows that the general solution of the inhomogeneous system associated with (3.29)

$$y' = Ay + b(x)\,, \qquad \text{con } b : I \to \mathbb{R}^n \text{ continuous,}$$

is given by

$$y(x) = \mathrm{e}^{xA}c + \mathrm{e}^{xA}\int^x \mathrm{e}^{-sA}b(s)\,\mathrm{d}s = \mathrm{e}^{xA}c + \int^x \mathrm{e}^{(x-s)A}b(s)\,\mathrm{d}s\,, \qquad c \in \mathbb{R}^n\,. \tag{3.39}$$

If we impose the initial condition $y(x_0) = y_0$, the solution thus obtained reads (cf. eq. (2.27))

$$\boxed{y(x) = \mathrm{e}^{A(x-x_0)}y_0 + \int_{x_0}^x \mathrm{e}^{(x-s)A}b(s)\,\mathrm{d}s}\,. \tag{3.40}$$

Example 3.6. Let us determine the canonical fundamental matrix at the origin of the system $y' = Ay$, where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{3.41}$$

directly from the definition (3.34). Since the matrix (3.41) satisfies $A^2 = -\mathbb{1}$, its powers are given by

$$A^{2k} = (-1)^k\mathbb{1}\,, \quad A^{2k+1} = (-1)^kA\,, \qquad k = 0, 1, \ldots\,.$$

Splitting the series for the exponential into even and odd powers of A and using the above expressions we obtain

$$\begin{aligned} \mathrm{e}^{xA} &= \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}(-1)^k\mathbb{1} + \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}(-1)^kA \\ &= \cos x\,\mathbb{1} + \sin x\,A = \begin{pmatrix} \cos x & 0 & \sin x & 0 \\ 0 & \cos x & 0 & -\sin x \\ -\sin x & 0 & \cos x & 0 \\ 0 & \sin x & 0 & \cos x \end{pmatrix} \end{aligned}$$

- In most cases it is not convenient to use directly the definition (3.34) to compute the canonical fundamental matrix e^{xA} of the system (3.29), since it is not easy to obtain a general expression for the powers of the matrix A, let alone summing the resulting series for the exponential. There are many different practical methods for computing the matrix exponential (see, e.g., [EDI2009]), most of which presuppose the knowledge of concepts in linear algebra that may not have been explained in the first year of the Degree. In the next section we will discuss some practical methods for computing the matrix exponential which require only a basic knowledge of linear algebra. In any case, if a fundamental matrix $Y(x)$ of the system (3.29) is known (obtained by any method), then it is always possible to compute the matrix e^{xA} using the formula

$$\boxed{\mathrm{e}^{xA} = Y(x)Y(0)^{-1}}\,. \tag{3.42}$$

Example 3.7. Let us determine the canonical fundamental matrix at the origin of the system

$$\begin{cases} y_1' = y_1 \\ y_2' = y_1 + 2y_2 \\ y_3' = y_1 - y_3 \,. \end{cases} \tag{3.43}$$

In other words, we have to compute e^{xA}, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Let us first find a fundamental matrix of the system. To this purpose, note that the equation for y_1 is uncoupled from the remaining ones, and it can be readily solved as it is a homogeneous equation with constant coefficients (the characteristic polynomial is $p_1(\lambda) = \lambda - 1$):

$$y_1(x) = c_1 \mathrm{e}^x \,, \qquad c_1 \in \mathbb{R} \,.$$

Substituting this expression into the equations for y_2 and y_3 we get the following inhomogeneous equations with constant coefficients:

$$\begin{cases} y_2' = 2y_2 + c_1 \mathrm{e}^x \\ y_3' = -y_3 + c_1 \mathrm{e}^x \,. \end{cases}$$

The solution for y_2 is of the form

$$y_2(x) = c_2 \mathrm{e}^{2x} + y_{2,\mathrm{p}}(x) \,, \qquad c_2 \in \mathbb{R} \,.$$

with $y_{2,\mathrm{p}}(x) = \alpha \mathrm{e}^x$ for some constant α (according to the method undetermined coefficients). Substituting $y_{2,\mathrm{p}}(x)$ into its corresponding equation we easily obtain $\alpha = -c_1$. Thus

$$y_2(x) = c_2 \mathrm{e}^{2x} - c_1 \mathrm{e}^x \,.$$

Similarly,

$$y_3(x) = c_3 \mathrm{e}^{-x} + \frac{c_1}{2} \mathrm{e}^x \,, \qquad c_3 \in \mathbb{R} \,.$$

In summary, the general solution of the system (3.43) is given by

$$y(x) \equiv \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = c_1 \mathrm{e}^x \begin{pmatrix} 1 \\ -1 \\ \frac{1}{2} \end{pmatrix} + c_2 \mathrm{e}^{2x} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \mathrm{e}^{-x} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = Y(x) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

so that

$$Y(x) = \begin{pmatrix} \mathrm{e}^x & 0 & 0 \\ -\mathrm{e}^x & \mathrm{e}^{2x} & 0 \\ \frac{1}{2} \mathrm{e}^x & 0 & \mathrm{e}^{-x} \end{pmatrix}$$

is a fundamental matrix. The canonical fundamental matrix of the system (3.43) at the origin is thus

$$\mathrm{e}^{xA} = Y(x) Y(0)^{-1} = \begin{pmatrix} \mathrm{e}^x & 0 & 0 \\ -\mathrm{e}^x & \mathrm{e}^{2x} & 0 \\ \frac{1}{2} \mathrm{e}^x & 0 & \mathrm{e}^{-x} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^x & 0 & 0 \\ \mathrm{e}^{2x} - \mathrm{e}^x & \mathrm{e}^{2x} & 0 \\ \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}) & 0 & \mathrm{e}^{-x} \end{pmatrix}.$$

3.3 Practical methods for computing the matrix exponential

In this section we shall present some practical methods for the computation of the canonical fundamental matrix e^{xA} of the system $y' = Ay$, with $A \in M_n(\mathbb{R})$. We shall distinguish two different cases, depending on whether the matrix A is diagonalizable or not. Let us start with a brief reminder of some basic notions of linear algebra.

- We say that $\lambda \in \mathbb{C}$ is an **eigenvalue** of the matrix A if there exists a nonzero vector $v \in \mathbb{C}^n$ such that $Av = \lambda v$. In this case, we say that v is an **eigenvector** of A with eigenvalue λ. The matrix A is **diagonalizable** if there exists an invertible constant matrix[2] P such that $J \equiv P^{-1}AP$ is a diagonal matrix. The elements of the main diagonal of J are the eigenvalues of the matrix A, and the columns of P are a basis of $\mathbb{C}^n$ formed by eigenvectors of A.

- It is well known that the eigenvalues $\lambda_1, \dots, \lambda_m$ of the matrix A (where we are assuming that $\lambda_i \neq \lambda_j$ if $i \neq j$) are the roots of the **characteristic polynomial** of A, defined as

$$\boxed{p_A(\lambda) = \det(\lambda \mathbb{1} - A)}\,. \tag{3.44}$$

In other words, the characteristic polynomial factors as

$$\boxed{p_A(\lambda) = \prod_{i=1}^{m} (\lambda - \lambda_i)^{r_i}}\,. \tag{3.45}$$

The integer $r_i \geqslant 1$ is called the **algebraic multiplicity** of the eigenvalue λ_i. Since p_A is an n-th degree polynomial, from (3.45) it follows that

$$\sum_{i=1}^{m} r_i = n\,. \tag{3.46}$$

- An elementary result in linear algebra states that A is diagonalizable if and only if the algebraic multiplicity r_i of each eigenvalue λ_i coincides with its **geometric multiplicity** s_i, defined as[3]

$$s_i = \dim \ker(A - \lambda_i)\,. \tag{3.47}$$

Thus s_i is the maximum number of linearly independent eigenvectors corresponding to the eigenvalue λ_i. It is well known that the geometric multiplicity cannot exceed the algebraic one, i.e., $s_i \leqslant r_i$ for all i. Therefore, when all eigenvalues are simple (i.e., if $r_i = 1$ for all i) the matrix A is always diagonalizable.

- Another criterion for determining whether a matrix A is diagonalizable is based on the notions of minimal polynomial and index of an eigenvalue. Recall that the **minimal polynomial** $\phi_A(\lambda)$ of a matrix A is the monic polynomial[4] of lowest degree annihilating this matrix (i.e., $\phi_A(A) = 0$). The existence of the minimal polynomial is a consequence of the Cayley–Hamilton theorem, according to which $p_A(A) = 0$. It can be proved that if the characteristic polynomial is given by (3.45), then the minimal polynomial is of the form

$$\boxed{\phi_A(\lambda) = \prod_{i=1}^{m} (\lambda - \lambda_i)^{d_i}\,, \qquad 1 \leqslant d_i \leqslant r_i}\,.$$

[2] In general, complex.
[3] For simplicity's sake, from now on we shall write $A - \lambda$ instead of $A - \lambda\mathbb{1}$.
[4] A polynomial is *monic* if the coefficient of the highest degree term appearing in that polynomial is equal to 1. For instance, from (3.44) or (3.45) it follows that the characteristic polynomial of any matrix is monic.

Note, in particular, that the minimal polynomial always divides the characteristic polynomial. The multiplicity d_i of λ_i as a root of the minimal polynomial is known as the **index** of the eigenvalue λ_i. It can be shown (see, e.g., [EDI2009]) that

$$r_i = s_i \quad \Longleftrightarrow \quad d_i = 1\,. \tag{3.48}$$

Thus

$$A \text{ diagonalizable} \quad \Longleftrightarrow \quad d_i = 1\,,\ \forall i = 1,\dots,m\,. \tag{3.49}$$

3.3.1 A diagonalizable

In this section we shall see that when the matrix A is diagonalizable it is straightforward to compute the matrix exponential e^{xA}. Indeed, let $\{v^1,\dots,v^n\}$ be a basis of $\mathbb{C}^n$ formed by eigenvectors of A, and let $\mu_1,\dots,\mu_n$ be their corresponding eigenvalues:

$$Av^i = \mu_i v^i\,, \qquad i = 1,\dots,n. \tag{3.50}$$

(Note that in this notation the eigenvalues μ_i may not be distinct from each other.) Thus, if

$$J = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}, \qquad P = \begin{pmatrix} v^1 & \cdots & v^n \end{pmatrix},$$

then

$$A = PJP^{-1}\,.$$

From the equation (3.38) it immediately follows that

$$\boxed{e^{xA} = P\,e^{xJ}\,P^{-1}}, \tag{3.51}$$

where

$$e^{xJ} = \sum_{k=0}^{\infty} \frac{x^k}{k!} J^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} \begin{pmatrix} \mu_1^k & & \\ & \ddots & \\ & & \mu_n^k \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{x^k \mu_1^k}{k!} & & \\ & \ddots & \\ & & \sum_{k=0}^{\infty} \frac{x^k \mu_n^k}{k!} \end{pmatrix} = \begin{pmatrix} e^{\mu_1 x} & & \\ & \ddots & \\ & & e^{\mu_n x} \end{pmatrix}.$$

- Since e^{xA} is a fundamental matrix of the homogeneous system (3.29) and P is an invertible matrix, multiplying the equality (3.51) from the right by P it follows that

$$\boxed{P\,e^{xJ} = \begin{pmatrix} e^{\mu_1 x} v^1 & \cdots & e^{\mu_n x} v^n \end{pmatrix}} \tag{3.52}$$

  is also a fundamental matrix of this system (see the remarks on page 25). In fact, it is usually easier to determine Pe^{xJ} than e^{xA}, since for the former it is not necessary to compute P^{-1}. However (unlike e^{xA}, which is always real when A is real) the matrix Pe^{xJ} may be complex, so that it might be necessary to take linear combinations of its columns in order to obtain a real fundamental matrix.

- If the matrix A is diagonalizable, from the previous remark it follows that

$$\left\{e^{\mu_1 x} v^1, \dots, e^{\mu_n x} v^n\right\} \tag{3.53}$$

  is a fundamental system of solutions of the homogeneous system. If an eigenvalue μ_i is real we can always choose v^i to be real, so that the corresponding solution $e^{\mu_i x} v^i$ is also real. On the other hand, if $\mu_i = a + \mathrm{i}b \in \mathbb{C}\backslash\mathbb{R}$, there exists another eigenvalue $\mu_j = \overline{\mu_i} = a - \mathrm{i}b$ with the same algebraic multiplicity as μ_i (since the characteristic polynomial is real), and we can always choose $v^j = \overline{v^i}$. If $v^i = u + \mathrm{i}v$ (with $u, w \in \mathbb{R}^n$), we can substitute the two complex solutions associated with μ_i, μ_j

$$\mathrm{e}^{(a\pm\mathrm{i}b)x}(u \pm \mathrm{i}w)$$

by the real and imaginary parts of any of them:

$$\mathrm{e}^{ax}\big(u\cos(bx) - w\sin(bx)\big), \qquad \mathrm{e}^{ax}\big(u\sin(bx) + w\cos(bx)\big). \tag{3.54}$$

(Note that the functions (3.54) are still solutions of the system, on account of the general remarks on page 22.)

Example 3.8. Let us determine the solution of the inhomogeneous system

$$\begin{cases} y_1' = 5y_1 - 6y_2 + 3\mathrm{e}^{2x} \\ y_2' = 3y_1 - 4y_2 \end{cases}$$

satisfying the initial condition $y_1(0) = 2$, $y_2(0) = 1$. We shall first compute the matrix exponential e^{xA}, where

$$A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$$

is the matrix of the homogeneous system, and then determine the sought-for solution making use of the equation (3.40). The characteristic polynomial of A is

$$p_A(\lambda) = \begin{vmatrix} \lambda - 5 & 6 \\ -3 & \lambda + 4 \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)\,.$$

Since the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$ are simple, the matrix A is diagonalizable. Thus the homogeneous system possesses a fundamental system of solutions of the form $\mathrm{e}^{\lambda_i x}v^i$ $(i = 1, 2)$, where v^i is an eigenvector with eigenvalue λ_i. The eigenvectors can be easily computed solving the linear systems $(A - \lambda_i)v^i = 0$ $(i = 1, 2)$, with the result

$$\lambda_1 = 2:\ v^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}; \qquad \lambda_2 = -1:\ v^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From the equation (3.51) it follows that the canonical fundamental matrix at the origin is given by

$$\mathrm{e}^{xA} = P\begin{pmatrix} \mathrm{e}^{2x} & 0 \\ 0 & \mathrm{e}^{-x} \end{pmatrix}P^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{2x} & 0 \\ 0 & \mathrm{e}^{-x} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2\mathrm{e}^{2x} - \mathrm{e}^{-x} & 2\mathrm{e}^{-x} - 2\mathrm{e}^{2x} \\ \mathrm{e}^{2x} - \mathrm{e}^{-x} & 2\mathrm{e}^{-x} - \mathrm{e}^{2x} \end{pmatrix}.$$

Finally, we determine the solution of the homogeneous system satisfying the given initial condition using equation (3.40) with $x_0 = 0$, $y_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = v^1$, $b(s) = \begin{pmatrix} 3\mathrm{e}^{2s} \\ 0 \end{pmatrix}$:

$$\begin{aligned} y(x) &= \mathrm{e}^{xA}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \int_0^x \mathrm{e}^{(x-s)A}\begin{pmatrix} 3\mathrm{e}^{2s} \\ 0 \end{pmatrix}\mathrm{d}s = \mathrm{e}^{2x}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3\int_0^x \begin{pmatrix} 2\mathrm{e}^{2x} - \mathrm{e}^{3s-x} \\ \mathrm{e}^{2x} - \mathrm{e}^{3s-x} \end{pmatrix}\mathrm{d}s \\ &= \begin{pmatrix} 2\mathrm{e}^{2x} \\ \mathrm{e}^{2x} \end{pmatrix} + 3\begin{pmatrix} \big[2\mathrm{e}^{2x}s - \frac{1}{3}\mathrm{e}^{3s-x}\big]_0^x \\ \big[\mathrm{e}^{2x}s - \frac{1}{3}\mathrm{e}^{3s-x}\big]_0^x \end{pmatrix} = \begin{pmatrix} (6x+1)\,\mathrm{e}^{2x} + \mathrm{e}^{-x} \\ 3x\,\mathrm{e}^{2x} + \mathrm{e}^{-x} \end{pmatrix}. \end{aligned}$$

Notice that in the second equality we have taken into account the following general property: if v is an eigenvector of A with eigenvalue λ, then

$$\mathrm{e}^{xA}v = \left(\sum_{k=0}^{\infty}\frac{x^k}{k!}A^k\right)v = \sum_{k=0}^{\infty}\frac{x^k}{k!}A^k v = \sum_{k=0}^{\infty}\frac{x^k}{k!}\lambda^k v = \mathrm{e}^{\lambda x}v\,.$$

3.3.2 A non-diagonalizable

In this section we shall present a general method for the computation of the exponential e^{xA}, valid for any A matrix (whether or not it is diagonalizable). The method is based on the following result, which is a direct consequence of the existence of the minimal polynomial of the matrix A:

Lemma 3.9. *Let $A \in M_n(\mathbb{R})$, and let $\phi_A(\lambda)$ be its minimal polynomial. Then each matrix element $u(x)$ of e^{xA} is a solution of the linear equation with constant coefficients $\phi_A(D)u = 0$.*

Proof. From the identity

$$D^k e^{xA} = A^k e^{xA} \tag{3.55}$$

(cf. eq. (3.31)), it immediately follows that $P(D)\, e^{xA} = P(A)\, e^{xA}$ for any polynomial P. In particular,

$$\phi_A(D)\, e^{xA} = \phi_A(A)\, e^{xA} = 0$$

by the definition of the minimal polynomial. In other words, each entry of e^{xA} is a solution of the scalar equation $\phi_A(D)u = 0$. □

The obvious question that arises in view of the previous lemma is which concrete solution of the equation $\phi_A(D)u = 0$ appears in each matrix element of e^{xA}, a question that we shall answer next. Let d be the degree of the minimal polynomial, and let $\{\varphi_1, \dots, \varphi_d\}$ be a fundamental system of solutions of the equation $\phi_A(D)u = 0$, which will be assumed to be real. By virtue of Lemma 3.9, each matrix element $(e^{xA})_{ij}$ is a linear combination of $\{\varphi_1, \dots, \varphi_d\}$, that is, there exist d real constants $c_{ij}^1, \dots, c_{ij}^d$ such that

$$(e^{xA})_{ij} = c_{ij}^1\, \varphi_1(x) + c_{ij}^2\, \varphi_2(x) + \cdots + c_{ij}^d\, \varphi_d(x)\,. \tag{3.56}$$

Equivalently, if

$$C_k = \begin{pmatrix} c_{11}^k & c_{12}^k & \dots & c_{1n}^k \\ c_{21}^k & c_{22}^k & \dots & c_{2n}^k \\ \vdots & \vdots & & \vdots \\ c_{n1}^k & c_{n2}^k & \dots & c_{nn}^k \end{pmatrix}, \qquad k = 1, \dots, d\,,$$

we can express the equality (3.56) in matrix form as

$$e^{xA} = \varphi_1(x) C_1 + \varphi_2(x) C_2 + \cdots + \varphi_d(x) C_d = \begin{pmatrix} \varphi_1(x) & \varphi_2(x) & \dots & \varphi_d(x) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_d \end{pmatrix}. \tag{3.57}$$

Differentiating repeatedly this expression taking into account equation (3.55), and evaluating the resulting expression at $x = 0$ we obtain

$$\varphi_1^{(k)}(0) C_1 + \varphi_2^{(k)}(0) C_2 + \cdots + \varphi_d^{(k)}(0) C_d = A^k\,, \qquad k = 0, \dots, d-1\,. \tag{3.58}$$

These d equations are equivalent to the formal matrix equation

$$\begin{pmatrix} \varphi_1(0) & \varphi_2(0) & \dots & \varphi_d(0) \\ \varphi_1'(0) & \varphi_2'(0) & \dots & \varphi_d'(0) \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(d-1)}(0) & \varphi_2^{(d-1)}(0) & \dots & \varphi_d^{(d-1)}(0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_d \end{pmatrix} = \begin{pmatrix} \mathbb{1} \\ A \\ \vdots \\ A^{d-1} \end{pmatrix},$$

i.e.,

$$\Phi(0)\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_d \end{pmatrix} = \begin{pmatrix} \mathbb{1} \\ A \\ \vdots \\ A^{d-1} \end{pmatrix}$$

where $\Phi(0)$ is the Wronski matrix of the solutions $\varphi_1, \ldots, \varphi_d$ evaluated at $x = 0$. Since $\Phi(0)$ is invertible (why?), from the latter equation it follows that

$$\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_d \end{pmatrix} = \Phi(0)^{-1}\begin{pmatrix} \mathbb{1} \\ A \\ \vdots \\ A^{d-1} \end{pmatrix}.$$

Substituting this expression into (3.57) we obtain the following formula for computing e^{xA}:

$$\boxed{e^{xA} = \begin{pmatrix} \varphi_1(x) & \varphi_2(x) & \ldots & \varphi_d(x) \end{pmatrix} \Phi(0)^{-1}\begin{pmatrix} \mathbb{1} \\ A \\ \vdots \\ A^{d-1} \end{pmatrix}}. \tag{3.59}$$

- A variant of the method just explained consists in using the characteristic polynomial $p_A(\lambda)$ instead of the minimal polynomial $\phi_A(\lambda)$. In that case we would obtain a formula analogous to (3.59) substituting d by n, and $\Phi(0)^{-1}$ would be the inverse of the Wronski matrix evaluated at $x = 0$ of a fundamental system of solutions $\{\varphi_1, \ldots, \varphi_n\}$ of the scalar equation $p_A(D)u = 0$. The advantage of this variant is that it is not necessary to determine the minimal polynomial, but if $n > d$ it has the disadvantage of having to compute higher powers of the matrix A and inverting $\Phi(0)$, which in this case would be an $n \times n$ matrix.

Example 3.10. Find the solution of the initial value problem

$$\begin{cases} y' = Ay \\ y(0) = y_0\,, \end{cases} \qquad \text{with} \qquad A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix},\ y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

From the equation (3.35) it follows that the sought-for solution is

$$y(x) = e^{xA}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

that is, the first column of e^{xA}. The characteristic polynomial of A is

$$p_A(\lambda) = \begin{vmatrix} \lambda - 3 & 4 \\ -1 & \lambda + 1 \end{vmatrix} = (\lambda - 1)^2\,,$$

so that A only has the eigenvalue $\lambda_1 = 1$ with algebraic multiplicity 2. Since A is not diagonalizable (any matrix with just an eigenvalue is diagonalizable if and only if it is a multiple of the identity) the index of λ_1 is 2, and the minimal and characteristic polynomials coincide. A fundamental system of solutions of the equation

$$\phi_A(D)u = (D - 1)^2 u = 0$$

consists of the functions $\varphi_1 = e^x$ and $\varphi_2 = xe^x$, whose corresponding Wronski matrix is

$$\Phi(x) = \begin{pmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{pmatrix}.$$

Thus

$$\Phi(0)^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Using the formula (3.59), we have

$$e^{xA} = \begin{pmatrix} e^x & xe^x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{1} \\ A \end{pmatrix} = \begin{pmatrix} (1-x)\,e^x & xe^x \end{pmatrix} \begin{pmatrix} \mathbb{1} \\ A \end{pmatrix} = (1-x)\,e^x\,\mathbb{1} + xe^x A = \begin{pmatrix} (1+2x)\,e^x & \cdot \\ xe^x & \cdot \end{pmatrix},$$

so that the sought-for solution is

$$y(x) = \begin{pmatrix} (1+2x)\,e^x \\ xe^x \end{pmatrix}.$$

Example 3.11. Let us apply the formula (3.59) to determine again e^{xA}, where A is the matrix (3.41) of the Example 3.6. The characteristic polynomial of A is

$$p_A(\lambda) = \begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & 1 \\ 1 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda \end{vmatrix} = \lambda^2(\lambda^2+1) + \lambda^2 + 1 = (\lambda^2+1)^2,$$

and thus its eigenvalues are $\pm\mathrm{i}$, both with algebraic multiplicity 2. Since the matrix A satisfies $A^2 = -\mathbb{1}$ and the minimal polynomial divides the characteristic one, in this case $\phi_A(\lambda) = \lambda^2 + 1$. (Note that the indices of the eigenvalues $\pm\mathrm{i}$ are both 1, so A is diagonalizable). A fundamental system of real solutions of the equation

$$\phi_A(D)u = (D^2+1)u = u'' + u = 0$$

is made up by the functions $\varphi_1 = \cos x$, $\varphi_2 = \sin x$, with Wronski matrix

$$\Phi(x) = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}.$$

From equation (3.59) it then follows that

$$e^{xA} = \begin{pmatrix} \cos x & \sin x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{1} \\ A \end{pmatrix} = \begin{pmatrix} \cos x & \sin x \end{pmatrix} \begin{pmatrix} \mathbb{1} \\ A \end{pmatrix} = \cos x\,\mathbb{1} + \sin x\,A,$$

in agreement with the expression previously obtained.

Example 3.12. Let us compute the canonical fundamental matrix at $x = 0$ of the system

$$y' = Ay, \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \tag{3.60}$$

The characteristic polynomial of the matrix A reads

$$p_A(\lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 \\ -2 & \lambda-1 & 1 \\ 0 & 1 & \lambda-1 \end{vmatrix} = (\lambda-1)(\lambda^2-2\lambda)+2(2-\lambda) = (\lambda-2)(\lambda^2-\lambda-2) = (\lambda+1)(\lambda-2)^2.$$

The eigenvalues of A are thus $\lambda_1 = -1$, $\lambda_2 = 2$, with respective algebraic multiplicities $r_1 = 1$, $r_2 = 2$. Since

$$(A+1)(A-2) = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ -2 & \cdot & \cdot \end{pmatrix} \neq 0,$$

the index of the eigenvalue λ_2 is $n_2 = 2$, so that the minimal and characteristic polynomials coincide and A is not diagonalizable. (Alternatively, since the rank of

$$A - 2 = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}$$

is clearly 2, it follows that

$$s_2 = \dim\ker(A-2) = 3 - \operatorname{rank}(A-2) = 1 < r_2\,,$$

so that A is non-diagonalizable and $n_2 = 2$.) A fundamental system of solutions of the equation

$$\phi_A(D)u = (D+1)(D-2)^2 u = 0$$

is made up by the functions $\varphi_1 = \mathrm{e}^{-x}$, $\varphi_2 = \mathrm{e}^{2x}$, $\varphi_3 = x\mathrm{e}^{2x}$, with corresponding Wronski matrix

$$\Phi(x) = \begin{pmatrix} \mathrm{e}^{-x} & \mathrm{e}^{2x} & x\mathrm{e}^{2x} \\ -\mathrm{e}^{-x} & 2\mathrm{e}^{2x} & (1+2x)\mathrm{e}^{2x} \\ \mathrm{e}^{-x} & 4\mathrm{e}^{2x} & 4(1+x)\mathrm{e}^{2x} \end{pmatrix}.$$

Thus

$$\Phi(0)^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 4 & 4 \end{pmatrix}^{-1} = \frac{1}{9}\begin{pmatrix} 4 & -4 & 1 \\ 5 & 4 & -1 \\ -6 & -3 & 3 \end{pmatrix}.$$

From the formula (3.59) it follows that

$$\begin{aligned}
\mathrm{e}^{xA} &= \frac{1}{9}\begin{pmatrix} \mathrm{e}^{-x} & \mathrm{e}^{2x} & x\mathrm{e}^{2x} \end{pmatrix}\begin{pmatrix} 4 & -4 & 1 \\ 5 & 4 & -1 \\ -6 & -3 & 3 \end{pmatrix}\begin{pmatrix} \mathbb{1} \\ A \\ A^2 \end{pmatrix} \\
&= \frac{1}{9}\begin{pmatrix} 4\mathrm{e}^{-x} + 5\mathrm{e}^{2x} - 6x\mathrm{e}^{2x} & -4\mathrm{e}^{-x} + 4\mathrm{e}^{2x} - 3x\mathrm{e}^{2x} & \mathrm{e}^{-x} - \mathrm{e}^{2x} + 3x\mathrm{e}^{2x} \end{pmatrix}\begin{pmatrix} \mathbb{1} \\ A \\ A^2 \end{pmatrix} \\
&= \frac{1}{9}\begin{pmatrix} 3\mathrm{e}^{-x} + 6\mathrm{e}^{2x} & 3\mathrm{e}^{2x} - 3\mathrm{e}^{-x} & 3\mathrm{e}^{2x} - 3\mathrm{e}^{-x} \\ 2(3x+2)\mathrm{e}^{2x} - 4\mathrm{e}^{-x} & 4\mathrm{e}^{-x} + (3x+5)\mathrm{e}^{2x} & 4\mathrm{e}^{-x} + (3x-4)\mathrm{e}^{2x} \\ 2(1-3x)\mathrm{e}^{2x} - 2\mathrm{e}^{-x} & 2\mathrm{e}^{-x} - (3x+2)\mathrm{e}^{2x} & 2\mathrm{e}^{-x} + (7-3x)\mathrm{e}^{2x} \end{pmatrix},
\end{aligned}$$

where in the last equality we have taken into account that

$$A^2 = \begin{pmatrix} 3 & 1 & 1 \\ 4 & 4 & 0 \\ -2 & -2 & 2 \end{pmatrix}.$$

Chapter 4

Analytic functions

4.1 Algebraic properties of complex numbers

Definition 4.1. $\mathbb{C} = \{\mathbb{R}^2, +, \cdot\}$, with **sum** and **multiplication** defined as

$$\boxed{\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\\ (x_1, y_1) \cdot (x_2, y_2) &= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).\end{aligned}}$$

Justification:

- The sum and product of pairs of the form $(x, 0)$ coincides with those of the real numbers $x \in \mathbb{R}$

  $\Longrightarrow$ we can identify the complex number $(x, 0)$ with the real number $x \in \mathbb{R}$

  $\Longrightarrow$ we can identify $\mathbb{R}$ with the subset $\{(x, 0) : x \in \mathbb{R}\} \subset \mathbb{C}$ (*real axis*)

  Note that for all $\lambda \in \mathbb{R}$ we have $\lambda(x, y) = (\lambda, 0)(x, y) = (\lambda x, \lambda y)$. Thus the set of complex numbers can be regarded as the vector space $\mathbb{R}^2$ (with the usual operations of sum of vectors and scalar multiplication) with an additional operation of multiplication of complex numbers).

- $\mathrm{i} \equiv (0, 1) \Longrightarrow \mathrm{i}^2 = \mathrm{i} \cdot \mathrm{i} = (0, 1) \cdot (0, 1) = (-1, 0) \equiv -1$

- $(x, y) = (x, 0) + y(0, 1) \equiv x + \mathrm{i}y$

  $$\Longrightarrow \quad \boxed{(x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) = (x_1 x_2 - y_1 y_2) + \mathrm{i}(x_1 y_2 + x_2 y_1)}\,,$$

  which is the usual rule for multiplying the complex numbers $x_1 + \mathrm{i}y_1$ and $x_2 + \mathrm{i}y_2$.

- If $z = x + \mathrm{i}y$ $(x, y \in \mathbb{R})$, we define

  $$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y$$

  (*real* and *imaginary* parts of the complex number z)

- Since $\mathbb{C} = \mathbb{R}^2$ (as *sets*), **equality** in $\mathbb{C}$ is defined as

  $$z \equiv x + \mathrm{i}y = w \equiv u + \mathrm{i}v \Longleftrightarrow x = u, y = v\,.$$

  In particular,

  $$z = x + \mathrm{i}y = 0 \Longleftrightarrow x = y = 0.$$

Proposition 4.2. *$\mathbb{C}$ is a* **field***: for all $z, w, s \in \mathbb{C}$ we have*

$$\begin{aligned}
z + w &= w + z & z\,w &= w\,z\\
z + (w + s) &= (z + w) + s & z\,(w\,s) &= (z\,w)\,s\\
z + 0 &= z & 1\,z &= z\\
\exists -z \in \mathbb{C} \text{ t.q. } z + (-z) &= 0 & z \neq 0 \Longrightarrow \exists z^{-1} \in \mathbb{C} \text{ t.q. } z\,z^{-1} &= 1
\end{aligned}$$

$$z(w + s) = z\,w + z\,s.$$

Proof. Obviously, $z = x + \mathrm{i}y \implies -z = -x - \mathrm{i}y$. The existence of the inverse with respect to the product for all $z = x + \mathrm{i}y \neq 0$ is deduced from the following calculation:

$$\begin{aligned}
z^{-1} = u + \mathrm{i}v \implies z\,z^{-1} &= (x\,u - y\,v) + \mathrm{i}\,(x\,v + y\,u) = 1 \\
&\iff \begin{cases} x\,u - y\,v = 1 \\ y\,u + x\,v = 0 \end{cases} \\
&\iff u = \frac{x}{x^2+y^2}, \quad v = -\frac{y}{x^2+y^2} \quad \text{(note that } z \neq 0 \implies x^2 + y^2 \neq 0\text{)} \\
&\iff z^{-1} = \frac{x}{x^2+y^2} - \mathrm{i}\,\frac{y}{x^2+y^2}.
\end{aligned}$$

The remaining properties can be easily checked from the definition of the operations in $\mathbb{C}$. □

- As in every field, the inverses $-z$ and z^{-1} (if $z \neq 0$) of the number $z \in \mathbb{C}$ with respect to the sum and product are *unique*.

Notation: $\dfrac{z}{w} \equiv z\,w^{-1}, \quad z^n = \underbrace{z \cdot z \cdot \cdots \cdot z}_{n \text{ times}} \quad (n \in \mathbb{N}).$

- $\mathbb{C}$ is *not* an ordered field: if it were so, then

$$\mathrm{i}^2 = \mathrm{i} \cdot \mathrm{i} = -1 \geqslant 0.$$

4.1.1 **Square roots** (algebraic method)

If $z = x + \mathrm{i}y$, let us find all $w \equiv u + \mathrm{i}v \in \mathbb{C}$ such that $w^2 = z$:

$$\begin{aligned}
w^2 = z &\iff u^2 - v^2 + 2\mathrm{i}uv = x + \mathrm{i}y \\
&\iff \begin{cases} u^2 - v^2 = x \\ \quad 2\,u\,v = y \end{cases} \\
&\implies x^2 + y^2 = (u^2 + v^2)^2 \implies u^2 + v^2 = \sqrt{x^2 + y^2} \\
&\implies u^2 = \frac{1}{2}\Big(x + \sqrt{x^2+y^2}\Big), \quad v^2 = \frac{1}{2}\Big(-x + \sqrt{x^2+y^2}\Big)
\end{aligned}$$

Since (by the second equation) the sign of uv must coincide with that of y, it follows that

$$w = \begin{cases} \pm\left(\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}} + \mathrm{i}\,\operatorname{sgn} y\,\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}\right), & y \neq 0 \\ \pm\sqrt{x}, & y = 0,\ x \geqslant 0 \\ \pm\mathrm{i}\sqrt{-x}, & y = 0,\ x < 0. \end{cases}$$

The square roots of a complex number $z \neq 0$ are therefore *two* distinct complex numbers (of opposite signs). The square roots of z are *real* if and only if $z \in \mathbb{R}^+ \cup \{0\}$, and **pure imaginary** if and only if $z \in \mathbb{R}^-$.

Example: The square roots of $3 - 4\,\mathrm{i}$ are

$$\pm\left(\sqrt{\frac{8}{2}} - \mathrm{i}\sqrt{\frac{2}{2}}\right) = \pm(2 - \mathrm{i}).$$

- Any quadratic equation with *complex* coefficients can be solved using the usual formula:

$$\boxed{a\,z^2 + b\,z + c = 0 \iff z = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)}\,, \qquad a, b, c \in \mathbb{C}\,,\ a \neq 0,$$

where $\pm\sqrt{b^2 - 4ac}$ denotes the two square roots of the complex number $b^2 - 4ac$. Indeed, it suffices to complete the square

$$a\,z^2 + b\,z + c = a\left(z + \frac{b}{2a}\right)^2 - \frac{1}{4a}\,(b^2 - 4ac)$$

and apply the previous result on the existence of square roots in $\mathbb{C}$.

Example 4.3. The solutions of the equation $z^2 - 8\mathrm{i}z - (19 - 4\mathrm{i}) = 0$ are the complex numbers

$$4\mathrm{i} \pm \sqrt{-16 + 19 - 4\mathrm{i}} = 4\mathrm{i} \pm \sqrt{3 - 4\mathrm{i}} = 4\mathrm{i} \pm (2 - \mathrm{i}) = \begin{cases} 2 + 3\mathrm{i} \\ -2 + 5\mathrm{i}\,. \end{cases}$$

- Newton's *binomial theorem* is valid in the complex case:

$$\boxed{(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}}\,, \qquad a, b \in \mathbb{C},\ n \in \mathbb{N}\,.$$

Indeed, as in the real case, the proof of this identity uses only the field properties of the complex numbers.

4.1.2 Modulus and conjugation

Geometrically, complex numbers can be regarded as points on the *plane* by identifying the complex number $z = x + \mathrm{i}\,y$ with the point with coordinates (x, y). Hence the set $\mathbb{C}$ is often called the **complex plane**. When using this geometric representation of $\mathbb{C}$ it is also common to refer to the horizontal axis the as the *real axis*, and to the vertical one as the *imaginary axis*, cf. fig. 4.1.

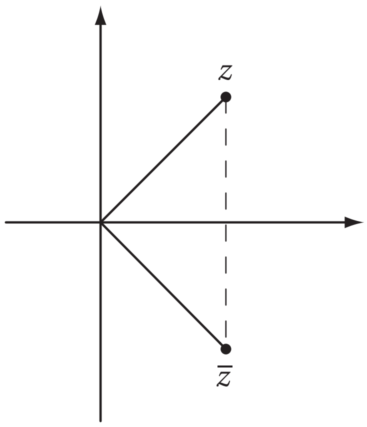

Figure 4.1: Complex plane.

- If $z = x + \mathrm{i}\,y \in \mathbb{C}$, its *modulus* and *complex conjugate* are respectively defined as

$$\begin{cases} |z| = \sqrt{x^2 + y^2} & \text{(distance of } z \text{ to the origin)} \\ \bar{z} = x - \mathrm{i}\,y & \text{(reflection of } z \text{ with respect to the real axis)} \end{cases}$$

$$\implies \boxed{\operatorname{Re} z = \frac{1}{2}(z+\overline{z}), \qquad \operatorname{Im} z = \frac{1}{2\mathrm{i}}(z-\overline{z})}.$$

The number $z \in \mathbb{C}$ is *real* if and only if $z = \overline{z}$, and *purely imaginary* if and only if $z = -\overline{z}$.

- *Properties:*

  i) $\overline{\overline{z}} = z$

  ii) $\overline{z+w} = \overline{z} + \overline{w}$

  iii) $\overline{z \cdot w} = \overline{z} \cdot \overline{w} \implies \overline{1/z} = 1/\overline{z}$ (if $z \neq 0$)

  iv) $|\overline{z}| = |z|$

  v) $z\overline{z} = |z|^2 \implies \begin{cases} z \neq 0 & \implies z^{-1} = \dfrac{\overline{z}}{|z|^2} \\ |z| = 1 & \iff \overline{z} = z^{-1} \end{cases}$

  vi) $|z \cdot w| = |z| \cdot |w|$ (square this equality to prove it) $\implies \left|z^{-1}\right| = |z|^{-1}$ (si $z \neq 0$)

  vii) $w \neq 0 \implies \overline{z/w} = \overline{z}/\overline{w}, \quad |z/w| = |z| \,/\, |w|$ (consequence of iii) and vi))

  viii) $|\operatorname{Re} z| \leqslant |z|, \quad |\operatorname{Im} z| \leqslant |z|$ (i.e., $-|z| \leqslant \operatorname{Re} z, \operatorname{Im} z \leqslant |z|$)

- *Triangle inequality:* $\boxed{|z+w| \leqslant |z| + |w|}$

  In fact:

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z+w}) = |z|^2 + |w|^2 + (z\overline{w} + \overline{z}w) = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w}) \\ &\leqslant |z|^2 + |w|^2 + 2\,|z\overline{w}| = |z|^2 + |w|^2 + 2\,|z|\,|w| = (|z| + |w|)^2. \end{aligned}$$

- Consequences:

  i) $||z| - |w|| \leqslant |z - w|$

  Indeed:

$$|z| = |(z-w) + w| \leqslant |z-w| + |w| \implies |z| - |w| \leqslant |z-w| \ ,$$

  and interchanging z and w one obtains the inequality $|w| - |z| \leqslant |z - w|$.

  ii) $|z| > |w| \implies \dfrac{1}{|z-w|} \leqslant \dfrac{1}{|z| - |w|}$

4.1.3 Argument

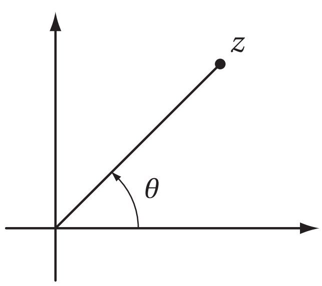

Figure 4.2: Definition of argument.

- If $0 \neq z \in \mathbb{C}$, there exists $\theta \in \mathbb{R}$ such that

$$z = |z|\,(\cos\theta + \mathrm{i}\sin\theta) \qquad \text{(cf. Fig. 4.2)}.$$

Geometrically, the number θ is the angle formed by the positive real axis with the vector z, and is thus defined up to an integer multiple of 2π. For instance,

$$z = \mathrm{i} \implies \theta \in \left\{\frac{\pi}{2}, \frac{\pi}{2} \pm 2\pi, \frac{\pi}{2} \pm 4\pi, \dots\right\} = \left\{\frac{\pi}{2} + 2k\pi : k \in \mathbb{Z}\right\}.$$

Definition 4.4. $\arg z$ (**argument** of z): *any* $\theta \in \mathbb{R}$ such that $z = |z|\,(\cos\theta + \mathrm{i}\sin\theta)$.

In other words, $\arg z$ is *any* of the oriented angles formed by the positive real axis with the vector z. Thus $\arg z$ takes *infinitely* many values, which differ from each other by an *integer multiple* of 2π. Note, in particular, that arg is ***not*** *a function*.

Example:

$$\arg \mathrm{i} \in \left\{\tfrac{\pi}{2} + 2k\pi : k \in \mathbb{Z}\right\}, \qquad \arg(-1-\mathrm{i}) \in \left\{\tfrac{5\pi}{4} + 2k\pi : k \in \mathbb{Z}\right\} = \left\{-\tfrac{3\pi}{4} + 2k\pi : k \in \mathbb{Z}\right\}.$$

- The argument θ can be made *unique* by imposing the additional condition that it belongs to a certain half-open interval I of length 2π (such as $[0, 2\pi)$, $(-\pi, \pi]$, $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, etc.). In so doing we say that we have chosen the **branch** I of the argument, denoted as $\arg_I$. In other words:

Definition 4.5. $\arg_I(z) \equiv$ *unique* value of $\arg z$ which belongs to the interval I.

Note, in particular, that

$$\arg_I : \mathbb{C} \setminus \{0\} \to I$$

is a function.

Example: $\arg_{[0,2\pi)}(-1-\mathrm{i}) = \frac{5\pi}{4}$, $\arg_{(-\pi,\pi]}(-1-\mathrm{i}) = -\frac{3\pi}{4}$.

- **Principal value** or **main branch** of the argument:

$$\boxed{\operatorname{Arg} \equiv \arg_{(-\pi,\pi]}}$$

Example:

z	1	$1+i$	i	$-1+\mathrm{i}$	-1	$-1-\mathrm{i}$	$-\mathrm{i}$	$1-\mathrm{i}$
$\operatorname{Arg} z$	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$-3\pi/4$	$-\pi/2$	$-\pi/4$

- Clearly, $\operatorname{Arg} : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ is a *discontinuous* function on $\mathbb{R}^- \cup \{0\}$. Likewise, $\arg_{[0,2\pi)}$ is discontinuous on $\mathbb{R}^+ \cup \{0\}$. In general, the branch $\arg_{[\theta_0,\theta_0+2\pi)}$ (or $\arg_{(\theta_0,\theta_0+2\pi]}$) is discontinuous on the closed half-line forming an angle θ_0 with the positive real axis.

- *Polar* or *trigonometric* form of a complex number:

$$z \neq 0 \implies z = r(\cos\theta + \mathrm{i}\sin\theta), \qquad r = |z|\,, \quad \theta = \arg z.$$

- $z, w \neq 0$; $z = w \iff \big(|z| = |w|\,, \quad \arg z = \arg w \mod 2\pi\big)$.

- Geometric interpretation of the product of complex numbers: if $z_k = r_k(\cos\theta_k + \mathrm{i}\sin\theta_k) \neq 0$ $(k = 1, 2)$ then

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)\, r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2) \\ &= r_1 r_2\,[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{i}(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)] \\ &= r_1 r_2\,[\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2)] \end{aligned}$$

From this calculation it follows that $|z_1 z_2| = |z_1|\,|z_2|$ (property vi) on page 52), together with

$$\boxed{\arg(z_1 z_2) = \arg z_1 + \arg z_2 \mod 2\pi}\,. \tag{4.1}$$

- Note that, in general, $\operatorname{Arg}(z_1 z_2) \neq \operatorname{Arg} z_1 + \operatorname{Arg} z_2$. For example,

$$\operatorname{Arg}(-\mathrm{i}) = -\frac{\pi}{2} \neq \operatorname{Arg}(-1) + \operatorname{Arg}\mathrm{i} = \frac{3\pi}{2}\,.$$

- Consequences: given nonzero z, w, then

$$\begin{aligned}
(zz^{-1} = 1 \Longrightarrow) \qquad & \arg(z^{-1}) = -\arg z \mod 2\pi \\
(z\bar{z} = |z|^2 > 0 \Longrightarrow) \qquad & \arg(\bar{z}) = -\arg z \mod 2\pi \\
\Longrightarrow \qquad & \arg(z/w) = \arg z - \arg w \mod 2\pi.
\end{aligned}$$

4.1.4 De Moivre's formula

- If $z = r(\cos\theta + \mathrm{i}\sin\theta)$, from (4.1) it may be proved by induction *de Moivre's formula*

$$\boxed{z^n = r^n\big[\cos(n\theta) + \mathrm{i}\sin(n\theta)\big], \quad n \in \mathbb{N}}\,.$$

- $z^{-1} = r^{-1}\big[\cos(-\theta) + \mathrm{i}\sin(-\theta)\big] \Longrightarrow$ the formula is valid for all $n \in \mathbb{Z}$.
- De Moivre's formula can be used to express $\cos(n\theta)$ or $\sin(n\theta)$ as a polynomial in $\cos\theta$ and $\sin\theta$. For instance:

$$\begin{aligned}
(\cos\theta + \mathrm{i}\sin\theta)^3 &= \cos(3\theta) + \mathrm{i}\sin(3\theta) \\
&= (\cos^3\theta - 3\cos\theta\sin^2\theta) + \mathrm{i}(3\cos^2\theta\sin\theta - \sin^3\theta)
\end{aligned}$$

$$\Longrightarrow \begin{cases} \cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta \\ \sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta. \end{cases}$$

4.1.5 n-th roots

If $z = r(\cos\theta + \mathrm{i}\sin\theta) \neq 0$ and $n \in \mathbb{N}$, the n-th roots of z are the solutions $w \in \mathbb{C}$ of the equation $w^n = z$:

$$\begin{aligned}
w \neq 0 \quad &\Longrightarrow \quad w = \rho(\cos\varphi + \mathrm{i}\sin\varphi) \\
w^n &= \rho^n\big[\cos(n\varphi) + \mathrm{i}\sin(n\varphi)\big] = r(\cos\theta + \mathrm{i}\sin\theta) \\
&\Longleftrightarrow \begin{cases} \rho^n = r \Longleftrightarrow \rho = \sqrt[n]{r} \equiv r^{1/n} \\ n\varphi = \theta + 2k\pi, \quad k \in \mathbb{Z} \end{cases}
\end{aligned}$$

$$\Longleftrightarrow \boxed{w = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + \mathrm{i}\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right], \quad k = 0, 1, \ldots, n-1} \tag{4.2}$$

(since k and $k + ln$, with $l \in \mathbb{Z}$, yield the same number w).

$\therefore$ *A nonzero complex number has n distinct n-th roots. In particular,* $\sqrt[n]{z}$ *is not a function.*

Example: the cube roots of i are the numbers

$$\begin{aligned}
w &= \cos\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right), \quad k = 0, 1, 2 \\
\Longleftrightarrow w &= \frac{1}{2}(\sqrt{3} + \mathrm{i}),\ \frac{1}{2}(-\sqrt{3} + \mathrm{i}),\ -\mathrm{i}.
\end{aligned}$$

- Geometrically, the n n-th roots of a number $z \neq 0$ are the vertices of a regular polygon with n sides inscribed in the circle of radius $\sqrt[n]{|z|}$ centered at the origin.
- In particular, the n n-th roots of unity ($z = 1$) are the numbers

$$\varepsilon_{n,k} = \cos\left(\frac{2k\pi}{n}\right) + \mathrm{i}\sin\left(\frac{2k\pi}{n}\right), \quad k = 0, 1, \dots, n-1.$$

- From de Moivre's formula it follows that $\varepsilon_{n,k} = (\varepsilon_n)^k$, where

$$\varepsilon_n \equiv \varepsilon_{n,1} = \cos\left(\frac{2\pi}{n}\right) + \mathrm{i}\sin\left(\frac{2\pi}{n}\right)$$

.

Example: the six sixth roots of unity are

$$\begin{aligned}\left[\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right]^k &= \frac{1}{2^k}(1+\mathrm{i}\sqrt{3})^k, \quad k = 0, 1, \dots, 5\\ &= 1,\ \frac{1}{2}(1+\mathrm{i}\sqrt{3}),\ \frac{1}{2}(-1+\mathrm{i}\sqrt{3}),\ -1,\ -\frac{1}{2}(1+\mathrm{i}\sqrt{3}),\ \frac{1}{2}(1-\mathrm{i}\sqrt{3}).\end{aligned}$$

Exercise. Let z be a nonzero complex number.

i) Show that its n-th roots are given by

$$\omega_0 \cdot (\varepsilon_n)^k, \quad k = 0, 1, \dots, n-1,$$

where ω_0 is any n-th root of z.

ii) Prove that the sum of all n-th roots of z is 0.

4.2 Elementary functions

4.2.1 Exponential function

If $t \in \mathbb{R}$,

$$\begin{aligned}\mathrm{e}^t &= \sum_{k=0}^{\infty} \frac{t^k}{k!}\\ \cos t &= \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!}\\ \sin t &= \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!}.\end{aligned}$$

If $z = x + \mathrm{i}y \in \mathbb{C}$ (with $x, y \in \mathbb{R}$), the identity $\mathrm{e}^{t_1+t_2} = \mathrm{e}^{t_1}\mathrm{e}^{t_2}$ suggests defining $\mathrm{e}^z = \mathrm{e}^x\mathrm{e}^{\mathrm{i}y}$. On the other hand, proceeding formally we obtain

$$\begin{aligned}\mathrm{e}^{\mathrm{i}y} &= \sum_{n=0}^{\infty} \mathrm{i}^n \frac{y^n}{n!} = \sum_{k=0}^{\infty} \mathrm{i}^{2k} \frac{y^{2k}}{(2k)!} + \mathrm{i}\sum_{k=0}^{\infty} \mathrm{i}^{2k} \frac{y^{2k+1}}{(2k+1)!}\\ &= \cos y + \mathrm{i}\sin y \quad (\text{ya que } \mathrm{i}^{2k} = (\mathrm{i}^2)^k = (-1)^k).\end{aligned}$$

Definition 4.6. Given $z = x + \mathrm{i}y \in \mathbb{C}$ (with $x, y \in \mathbb{R}$), we define

$$\boxed{\mathrm{e}^z = \mathrm{e}^x(\cos y + \mathrm{i}\sin y)}\,.$$

Remark: If $z \in \mathbb{R}$, the complex exponential obviously reduces to the real one.

Particular cases:
$$\boxed{\mathrm{e}^0 = 1, \quad \mathrm{e}^{\pi\mathrm{i}/2} = \mathrm{i}, \quad \mathrm{e}^{\pi\mathrm{i}} = -1, \quad \mathrm{e}^{3\pi\mathrm{i}/2} = -\mathrm{i}, \quad \mathrm{e}^{2\pi\mathrm{i}} = 1}\,.$$

Properties: For all $z, w \in \mathbb{C}$ we have

i) $|\mathrm{e}^z| = \mathrm{e}^{\operatorname{Re} z}, \quad \arg(\mathrm{e}^z) = \operatorname{Im} z \mod 2\pi, \quad \overline{\mathrm{e}^z} = \mathrm{e}^{\overline{z}}$.

ii) $\mathrm{e}^{z+w} = \mathrm{e}^z \mathrm{e}^w$.

iii) $\mathrm{e}^z \neq 0$, for all $z \in \mathbb{C}$.

iv) $\mathrm{e}^z = 1 \iff z = 2k\pi\mathrm{i}$, with $k \in \mathbb{Z}$.

v) e^z is a periodic function with periods $2k\pi\mathrm{i}$, where $k \in \mathbb{Z}$.

Proof:

i) Immediate consequence of the definition.

ii) If $z = x + \mathrm{i}y$, $w = u + \mathrm{i}v$, from the previous property and equation (4.1) it follows that
$$\mathrm{e}^z \mathrm{e}^w = \mathrm{e}^x \mathrm{e}^u \left[\cos(y+v) + \mathrm{i}\sin(y+v)\right] = \mathrm{e}^{x+u}\left[\cos(y+v) + \mathrm{i}\sin(y+v)\right] = \mathrm{e}^{z+w}.$$

iii) $\mathrm{e}^z \mathrm{e}^{-z} = \mathrm{e}^0 = 1 \implies (\mathrm{e}^z)^{-1} = \mathrm{e}^{-z}$.

iv) $\mathrm{e}^z = \mathrm{e}^x(\cos y + \mathrm{i}\sin y) = 1 \iff \mathrm{e}^x = 1, \; y = 0 \mod 2\pi \iff x = 0, \; y = 2k\pi \; (k \in \mathbb{Z})$.

v) $\mathrm{e}^z = \mathrm{e}^{z+w} \iff \mathrm{e}^w = 1 \iff w = 2k\pi\mathrm{i} \; (k \in \mathbb{Z})$.

- $z = |z|\,\mathrm{e}^{\mathrm{i}\arg z}$.
- From the definition of the complex exponential and the formula (4.2) it follows that the n-th roots of a nonzero number z are given by
$$\boxed{\sqrt[n]{|z|}\,\mathrm{e}^{\frac{\mathrm{i}}{n}(\arg z + 2k\pi)}, \qquad k = 0, 1, \ldots, n-1.} \tag{4.3}$$

4.2.2 Trigonometric and hyperbolic functions

If y is real then
$$\mathrm{e}^{\mathrm{i}y} = \cos y + \mathrm{i}\sin y, \;\; \mathrm{e}^{-\mathrm{i}y} = \cos y - \mathrm{i}\sin y \implies \cos y = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}y} + \mathrm{e}^{-\mathrm{i}y}\right), \; \sin y = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}y} - \mathrm{e}^{-\mathrm{i}y}\right).$$

Definition 4.7. For all $z \in \mathbb{C}$ we define
$$\boxed{\cos z = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}\right), \quad \sin z = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}\right)}\,.$$

Note that for real z, both $\cos z$ and $\sin z$ reduce to the corresponding real-valued functions.

Properties: for all $z, w \in \mathbb{C}$ we have

i) $\cos(-z) = \cos(z), \quad \sin(-z) = -\sin z$.

ii) $\overline{\cos z} = \cos\overline{z}, \quad \overline{\sin z} = \sin\overline{z}$.

iii) $\cos(z+w) = \cos z \cos w - \sin z \sin w\,, \quad \sin(z+w) = \sin z \cos w + \cos z \sin w.$

iv) $\cos z = \sin\left(\frac{\pi}{2} \pm z\right).$

v) $\cos^2 z + \sin^2 z = 1.$

vi) $\sin z = 0 \iff z = k\pi \ (k \in \mathbb{Z}),\ \cos z = 0 \iff z = \frac{\pi}{2} + k\pi \ (k \in \mathbb{Z}).$

vii) $\cos z$ y $\sin z$ are periodic functions with periods $2k\pi$, where $k \in \mathbb{Z}$.

Proof:

i) Immediate.

ii) Consequence of $\overline{e^w} = e^{\overline{w}}$.

iii) For instance,

$$\begin{aligned}\cos z \cos w - \sin z \sin w &= \frac{1}{4}\left(e^{iz} + e^{-iz}\right)\left(e^{iw} + e^{-iw}\right) + \frac{1}{4}\left(e^{iz} - e^{-iz}\right)\left(e^{iw} - e^{-iw}\right)\\ &= \frac{1}{2}(e^{iz}e^{iw} + e^{-iz}e^{-iw}) = \cos(z+w).\end{aligned}$$

iv) Particular case of iii).

v) Take $w = -z$ in the formula for $\cos(z+w)$.

vi) $\sin z = 0 \iff e^{iz} - e^{-iz} = 0 \iff e^{2iz} = 1 \iff 2iz = 2k\pi i \ (k \in \mathbb{Z}) \iff z = k\pi \ (k \in \mathbb{Z}).$ From item iv) it follows the corresponding formula for the zeros of cos.

vii) By item iv), it suffices to prove the statement for the sine function. From the identity

$$\sin(z+w) - \sin z \equiv \sin\left(z + \frac{w}{2} + \frac{w}{2}\right) - \sin\left(z + \frac{w}{2} - \frac{w}{2}\right) = 2\sin\left(\frac{w}{2}\right)\cos\left(z + \frac{w}{2}\right)$$

it follows that $\sin(z+w) - \sin z = 0$ for *all* z if and only if $\sin(w/2) = 0$ (take $z = -w/2$). By the previous item, this condition is fulfilled if and only if w is an integer multiple of 2π.

As in the real case, the remaining trigonometric functions are defined in terms of sin and cos:

$$\begin{aligned}\tan z &= \frac{\sin z}{\cos z}\,, \quad \sec z = \frac{1}{\cos z} \qquad (z \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z});\\ \cot z &= \frac{\cos z}{\sin z} = \frac{1}{\tan z}\,, \quad \csc z = \frac{1}{\sin z} \qquad (z \neq k\pi,\ k \in \mathbb{Z})\,.\end{aligned}$$

Hyperbolic functions: for all $z \in \mathbb{C}$ we define

$$\boxed{\cosh z = \frac{1}{2}\left(e^z + e^{-z}\right), \quad \sinh z = \frac{1}{2}\left(e^z - e^{-z}\right)}\,.$$

$\implies$ $\boxed{\cosh z = \cos(iz), \quad \sinh z = -i\sin(iz)}$

- The properties of the hyperbolic functions can be deduced from the latter equalities. For instance:

$$\cosh^2 z - \sinh^2 z = \cos^2(iz) + \sin^2(iz) = 1\,.$$

- The remaining hyperbolic functions are defined just as in the trigonometric case. For instance,

$$\tanh z \equiv \frac{\sinh z}{\cosh z} = -i\tan(iz) \quad (z \neq \frac{\pi i}{2} + k\pi i,\ k \in \mathbb{Z}), \quad \text{etc.}$$

- $\sin z = \sin(x + \mathrm{i}y) = \sin x \cos(\mathrm{i}y) + \cos x \sin(\mathrm{i}y) = \sin x \cosh y + \mathrm{i} \cos x \sinh y$.

  In particular, notice that $\sin z$ is real if z is real, or if $z = \frac{\pi}{2} + \mathrm{i}y + k\pi$ with arbitrary $y \in \mathbb{R}$ and $k \in \mathbb{Z}$. Likewise, $\cos z$ is real if $z \in \mathbb{R}$, or if $z = \mathrm{i}y + k\pi$ with arbitrary $y \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Exercise. If $z = x + \mathrm{i}y$ (with $x, y \in \mathbb{R}$), show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y\,, \qquad |\cos z|^2 = \cos^2 x + \sinh^2 y\,.$$

Deduce the inequalities $|\sinh y| \leqslant |\sin z| \leqslant \cosh y$ and $|\sinh y| \leqslant |\cos z| \leqslant \cosh y$. In particular, note that sin and cos *are not bounded* on $\mathbb{C}$.

4.2.3 Logarithms

- In the real case, $\exp : \mathbb{R} \to \mathbb{R}^+$ (where $\exp(t) \equiv \mathrm{e}^t$) is a bijection. Its inverse is the function $\log : \mathbb{R}^+ \to \mathbb{R}$. By definition

$$\log x = y \iff x = \mathrm{e}^y \quad (\Longrightarrow x > 0).$$

- In the complex case, exp is *not* invertible for it is not injective (since it is periodic). By definition, the logarithms of $z \in \mathbb{C}$ are *all* complex numbers w such that $\mathrm{e}^w = z$. We then have:

$$\begin{aligned}
\mathrm{e}^w = z \quad &\Longrightarrow \quad z \neq 0;\\
w = u + \mathrm{i}v \quad &\Longrightarrow \quad \mathrm{e}^u(\cos v + \mathrm{i}\sin v) = z \neq 0\\
&\iff \begin{cases} \mathrm{e}^u = |z| \iff u = \log|z| \\ \ v = \arg z \mod 2\pi \end{cases}\\
&\iff w = \log|z| + \mathrm{i}\arg z \mod 2\pi\mathrm{i}.
\end{aligned}$$

  If $z \neq 0$, the equation $\mathrm{e}^w = z$ thus have *infinitely many* solutions, differing from each other by an integer multiple of $2\pi\mathrm{i}$. These solutions w are called the **logarithms** of $z \neq 0$. In other words,

$$\boxed{z \neq 0 \quad \Longrightarrow \quad \log z = \log|z| + \mathrm{i}\arg z + 2k\pi\mathrm{i}, \quad k \in \mathbb{Z}}\,.$$

  Note, in particular, that log (just as arg) is *not* a function.

- Example:

$$\log(-2\mathrm{i}) = \log 2 - \frac{\pi\mathrm{i}}{2} + 2k\pi\mathrm{i}\,, \quad k \in \mathbb{Z}\,,$$

  where $\log 2 \in \mathbb{R}$ is the *real* logarithm of 2.

Notation: In general, if $x \in \mathbb{R}^+$ we shall denote by $\log x$ the real logarithm of x, whereas $\log_{\mathbb{C}} x = \log x + 2k\pi\mathrm{i}$ (with $k \in \mathbb{Z}$) will denote its complex logarithms.

Definition 4.8. Given a half-open interval I of length 2π, the **branch** I of the logarithm is defined as

$$\boxed{\log_I z = \log|z| + \mathrm{i}\arg_I z, \qquad \forall z \neq 0}\,.$$

For instance, $\log_{[0,2\pi)}(-2\mathrm{i}) = \log 2 + \frac{3\pi\mathrm{i}}{2}$.

- Note that $\log_I : \mathbb{C} \setminus \{0\} \to \{s \in \mathbb{C} : \operatorname{Im} s \in I\} \equiv \mathbb{R} \times I$ is a *function.*

- The **principal branch** of the logarithm is defined by

$$\boxed{\mathrm{Log} = \log_{(-\pi,\pi]}}\,.$$

Example: $\operatorname{Log}(-2\mathrm{i}) = \log 2 - \dfrac{\pi \mathrm{i}}{2}$, $\operatorname{Log}(-1) = \pi\mathrm{i}$, $\operatorname{Log}(-1-\mathrm{i}) = \dfrac{1}{2}\log 2 - \dfrac{3\pi\mathrm{i}}{4}$.

Properties:

i) For all $z \neq 0$, $\mathrm{e}^{\log_I z} = z$.

ii) $\log_I(\mathrm{e}^w) = w \mod 2\pi\mathrm{i}$. In particular, $\log_I(\mathrm{e}^w) = w \iff \operatorname{Im} w \in I$.

iii) $\log_I : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times I$ is a bijection, with inverse function given by $\exp : \mathbb{R} \times I \to \mathbb{C} \setminus \{0\}$, where $\exp(z) = \mathrm{e}^z$.

iv) $z, w \neq 0 \implies \log_I(z \cdot w) = \log_I z + \log_I w \mod 2\pi\mathrm{i}$.

Proof:

i) $z \neq 0 \Longrightarrow \mathrm{e}^{\log_I z} = \mathrm{e}^{\log|z| + \mathrm{i}\arg_I z} = \mathrm{e}^{\log|z|}\mathrm{e}^{\mathrm{i}\arg_I z} = |z|\,\mathrm{e}^{\mathrm{i}\arg_I z} = z$.

ii) If $w = u + \mathrm{i}v$ then

$$\log_I(\mathrm{e}^w) = \log(\mathrm{e}^u) + \mathrm{i}\arg_I(\mathrm{e}^w) = u + \mathrm{i}v \equiv w \mod 2\pi\mathrm{i}\,.$$

since $|\mathrm{e}^w| = \mathrm{e}^u$, $\arg_I(\mathrm{e}^w) = \operatorname{Im} w \mod 2\pi$. On the other hand, from the previous calculation it follows that

$$\log_I(\mathrm{e}^w) = w \iff \arg_I(\mathrm{e}^w) = v \iff v \equiv \operatorname{Im} w \in I\,.$$

iii) In order to establish that $\log_I$ is a bijection, one must show that for any w with $\operatorname{Im} w \in I$ there is a unique $z \in \mathbb{C} \setminus \{0\}$ such that $\log_I z = w$. But this clearly holds in view of the previous properties, with $z = \mathrm{e}^w \equiv \exp(w)$.

iv) The exponentials of both sides of the equality coincide, so that this property follows from ii). As an alternative proof observe that

$$\begin{aligned}
\log_I(zw) &= \log|zw| + \mathrm{i}\arg_I(zw) \\
&= \log|z| + \log|w| + \mathrm{i}(\arg_I z + \arg_I w) \mod 2\pi\mathrm{i} \\
&= (\log|z| + \mathrm{i}\arg_I z) + (\log|w| + \mathrm{i}\arg_I w) \mod 2\pi\mathrm{i} \\
&\equiv \log_I z + \log_I w \mod 2\pi\mathrm{i}.
\end{aligned}$$

Note: In general, $\operatorname{Log}(zw) \neq \operatorname{Log} z + \operatorname{Log} w$. For example,

$$\operatorname{Log}(-\mathrm{i}) = -\frac{\pi\mathrm{i}}{2} \neq \operatorname{Log}(-1) + \operatorname{Log}\mathrm{i} = \pi\mathrm{i} + \frac{\pi\mathrm{i}}{2} = \frac{3\pi\mathrm{i}}{2}.$$

4.2.4 Complex powers

If $a, b \in \mathbb{C}$ and $a \neq 0, \mathrm{e}$, we define

$$\boxed{a^b = \mathrm{e}^{b\log a}, \quad \text{where} \quad \log a = \log_I a + 2k\pi\mathrm{i},\ \ k \in \mathbb{Z}}\,.$$

Therefore, in general a^b denotes a *collection* of complex numbers:

$$a^b = \mathrm{e}^{2kb\pi\mathrm{i}}\mathrm{e}^{b\log_I a}, \quad k \in \mathbb{Z}.$$

More precisely, it can be shown that:

i) $b \in \mathbb{Z} \implies a^b$ takes a *single* value:

$$\begin{cases} \underbrace{a \cdot a \cdot \cdots \cdot a}_{b \text{ times}} & \text{if } b > 0, \\ 1\,, & \text{if } b = 0 \\ \underbrace{a^{-1} \cdot a^{-1} \cdot \cdots \cdot a^{-1}}_{-b \text{ times}}\,, & \text{if } b < 0\,. \end{cases}$$

ii) If $b = p/q \in \mathbb{Q}$, with $p \in \mathbb{Z}$ and $1 < q \in \mathbb{N}$ coprime, then $a^b = a^{p/q}$ takes exactly q values (the q q-th roots of a^p).

iii) If $b \in \mathbb{C} \setminus \mathbb{Q}$, a^b takes infinitely many values which differ from each other by a factor of the form $\mathrm{e}^{2kb\pi\mathrm{i}}$, with $k \in \mathbb{Z}$.

Example:

$$\begin{aligned}(-1+\mathrm{i})^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}[\mathrm{Log}(-1+\mathrm{i})+2k\pi\mathrm{i}]} &= \mathrm{e}^{-2k\pi}\mathrm{e}^{\mathrm{i}\left(\frac{1}{2}\log 2+\frac{3\pi\mathrm{i}}{4}\right)} \quad (k \in \mathbb{Z}) \\ &= \mathrm{e}^{\frac{5\pi}{4}+2n\pi}\mathrm{e}^{\frac{\mathrm{i}}{2}\log 2} \quad (n \in \mathbb{Z}).\end{aligned}$$

- If $a \neq 0, \mathrm{e}$, each branch of log defines a *function* $a_I^z \equiv \mathrm{e}^{z \log_I a}$.

Exercise. Given $a, b, c \in \mathbb{C}$ with $a \neq 0, \mathrm{e}$, discuss the validity of the equality

$$a^{b+c} = a^b a^c.$$

4.3 Cauchy–Riemann equations

4.3.1 Basic topological concepts

i) A **neighborhood** of $a \in \mathbb{C}$ is any open disc centered at a with radius $r > 0$, i.e.,

$$D(a;r) = \{z \in \mathbb{C} : |z-a| < r\}.$$

We shall denote by $\overline{D}(a;r) = \{z \in \mathbb{C} : |z-a| \leqslant r\}$ the corresponding closed disc.

ii) **Punctured neighborhood** of $a \in \mathbb{C} \equiv D(a;r) - \{a\} = \{z \in \mathbb{C} : 0 < |z-a| < r\}$.

iii) A set $A \subset \mathbb{C}$ is **open** if it contains a neighborhood of each of its points:

$$\forall a \in A,\ \exists r > 0 \quad \text{s.t.} \quad D(a;r) \subset A.$$

iv) A set $A \subset \mathbb{C}$ is **closed** $\Longleftrightarrow$ its complement $\mathbb{C} \setminus A$ is open.

v) A set $A \subset \mathbb{C}$ is **compact** $\Longleftrightarrow$ A is closed and **bounded** (A is bounded if $A \subset D(0;R)$ for some $R > 0$).

vi) An *open* set $A \subset \mathbb{C}$ is **connected** if for any two points $z, w \in A$ there is a *continuous* curve $\gamma : [0,1] \to A$ such that $\gamma(0) = z$, $\gamma(1) = w$. [*Note:* it can be shown that in the latter definition the term “continuous” may replaced by “differentiable” or even C^∞.]

vii) A **region** is a non-empty connected open subset of $\mathbb{C}$.

4.3.2 Limits

Notation:

$$\begin{cases} f : \mathbb{C} \to \mathbb{C} \\ z = x + \mathrm{i}y \mapsto f(z) \equiv u(x, y) + \mathrm{i}v(x, y). \end{cases}$$

Note: The notation $f : \mathbb{C} \to \mathbb{C}$ does *not* imply that f be defined on all of $\mathbb{C}$.

- $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ (the real and imaginary parts of f, respectively) are scalar *real-valued* functions

Definition 4.9. If $f : \mathbb{C} \to \mathbb{C}$ is defined on a punctured neighborhood of $a \in \mathbb{C}$ and $l \in \mathbb{C}$, we shall say that $\lim_{z\to a} f(z) = l$ if

$$\forall\varepsilon > 0\, \exists\delta > 0 \text{ s.t. } 0 < |z - a| < \delta \implies |f(z) - l| < \varepsilon.$$

- Since the modulus of the complex number $w = u + \mathrm{i}v$ is equal to the norm of the vector $(u, v) \in \mathbb{R}^2$, the latter definition of limit coincides with the usual one for a function $f : \mathbb{R}^2 \to \mathbb{R}^2$.

Properties:

i) If the limit $\lim_{z\to a} f(z)$ exists, it is necessarily unique.

ii) $\lim_{z\to a} f(z) = l \iff \lim_{(x,y)\to a} u(x, y) = \operatorname{Re} l$ and $\lim_{(x,y)\to a} v(x, y) = \operatorname{Im} l$.

iii) $\exists \lim_{z\to a} f(z), \lim_{z\to a} g(z) \implies \lim_{z\to a} [f(z) + g(z)] = \lim_{z\to a} f(z) + \lim_{z\to a} g(z)$.

iv) $\exists \lim_{z\to a} f(z), \lim_{z\to a} g(z) \implies \lim_{z\to a} [f(z)g(z)] = \lim_{z\to a} f(z) \cdot \lim_{z\to a} g(z)$.

v) $\exists \lim_{z\to a} g(z) \neq 0 \implies \lim_{z\to a} \dfrac{1}{g(z)} = \dfrac{1}{\lim_{z\to a} g(z)}$.

Proof:

i)–iii) are well-known properties of the limits of functions $\mathbb{R}^2 \to \mathbb{R}^2$

iv)–v) are demonstrated as in the real case, replacing the absolute value by the modulus.

4.3.3 Continuity

Definition 4.10. Let $f : \mathbb{C} \to \mathbb{C}$ be defined on a neighborhood of $a \in \mathbb{C}$. We shall say that f is **continuous** at a if

$$\lim_{z\to a} f(z) = f(a)\,.$$

We shall say that $f : \mathbb{C} \to \mathbb{C}$ is continuous on a set $A \subset \mathbb{C}$ if and only if f is continuous at every point of A.

Properties:

i) f and g continuous at $a \implies f + g$ y fg continuous at a.

ii) If, in addition, $g(a) \neq 0$, then f/g is continuous at a.

iii) $f : \mathbb{C} \to \mathbb{C}$ continuous at a and $h : \mathbb{C} \to \mathbb{C}$ continuous at $f(a) \implies h \circ f$ continuous at a.

Proof:

i)–ii) are an immediate consequence of the properties iii)–v) of limits, whereas iii) is proved as in the case of functions $\mathbb{R} \to \mathbb{R}$.

- A *polynomial* or a *rational function* is continuous at all points of its domain.

4.3.4 Differentiability

Definition 4.11.

- A function $f:\mathbb{C}\to\mathbb{C}$ defined on a *neighborhood* of $a\in\mathbb{C}$ is **differentiable** at a if there exists

$$\boxed{\lim_{z\to a}\frac{f(z)-f(a)}{z-a}\equiv f'(a)}\,.$$

  The number $f'(a)\in\mathbb{C}$ is called the **derivative** of f at a.

- $f:\mathbb{C}\to\mathbb{C}$ is **analytic** (or **holomorphic**) on an *open* set A if it differentiable at each point of A.

- f is analytic on an *arbitrary* set B if it analytic on an open set $A\supset B$, or, equivalently, if it is analytic on a neighborhood of each point of B.

In particular, f is analytic at a point $a\in\mathbb{C}$ if it is differentiable on a *neighborhood* of a. Therefore, f *analytic at* a is a stronger condition than f *differentiable at* a.

Proposition 4.12. *$f:\mathbb{C}\to\mathbb{C}$ differentiable at $a\in A$ $\implies$ f continuous at a.*

Proof. Indeed,

$$\lim_{z\to a}[f(z)-f(a)]=\lim_{z\to a}\left[\frac{f(z)-f(a)}{z-a}\cdot(z-a)\right]=\lim_{z\to a}\frac{f(z)-f(a)}{z-a}\cdot\lim_{z\to a}(z-a)=f'(a)\cdot 0=0.$$

□

Algebraic properties:
If $f:\mathbb{C}\to\mathbb{C}$ and $g:\mathbb{C}\to\mathbb{C}$ are differentiable at $z\in\mathbb{C}$, and $a,b\in\mathbb{C}$, then:

i) $af+bg$ is differentiable at z, with $(af+bg)'(z)=af'(z)+bg'(z)$ (*linearity*).

ii) fg is differentiable at z, with $(fg)'(z)=f'(z)g(z)+f(z)g'(z)$ (*Leibniz rule*).

iii) If $g(z)\neq 0$, then f/g is differentiable at z, with

$$(f/g)'(z)=\frac{g(z)f'(z)-f(z)g'(z)}{g(z)^2}.$$

- Polynomials and rational functions are differentiable at all points of their domain, and their derivatives are computed as in the real case.

4.3.5 Cauchy–Riemann equations

- If $a=a_1+\mathrm{i}a_2\in\mathbb{C}$, let $M_a:\mathbb{R}^2\to\mathbb{R}^2$ be the linear mapping

$$M_a\cdot z=a\,z\,,\qquad \forall z\in\mathbb{R}^2\equiv\mathbb{C}\,.$$

  Since

$$M_a\cdot(1,0)\equiv M_a\cdot 1=a\equiv(a_1,a_2)\,,\qquad M_a\cdot(0,1)\equiv M_a\cdot\mathrm{i}=\mathrm{i}a=-a_2+\mathrm{i}a_1\equiv(-a_2,a_1)\,,$$

  the matrix representing M_a in the canonical basis of $\mathbb{R}^2$ is given by $\begin{pmatrix}a_1 & -a_2\\ a_2 & a_1\end{pmatrix}$.

- Recall that a function $f : \mathbb{C} \to \mathbb{C}$ defined on a neighborhood of $z_0 \in \mathbb{C}$ is differentiable *in the real sense* at z_0 if there is a linear mapping $Df(z_0) : \mathbb{R}^2 \equiv \mathbb{C} \to \mathbb{R}^2 \equiv \mathbb{C}$ such that

$$\boxed{\lim_{z\to z_0} \frac{|f(z) - f(z_0) - Df(z_0)\cdot(z - z_0)|}{|z - z_0|} = 0}\,.$$

(Notice again that the *modulus* of $z = x + \mathrm{i}y \in \mathbb{C}$ is the *norm* of the corresponding vector $(x, y) \in \mathbb{R}^2$.) The linear mapping $Df(z_0)$ is called the *derivative in the real sense* of f at z_0. The matrix representing $Df(z_0)$ in the canonical basis of $\mathbb{R}^2$, called the *Jacobian matrix* of f at z_0, is given by

$$Jf(z_0) = \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix},$$

where we have used the customary notation $u_x \equiv \dfrac{\partial u}{\partial x}$, and similarly for u_y, v_x, v_y.

Theorem 4.13. *Let $f = u + \mathrm{i}v : \mathbb{C} \to \mathbb{C}$ be defined on a neighborhood of $z_0 = x_0 + \mathrm{i}y_0 \in \mathbb{C}$. Then f is differentiable at z_0* if and only if *the following* two *conditions hold:*

i) *f is differentiable* in the real sense *at* (x_0, y_0).

ii) *The* **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \qquad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

are satisfied.

Proof.
$\Longrightarrow$) f is differentiable in the real sense at $z_0 = (x_0, y_0)$ with derivative $Df(z_0) = M_{f'(z_0)}$, since

$$\lim_{z\to z_0} \frac{|f(z) - f(z_0) - f'(z_0)(z - z_0)|}{|z - z_0|} = \lim_{z\to z_0} \left| \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{z - z_0} \right|$$
$$= \lim_{z\to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| = 0\,,$$

for f is (by hypothesis) differentiable at z_0. Let us denote $\dfrac{\partial u}{\partial x}(x_0, y_0)$ as u_x, and similarly for the other partial derivatives of u and v at (x_0, y_0). Equating the matrix representing $Df(z_0)$ in the canonical basis of $\mathbb{R}^2$ —that is, the Jacobian matrix $Jf(z_0)$— with that of $M_{f'(z_0)}$, one obtains

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \operatorname{Re} f'(z_0) & -\operatorname{Im} f'(z_0) \\ \operatorname{Im} f'(z_0) & \operatorname{Re} f'(z_0) \end{pmatrix},$$

which yield the Cauchy–Riemann equations, together with the relations

$$f'(z_0) = u_x + \mathrm{i}v_x = \frac{1}{\mathrm{i}}(u_y + \mathrm{i}v_y).$$

$\Longleftarrow$) From the Cauchy–Riemann equations it follows that the Jacobian matrix of f at z_0 is of the form

$$\begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix},$$

which thus corresponds to the matrix representing the linear operator M_c, with $c \equiv u_x + \mathrm{i}v_x$. This implies that $Df(z_0) = M_c$, that is, $Df(z_0)\cdot(z - z_0) = c(z - z_0)$, so that

$$0 = \lim_{z\to z_0} \frac{|f(z) - f(z_0) - c(z - z_0)|}{|z - z_0|} = \lim_{z\to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} - c \right| \implies \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = c\,.$$

This shows that f is differentiable (in the complex sense) at z_0, and

$$f'(z_0) = c \equiv u_x + \mathrm{i}v_x = \frac{1}{\mathrm{i}}(u_y + \mathrm{i}v_y),$$

where the last equality follows again from the Cauchy–Riemann equations. □

• From the proof of the previous theorem it follows that if $f = u + \mathrm{i}v$ is differentiable at $z_0 = x_0 + \mathrm{i}y_0$ then

$$\boxed{\begin{aligned} f'(z_0) &= u_x(x_0, y_0) + \mathrm{i}v_x(x_0, y_0) \equiv \frac{\partial f}{\partial x}(z_0) \\ &= \frac{1}{\mathrm{i}}\big(u_y(x_0, y_0) + \mathrm{i}v_y(x_0, y_0)\big) \equiv \frac{1}{\mathrm{i}}\frac{\partial f}{\partial y}(z_0). \end{aligned}}$$

These equalities can also be easily deduced from the definition of derivative of f in Definition 4.11 (exercise). Note also that the Cauchy–Riemann equations are equivalent to the relation

$$\boxed{\frac{\partial f}{\partial x}(z_0) = \frac{1}{\mathrm{i}}\frac{\partial f}{\partial y}(z_0)}.$$

• Theorem 4.13 can also be rephrased in the following alternative form:

Theorem 4.14. *A function $f : \mathbb{C} \to \mathbb{C}$ defined on a neighborhood of $z_0 = x_0 + \mathrm{i}y_0 \in \mathbb{C}$ is differentiable at z_0 if and only if the following two conditions hold:*

i) f is differentiable in the real sense at (x_0, y_0)

ii) There is a complex number c such that $Df(x_0, y_0) = M_c$.

In addition, if the above conditions are satisfied then $f'(z_0) = c$.

An immediate consequence of Theorem 4.13 is the following

Proposition 4.15. *If $f : \mathbb{C} \to \mathbb{C}$ is analytic on a* region *A, and $f'(z) = 0$ for all $z \in A$, then f is constant on A.*

Proof. Indeed, f differentiable (in the complex sense) at $z \in A$ implies that f is differentiable in the real sense at this point, with $Df(z) = M_{f'(z)} = 0$. The statement then follows from the analogous result for functions $\mathbb{R}^n \to \mathbb{R}^m$. □

4.3.6 Derivatives of the elementary functions

Derivative of the exponential

$f(z) = \mathrm{e}^z \implies u(x, y) = \mathrm{e}^x \cos y,\ \ v(x, y) = \mathrm{e}^x \sin y \implies u$ and v are of class $C^\infty(\mathbb{R}^2) \implies f$ differentiable in the real sense on all of $\mathbb{R}^2$. Moreover,

$$u_x = \mathrm{e}^x \cos y = v_y, \quad u_y = -\mathrm{e}^x \sin y = -v_x.$$

Thus e^z is differentiable (in the complex sense) on $\mathbb{C}$, with

$$\boxed{(\mathrm{e}^z)' = u_x + \mathrm{i}v_x = \mathrm{e}^x \cos y + \mathrm{i}\mathrm{e}^x \sin y = \mathrm{e}^z}, \quad \forall z \in \mathbb{C}.$$

Chain rule:

Proposition 4.16. *If $f : \mathbb{C} \to \mathbb{C}$ is differentiable at z and $g : \mathbb{C} \to \mathbb{C}$ is differentiable at $f(z)$, then $g \circ f$ is differentiable at z, with*

$$(g \circ f)'(z) = g'\big(f(z)\big) \cdot f'(z). \tag{4.4}$$

Proof. Indeed, using the continuity of f at z and the fact that g is defined on a neighborhood of $f(z)$ (for it is differentiable at this point), it is easy to check that $g \circ f$ is defined on a neighborhood of z. On the other hand, by the previous theorem f y g are differentiable in the real sense at z and $f(z)$, respectively, with

$$Df(z) = M_{f'(z)}, \qquad Dg\big(f(z)\big) = M_{g'(f(z))}.$$

By the chain rule for functions $\mathbb{R}^n \to \mathbb{R}^m$, it follows that $g \circ f$ is differentiable in the real sense at z, and we have:

$$D(g \circ f)(z) = Dg\big(f(z)\big) \cdot Df(z) = M_{g'(f(z))} \cdot M_{f'(z)} = M_{g'(f(z))f'(z)},$$

which implies (4.4) by Theorem 4.14. ☐

Derivatives of the trigonometric and hyperbolic functions.

From the properties of the complex derivative (linearity and chain rule) and the derivative of the exponential function $f(z) = \mathrm{e}^z$, it follows that sin and cos are differentiable on $\mathbb{C}$, with

$$\boxed{(\sin z)' = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}z} + \mathrm{i}\mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}} = \cos z, \quad (\cos z)' = \frac{1}{2}(\mathrm{i}\mathrm{e}^{\mathrm{i}z} - \mathrm{i}\mathrm{e}^{-\mathrm{i}z}) = -\sin z}\,.$$

From the latter formulas we deduce that the remaining trigonometric functions are differentiable on their respective domains. For instance,

$$(\tan z)' = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \sec^2 z, \quad \forall z \neq \frac{\pi}{2} + k\pi \ \ (k \in \mathbb{Z}).$$

As in the real case, the derivative of the exponential together with the chain rule immediately yield the derivatives of the functions sinh y cosh:

$$\boxed{(\sinh z)' = \cosh z\,, \qquad (\cosh z)' = \sinh z}\,.$$

Again, from these formulas one can readily deduce the expressions for the derivatives of the remaining hyperbolic functions. For instance,

$$(\tanh z)' = \frac{\cosh^2 z - \sinh^2 z}{\cosh^2 z} = \operatorname{sech}^2 z, \quad \forall z \neq \frac{\pi\mathrm{i}}{2} + k\pi\mathrm{i} \ \ (k \in \mathbb{Z}).$$

Inverse function theorem:

Theorem 4.17. *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic on the open set A (with f' continuous on A). If $a \in A$ and $f'(a) \neq 0$, there are two open sets $U \ni a$ and $V \ni f(a)$ such that $U \subset A$, f'does not vanish on U and $f : U \to V$ is* bijective. *Moreover, $f^{-1} : V \to U$ is* analytic *on V, with*

$$(f^{-1})'(w) = \frac{1}{f'\big(f^{-1}(w)\big)}, \qquad \forall w \in V.$$

Remark. We shall see later on (Section 5.3.3) that if f is analytic on A then f' is automatically continuous on A.

Proof. f is differentiable in the real sense for all $z \in A$, and the determinant of its Jacobian matrix

$$Jf(z) = \begin{pmatrix} u_x(z) & -v_x(z) \\ v_x(z) & u_x(z) \end{pmatrix}$$

is $u_x^2(z) + v_x^2(z) = |f'(z)|^2$. In particular, $\det Df(a) = |f'(a)|^2 \neq 0$. By the inverse function theorem for functions $\mathbb{R}^2 \to \mathbb{R}^2$ (notice that the continuity of f' implies the continuity of the partial derivatives of u and v), there are two open sets $U \ni a$ and $V \ni f(a)$ such that $U \subset A$, $f : U \to V$ is a bijection, Df is invertible on U and $f^{-1} : V \to U$ is differentiable in the real sense on V, with

$$D(f^{-1})(w) = \left[Df\left(f^{-1}(w)\right)\right]^{-1}, \quad \forall w \in V.$$

Note that f' does not vanish on U, since $|f'(z)|^2 = \det Df(z)$. Calling $z = f^{-1}(w)$ and using Theorem 4.14, we have

$$D(f^{-1})(w) = [Df(z)]^{-1} = M^{-1}_{f'(z)} = M_{1/f'(z)}\,.$$

Again by Theorem 4.14, it follows that f^{-1} is differentiable in the complex sense at w, with derivative $1/f'(z)$. □

Derivative of $\log_I$.

- $\mathrm{Log} : \mathbb{C} \setminus \{0\} \to \{z \in \mathbb{C} : -\pi < \mathrm{Im}\, z \leqslant \pi\}$ is discontinuous on $\mathbb{R}^- \cup \{0\}$ (due to the discontinuity of Arg), and thus it is *not* differentiable on the latter set.

- However, Log is differentiable on the open set $B = \mathbb{C} \setminus (\mathbb{R}^- \cup \{0\})$. Indeed, Log is the *global* inverse of
$$\exp : A = \{z \in \mathbb{C} : -\pi < \mathrm{Im}\, z < \pi\} \to B,$$
and exp satisfies the conditions of the inverse function theorem at each point of A (since $\exp' = \exp$ does not vanish and is continuous on A)

- If $z \in A$ and $w = \mathrm{e}^z \in B$, there are two open sets $U \ni z$ and $V \ni w$ such that $\exp : U \subset A \to V$ is invertible on U, and
$$(\exp^{-1})'(w) = \frac{1}{\exp'(z)} = \frac{1}{\mathrm{e}^z} = \frac{1}{w}.$$
Since $U \subset A$ we have $\exp^{-1} = \mathrm{Log}$, and thus
$$\boxed{(\mathrm{Log}\, w)' = \frac{1}{w}, \quad \forall w \in \mathbb{C} \setminus (\mathbb{R}^- \cup \{0\})}\,.$$
In the same way, one can prove that the derivative of $\log_I$ (with $I = [y_0, y_0+2\pi)$ or $(y_0, y_0+2\pi]$) on the open set $\mathbb{C} \setminus (\{w : \arg w = y_0 \mod 2\pi\} \cup \{0\})$ is also given by $\log_I'(w) = 1/w$.

4.3.7 Harmonic functions

Definition 4.18. A function $u : \mathbb{R}^2 \to \mathbb{R}$ is **harmonic** on the open set $A \subset \mathbb{R}^2$ if $u \in C^2(A)$ and it satisfies Laplace's equation

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

at each point of A.

Proposition 4.19. *If $f : A \to \mathbb{C}$ is analytic on an open set A, then $u = \mathrm{Re}\, f$ and $v = \mathrm{Im}\, f$ are harmonic on A. (The functions u and v are then said to be* **harmonic conjugates** *on A).*

Proof. Indeed, we shall see later on (Section 5.3.3) that f analytic on $A \Longrightarrow u, v \in C^\infty(A)$. From the Cauchy–Riemann equations it then follows that

$$u_{xx} = \frac{\partial v_y}{\partial x} = v_{yx} = v_{xy} = -\frac{\partial u_y}{\partial y} = -u_{yy},$$

and similarly for v. (Note that $v_{xy} = v_{yx}$, since v is of class $C^2(A)$.) $\square$

Proposition 4.20. *If* $u : \mathbb{R}^2 \equiv \mathbb{C} \to \mathbb{R}$ *is harmonic on the open set* A, $z_0 \in A$ *and* $U \subset A$ *is a neighborhood of* z_0, *there is a function* $f : U \to \mathbb{C}$ *analytic on* U *such that* $\operatorname{Re} f = u$.

Proof. Indeed, if $z = x + \mathrm{i}\,y \in U$ then $v = \operatorname{Im} f$ should satisfy:

$$\begin{aligned}
&v_y = u_x \quad\Longrightarrow\quad v(x,y) = \int_{y_0}^{y} u_x(x,t)\,dt + h(x);\\
&v_x(x,y) = \int_{y_0}^{y} u_{xx}(x,t)\,dt + h'(x) = -\int_{y_0}^{y} u_{yy}(x,t)\,dt + h'(x)\\
&\quad = -u_y(x,y) + u_y(x,y_0) + h'(x) = -u_y(x,y) \Longleftrightarrow h'(x) = -u_y(x,y_0)\\
&\quad\Longrightarrow\quad h(x) = -\int_{x_0}^{x} u_y(t,y_0)\,dt + c \qquad (c \in \mathbb{R})\\
&\quad\Longrightarrow\quad \boxed{v(x,y) = \int_{y_0}^{y} u_x(x,t)\,dt - \int_{x_0}^{x} u_y(t,y_0)\,dt + c}\,, \qquad \forall (x,y) \in U\,.
\end{aligned} \tag{4.5}$$

If v is given by the above formula, the function $f = u + \mathrm{i}v$ satisfies by construction the Cauchy–Riemann equations in U, and is differentiable in the real sense in that set (since u, and thus v, are of class C^2 on U) $\Longrightarrow f$ is analytic on U. $\square$

- The previous proposition guarantees the existence of a *harmonic conjugate of* u in any open disk contained in A (although not necessarily in all A, as we shall see below).
- In a *region*, the harmonic conjugate v (if it exists) is determined up to a constant. In fact, if v_1 and v_2 are harmonic conjugates of a harmonic function u in a region A, then the functions $f_1 = u + \mathrm{i}v_1$ and $f_2 = u + \mathrm{i}v_2$ are analytic on A, so that $f = f_1 - f_2 = \mathrm{i}(v_1 - v_2)$ is also analytic on A. Since $\operatorname{Re} f = 0$ in A, the Cauchy–Riemann equations imply that the partial derivatives of $\operatorname{Im} f$ vanish in A. Since A is a region, $\operatorname{Im} f = v_1 - v_2$ must be constant in A.
- We may rewrite the formula (4.5) for the harmonic conjugate v as

$$v(z) = \int_{\gamma_0} (u_x\ \mathrm{d}y - u_y\ \mathrm{d}x) + c \equiv \int_{\gamma_0} (-u_y, u_x)\cdot \mathbf{dr} + c\,, \qquad \forall z \in U\,,$$

  where $\mathbf{dr} \equiv (\mathrm{d}x, \mathrm{d}y)$ and γ_0 is the broken line formed by the horizontal segment joining $z_0 \equiv x_0 + \mathrm{i}\,y_0$ with $x + \mathrm{i}\,y_0$ and the vertical segment joining the latter point with $z \equiv x + \mathrm{i}\,y$. Since the vector field $(-u_y, u_x)$ is *conservative* (since u is harmonic), this line integral is *independent of the path*, so we can also write

$$v(z) = \int_{\gamma} (u_x\ \mathrm{d}y - u_y\ \mathrm{d}x) + c\,, \qquad \forall z \in U\,,$$

  where γ is *any* (piecewise C^1) curve contained in U joining z_0 with z.

- The existence of the harmonic conjugate of a harmonic function on an open set A is not *globally* guaranteed in A. Consider, for instance, the function $u : A = \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ defined by $u(x, y) = \frac{1}{2}\log(x^2 + y^2)$. If U is any open disc contained in A, then the function $\log_I z = \log|z| + \mathrm{i}\, \arg_I z$ is analytic on U if the branch I is chosen so that the ray on which $\arg_I$ is discontinuous does not cut U. Thus $\operatorname{Re}\log_I = u$ is harmonic on U, and $v = \arg_I z + c$ (with $c \in \mathbb{R}$) is a harmonic conjugate of u in the disc U. This shows, in particular, that u is harmonic in all A, as can be easily checked by computing its partial derivatives.

  Let us now see that u *cannot* admit a harmonic conjugate defined on all A. Indeed, if there existed an analytic function f on A with $\operatorname{Re} f = u$, then f and (for instance) Log would differ by a (purely imaginary) constant in the region $B = \mathbb{C} \setminus (\mathbb{R}^- \cup \{0\}) \subset A$ (since Log is analytic in B, and $\operatorname{Re}\operatorname{Log} z = u(z)$). But this is impossible, due to the fact that for $x < 0$ one would have (since $f = \operatorname{Log} + c$ in B and f is continuous on A)

$$2\pi\mathrm{i} = \lim_{y\to 0+}\big[\operatorname{Log}(x+\mathrm{i}y) - \operatorname{Log}(x-\mathrm{i}y)\big] = \lim_{y\to 0+}\big[f(x+\mathrm{i}y) - f(x-\mathrm{i}y)\big] = f(x) - f(x) = 0\,.$$

Chapter 5

Cauchy's theorem

5.1 Contour integrals

- If $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ are integrable (e.g., continuous) on an interval $[a, b] \subset \mathbb{R}$ and $h = h_1 + \mathrm{i}h_2 : \mathbb{R} \to \mathbb{C}$, we define
$$\int_a^b h \equiv \int_a^b h(t)\ \mathrm{d}t = \int_a^b h_1(t)\ \mathrm{d}t + \mathrm{i}\int_a^b h_2(t)\ \mathrm{d}t \in \mathbb{C}.$$
Example: $\displaystyle\int_0^\pi \mathrm{e}^{it}\ \mathrm{d}t = \int_0^\pi \cos t\ \mathrm{d}t + \mathrm{i}\int_0^\pi \sin t\ \mathrm{d}t = 2\mathrm{i}.$

- A **continuous curve** or **contour** is an application $\gamma : [a, b] \to \mathbb{C}$ which is continuous on $[a, b]$ (i.e., $\operatorname{Re}\gamma$ and $\operatorname{Im}\gamma$ are both continuous on $[a, b]$).

- A continuous curve γ is **piecewise C^1** if there is a (finite) partition $a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$ of $[a, b]$ such that γ' exists and is continuous on each subinterval $[a_{i-1}, a_i]$ $(1 \leqslant i \leqslant n)$.
In other words, γ is continuous on $[a, b]$ and C^1 on $[a, b]\setminus\{a_0, \dots, a_n\}$, and the limits $\lim_{t\to a+}\gamma'(t)$, $\lim_{t\to b-}\gamma'(t)$ and $\lim_{t\to a_i\pm}\gamma'(t)$, $i = 1, \dots, n-1$, exist, although the left and right limits at each point a_i do not necessarily coincide.

- In what follows, we shall refer to continuous piecewise C^1 curves simply as **arcs**.

Definition 5.1. If $f : A \subset \mathbb{C} \to \mathbb{C}$ is continuous on the open set A and $\gamma : [a, b] \to \mathbb{C}$ is an arc such that $\gamma([a, b]) \subset A$, we define
$$\boxed{\int_\gamma f \equiv \int_\gamma f(z)\,\mathrm{d}z = \int_a^b f\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t \equiv \sum_{i=1}^n \int_{a_{i-1}}^{a_i} f\big(\gamma(t)\big)\gamma'(t)\,\mathrm{d}t \in \mathbb{C}}\,.$$
Note that $f\big(\gamma(t)\big)\gamma'(t)$ is continuous on each subinterval $[a_{i-1}, a_i]$, so that all integrals in the latter formula are well defined.

- If $f = u + \mathrm{i}v$ and $\gamma(t) = x(t) + \mathrm{i}y(t)$, then
$$\begin{aligned}\int_\gamma f &= \int_a^b \big[u\big(x(t), y(t)\big) + \mathrm{i}v\big(x(t), y(t)\big)\big]\big[x'(t) + \mathrm{i}y'(t)\big]\,\mathrm{d}t \\ &= \int_a^b \big[u\big(x(t), y(t)\big)\,x'(t) - v\big(x(t), y(t)\big)\,y'(t)\big]\,\mathrm{d}t \\ &\quad + \mathrm{i}\int_a^b \big[u\big(x(t), y(t)\big)\,y'(t) + v\big(x(t), y(t)\big)\,x'(t)\big]\,\mathrm{d}t \\ &= \boxed{\int_\gamma (u\ \mathrm{d}x - v\ \mathrm{d}y) + \mathrm{i}\int_\gamma (v\ \mathrm{d}x + u\ \mathrm{d}y)}\,.\end{aligned}$$

5.1.1 Properties of $\int_\gamma f$

Linearity. For all $\lambda, \mu \in \mathbb{C}$ we have

$$\boxed{\int_\gamma (\lambda f + \mu g) = \lambda \int_\gamma f + \mu \int_\gamma g}.$$

Chains. If $\gamma : [a,b] \to \mathbb{C}$ is an arc, the **opposite arc** $-\gamma : [a,b] \to \mathbb{C}$ is defined as

$$(-\gamma)(t) = \gamma(a+b-t), \qquad \forall t \in [a,b].$$

In other words, $-\gamma$ is just the arc γ traversed in the opposite sense. If $\gamma([a,b])$ is contained in the open set A and $f : A \to \mathbb{C}$ is continuous on A we have

$$\begin{aligned}\int_{-\gamma} f &= \int_a^b f\big(\gamma(a+b-t)\big)\big(-\gamma'(a+b-t)\big)\,\mathrm{d}t \overset{s=a+b-t}{=} \int_b^a f\big(\gamma(s)\big)\gamma'(s)\,\mathrm{d}s \\ &= -\int_a^b f\big(\gamma(s)\big)\gamma'(s)\,\mathrm{d}s = -\int_\gamma f. \end{aligned} \tag{5.1}$$

If $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [c,d] \to \mathbb{C}$ are two arcs satisfying $\gamma_1(b) = \gamma_2(c)$, we define their **sum** $\gamma_1 + \gamma_2 : [a, b+d-c] \to \mathbb{C}$ as

$$(\gamma_1+\gamma_2)(t) = \begin{cases} \gamma_1(t), & t \in [a,b] \\ \gamma_2(c-b+t), & t \in [b, b+d-c]. \end{cases}$$

If $\gamma_1([a,b]), \gamma_2([c,d]) \subset A$ and $f : A \to \mathbb{C}$ is continuous on the open set A, it is immediate to check that

$$\boxed{\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f}. \tag{5.2}$$

The general sum $\gamma_1 + \cdots + \gamma_n$ is defined similarly provided the final endpoint of the arc γ_i coincides with the initial one of the following arc γ_{i+1}, and we have again

$$\int_{\gamma_1+\cdots+\gamma_n} f = \sum_{i=1}^n \int_{\gamma_i} f.$$

Likewise, if the final endpoints of the arcs γ_1 and γ_2 coincide, we define $\gamma_1 - \gamma_2 = \gamma_1 + (-\gamma_2)$. From equations (5.1) and (5.2) it follows that

$$\int_{\gamma_1-\gamma_2} f \equiv \int_{\gamma_1+(-\gamma_2)} f = \int_{\gamma_1} f + \int_{-\gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f.$$

The above considerations can be summarized as

$$\boxed{\int_{\varepsilon_1\gamma_1+\cdots+\varepsilon_n\gamma_n} f = \sum_{i=1}^n \varepsilon_i \int_{\gamma_i} f},$$

where $\varepsilon_i = \pm 1$ for all i, and it is assumed that the final endpoint of the arc $\varepsilon_i\gamma_i$ coincides with the initial one of $\varepsilon_{i+1}\gamma_{i+1}$. The expression $\varepsilon_1\gamma_1 + \cdots + \varepsilon_n\gamma_n$ shall be referred to as a **chain**.

Invariance under reparametrization. Given an arc $\gamma : [a,b] \to \mathbb{C}$, a **reparametrization** thereof is a curve $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ of the form $\tilde{\gamma} = \gamma \circ \phi$, where $\phi : [\tilde{a}, \tilde{b}] \to \phi([\tilde{a}, \tilde{b}]) = [a,b]$ is a C^1 function with positive derivative on $(\tilde{a}, \tilde{b})$.

Note that since $\phi' > 0$ on $(\tilde{a}, \tilde{b})$, the change of parameter ϕ is an increasing function on $[\tilde{a}, \tilde{b}]$. Since ϕ is also surjective by hypothesis, it follows that $\phi(\tilde{a}) = a$, $\phi(\tilde{b}) = b$. Moreover, if γ is piecewise C^1 so is $\tilde{\gamma}$, and $\gamma([a,b]) = \tilde{\gamma}([\tilde{a}, \tilde{b}])$. Thus $\tilde{\gamma}$ is an arc with the same range as γ. In addition, since ϕ is an increasing function, the arcs γ and $\tilde{\gamma}$ have the same orientation.

Example: $\tilde{\gamma}(s) = \mathrm{e}^{\mathrm{i}s}$ $\left(s \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]\right)$ is a reparametrization of $\gamma(t) = -t + \mathrm{i}\sqrt{1-t^2}$ $\left(t \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right)$. Indeed, $\tilde{\gamma}(s) = \gamma(-\cos s)$, so that in this case $\phi(s) = -\cos s$ is of class C^1 and $\phi'(s) = \sin s > 0$ on $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$.

Proposition 5.2. *If $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ is a reparametrization of $\gamma : [a,b] \to \mathbb{C}$, $\gamma([a,b]) \subset A$, and $f : A \to \mathbb{C}$ is continuous on the open set A, then*

$$\int_{\tilde{\gamma}} f = \int_{\gamma} f.$$

Proof. Let $\tilde{\gamma} = \gamma \circ \phi$, with $\phi : [\tilde{a}, \tilde{b}] \to [a,b] \equiv [\phi(\tilde{a}), \phi(\tilde{b})]$. We then have:

$$\int_{\tilde{\gamma}} f = \int_{\tilde{a}}^{\tilde{b}} f\big(\tilde{\gamma}(s)\big)\tilde{\gamma}'(s)\ \mathrm{d}s = \int_{\tilde{a}}^{\tilde{b}} f\big(\gamma(\phi(s))\big)\gamma'\big(\phi(s)\big)\phi'(s)\ \mathrm{d}s \overset{t=\phi(s)}{=} \int_a^b f\big(\gamma(t)\big)\gamma'(t)\ \mathrm{d}t = \int_{\gamma} f.$$

□

5.1.2 Integral with respect to the arc length

If $f : A \to \mathbb{C}$ is continuous on the open set A and $\gamma : [a,b] \to \mathbb{C}$ is an arc with $\gamma([a,b]) \subset A$, we define

$$\boxed{\int_{\gamma} f(z)\ |\mathrm{d}z| = \int_a^b f\big((\gamma(t)\big)\left|\gamma'(t)\right|\ \mathrm{d}t}\,.$$

- If $\gamma(t) = x(t) + \mathrm{i}\,y(t)$ then $|\gamma'(t)|\ \mathrm{d}t = \sqrt{x'^2(t) + y'^2(t)}\ \ \mathrm{d}t$ is the element $\mathrm{d}s$ of arc length along the curve γ. If $f = u + \mathrm{i}v$, it follows that

$$\boxed{\int_{\gamma} f(z)\ |\mathrm{d}z| = \int_{\gamma} u\ \mathrm{d}s + \mathrm{i}\int_{\gamma} v\ \mathrm{d}s}\,.$$

- In particular,

$$\boxed{\int_{\gamma} |\mathrm{d}z| = \int_{\gamma} \mathrm{d}s = l(\gamma) \equiv \textit{length of}\ \gamma}\,.$$

Properties:

i) $\displaystyle\int_{\gamma} \big(\lambda f(z) + \mu g(z)\big)\ |\mathrm{d}z| = \lambda \int_{\gamma} f(z)\ |\mathrm{d}z| + \mu \int_{\gamma} g(z)\ |\mathrm{d}z|\,, \quad \forall \lambda, \mu \in \mathbb{C}.$

ii) $\displaystyle\int_{-\gamma} f(z)\ |\mathrm{d}z| = \int_{\gamma} f(z)\ |\mathrm{d}z|.$

iii) $\displaystyle\int_{\varepsilon_1\gamma_1 + \cdots + \varepsilon_n\gamma_n} f(z)\ |\mathrm{d}z| = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\ |\mathrm{d}z|.$

iv) If $\tilde{\gamma}$ is a reparametrization of γ, then $\displaystyle\int_{\tilde{\gamma}} f(z)\ |\mathrm{d}z| = \int_{\gamma} f(z)\ |\mathrm{d}z|.$

Fundamental inequality: $\boxed{\left|\int_\gamma f(z)\ \mathrm{d}z\right| \leqslant \int_\gamma |f(z)|\,|\mathrm{d}z|}$.

In particular, if $\max_{t\in[a,b]} \left|f\big(\gamma(t)\big)\right| = M$ then

$$\left|\int_\gamma f(z)\ \mathrm{d}z\right| \leqslant M\, l(\gamma).$$

Indeed, the second inequality is a consequence of the first one (by the properties of the integral of real-valued functions of a real variable). If $\int_\gamma f = 0$, the first inequality trivially holds. Otherwise, calling $\theta = \operatorname{Arg}(\int_\gamma f)$ we have:

$$\begin{aligned}\left|\int_\gamma f\right| &= \mathrm{e}^{-\mathrm{i}\theta}\int_\gamma f = \operatorname{Re}\left(\mathrm{e}^{-\mathrm{i}\theta}\int_\gamma f\right) = \int_a^b \operatorname{Re}\left[\mathrm{e}^{-\mathrm{i}\theta} f\big(\gamma(t)\big)\gamma'(t)\right]\mathrm{d}t\\ &\leqslant \int_a^b \left|\mathrm{e}^{-\mathrm{i}\theta} f\big(\gamma(t)\big)\gamma'(t)\right|\mathrm{d}t = \int_a^b \left|f\big(\gamma(t)\big)\right|\left|\gamma'(t)\right|\ \mathrm{d}t \equiv \int_\gamma |f(z)|\,|\mathrm{d}z|\,.\end{aligned}$$

5.1.3 Fundamental theorem of calculus. Path independence

Lemma 5.3. *If $\gamma : \mathbb{R} \to \mathbb{C}$ is differentiable at $t \in \mathbb{R}$ (that, if $\operatorname{Re}\gamma, \operatorname{Im}\gamma : \mathbb{R} \to \mathbb{R}$ are both differentiable at t) and $f : \mathbb{C} \to \mathbb{C}$ is differentiable at $\gamma(t)$, then $f \circ \gamma$ is differentiable at t, with derivative*

$$(f \circ \gamma)'(t) = f'\big(\gamma(t)\big)\,\gamma'(t)\,.$$

Proof. By Theorem 4.13, the function f is differentiable in the real sense at $\gamma(t)$, with $Df(\gamma(t)) = M_{f'(\gamma(t))}$. The chain rule for functions $\mathbb{R}^n \to \mathbb{R}^m$ implies that the composite function $f \circ \gamma$ is differentiable at t, with derivative

$$(f \circ \gamma)'(t) = Df\big(\gamma(t)\big)\,\gamma'(t) = M_{f'(\gamma(t))}\,\gamma'(t) \equiv f'\big(\gamma(t)\big)\,\gamma'(t)\,.$$

□

Fundamental theorem of calculus. *Let $F : A \to \mathbb{C}$ be analytic on the open set A (with F' continuous on A). If $\gamma : [a,b] \to \mathbb{C}$ is an arc such that $\gamma([a,b]) \subset A$, then*

$$\int_\gamma F' = F\big(\gamma(b)\big) - F\big(\gamma(a)\big).$$

In particular, if the arc γ is **closed** *(i.e., $\gamma(b) = \gamma(a)$) then*

$$\int_\gamma F' = 0.$$

Note: We shall see in Section 5.3.3 that if F is analytic on A then F' is automatically continuous on A.

Proof.

$$\int_\gamma F' = \int_a^b F'\big(\gamma(t)\big)\,\gamma'(t)\ \mathrm{d}t = \int_a^b (F\circ\gamma)'(t)\ \mathrm{d}t = F\big(\gamma(b)\big) - F\big(\gamma(a)\big),$$

by the fundamental theorem of Calculus for function $\mathbb{R} \to \mathbb{R}$ applied to the real and imaginary parts of $F \circ \gamma$. □

Path independence theorem. *If* $f : A \to \mathbb{C}$ *is continuous on a* region A*, the following statements are* equivalent*:*

i) $\int_\gamma f$ *is* **path independent***:* $\int_{\gamma_1} f = \int_{\gamma_2} f$ *for every pair of arcs* γ_1 *and* γ_2 *contained in* A *having the same endpoints.*

ii) $\int_\Gamma f = 0$ *for any* closed *arc* Γ *contained in* A*.*

iii) f *admits an* **antiderivative** (*or* **primitive function**) *on* A*, that is, there is a function* $F : A \to \mathbb{C}$ *analytic on* A *such that* $F'(z) = f(z)$ *for all* $z \in A$*.*

Proof.

ii) $\Longrightarrow$ i) If γ_1 and γ_2 are two arcs contained in A joining $z_1 \in A$ with $z_2 \in A$ then $\Gamma = \gamma_1 - \gamma_2$ is a closed arc, so that

$$\int_{\gamma_1} f - \int_{\gamma_2} f = \int_{\gamma_1 - \gamma_2} f = \int_\Gamma f = 0\,.$$

i) $\Longrightarrow$ ii) Since Γ is closed, its opposite arc $-\Gamma$ has the same endpoints as Γ. From i) it follows that

$$\int_\Gamma f = \int_{-\Gamma} f = -\int_\Gamma f \quad \Longrightarrow \quad \int_\Gamma f = 0\,.$$

iii) $\Longrightarrow$ i) By the fundamental theorem of calculus (since $F' = f$ is continuous by hypothesis).

i) $\Longrightarrow$ iii) Let z_0 be an arbitrary (fixed) point in A. If z is any point in A, since A is a *region* there is an arc $\gamma \subset A$ joining z_0 with z, and let

$$F(z) = \int_\gamma f.$$

By virtue of i) the function F does *not* depend on the arc $\gamma \subset A$ used to connect z_0 with z.

Let us finally prove that F is differentiable at every point $z \in A$, with $F'(z) = f(z)$. If $\varepsilon > 0$, since A is open and f is continuous on A, there exists $\delta > 0$ such that $|f(\zeta) - f(z)| < \varepsilon$ if $\zeta \in D(z;\delta) \subset A$. Given any point $w \in D(z;\delta)$ distinct from z, let $L \subset D(z;\delta) \subset A$ be the segment joining z with w. We then have:

$$F(w) - F(z) = \int_{\gamma+L} f - \int_\gamma f = \int_L f.$$

By the fundamental theorem of calculus, $w - z = \int_L d\zeta$ (since $1 = \zeta'$), and thus $(w-z)f(z) = f(z)\int_L d\zeta = \int_L f(z)\,d\zeta$. Hence

$$\left|\frac{F(w)-F(z)}{w-z} - f(z)\right| = \frac{|F(w)-F(z)-(w-z)f(z)|}{|w-z|} = \frac{\left|\int_L f(\zeta)\,d\zeta - \int_L f(z)\,d\zeta\right|}{|w-z|}$$
$$= \frac{\left|\int_L [f(\zeta)-f(z)]\,d\zeta\right|}{|w-z|} < \frac{\varepsilon\, l(L)}{|w-z|} = \varepsilon.$$

□

5.2 Cauchy's theorem

5.2.1 The Cauchy–Goursat theorem

- A closed arc $\gamma : [a,b] \to \mathbb{C}$ is **simple** if $a < s < t < b \Longrightarrow \gamma(s) \neq \gamma(t)$.

Cauchy's theorem (original version). *If* γ *is a simple closed arc and* $f : \mathbb{C} \to \mathbb{C}$ *is analytic* with continuous derivative *on and inside* γ*, then* $\displaystyle\int_\gamma f = 0$.

Proof. By Green's theorem (traversing the curve counterclockwise, so that the interior $\mathring{D}$ of γ lies to the left of γ), if $f = u + \mathrm{i}v$ we have

$$\int_\gamma f \,\mathrm{d}z = \int_\gamma (u\,\mathrm{d}x - v\,\mathrm{d}y) + \mathrm{i}\int_\gamma (v\,\mathrm{d}x + u\,\mathrm{d}y) = -\int_D (u_y + v_x)\,\mathrm{d}x\,\mathrm{d}y + \mathrm{i}\int_D (u_x - v_y)\,\mathrm{d}x\,\mathrm{d}y = 0,$$

by virtue of the Cauchy–Riemann equations. □

- This result is insufficient for our purposes, since it is not necessary to assume that f' is *continuous* (we shall prove that this assumption follows from the *analyticity* of f). Moreover, the result holds for much more general arcs than simple closed ones.

Cauchy–Goursat theorem for a rectangle. *Let R be a* closed *rectangle with sides parallel to the axes, and let ∂R be the boundary of R. If $f : \mathbb{C} \to \mathbb{C}$ is analytic on R then*

$$\int_{\partial R} f = 0.$$

Proof. Assume that ∂R is traversed counterclockwise (the result is obviously independent from the orientation of ∂R). If we divide R into four congruent subrectangles $R^{(i)}$ ($i = 1, \dots, 4$) (also oriented counterclockwise) then

$$\int_{\partial R} f = \sum_{i=1}^{4} \int_{\partial R^{(i)}} f\,,$$

since the integrals along the inner sides of the rectangles $R^{(i)}$ cancel each other. Thus, there is some $k \in \{1, \dots, 4\}$ such that

$$\left|\int_{\partial R} f\right| \leqslant 4 \left|\int_{\partial R^{(k)}} f\right|.$$

Let us call $R_1 = R^{(k)}$. Repeating indefinitely this bisection process, we obtain a sequence of closed nested rectangles $R_0 \equiv R \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset R_{n+1} \supset \dots$ such that

$$\left|\int_{\partial R_{n-1}} f\right| \leqslant 4 \left|\int_{\partial R_n} f\right| \implies \left|\int_{\partial R} f\right| \leqslant 4^n \left|\int_{\partial R_n} f\right|, \quad \forall n \in \mathbb{N}.$$

Moreover, if P_i y D_i respectively denote the perimeter and the diagonal of the i-th rectangle and $P \equiv P_0$, $D \equiv D_0$, we have:

$$P_i = \frac{P}{2^i}, \quad D_i = \frac{D}{2^i}, \qquad \forall i \in \mathbb{N}.$$

From Cantor's intersection theorem it follows that $\bigcap\limits_{n\in\mathbb{N}} R_n = \{a\}$, with $a \in R$ (for $R_n \subset R$ for all n). Notice that

$$z \in R_n \implies |z - a| \leqslant D_n = 2^{-n} D,$$

since $a \in R_n$ for every $n \in \mathbb{N}$. Given $\varepsilon > 0$, take $\delta > 0$ sufficiently small so that f be analytic on $D(a;\delta)$ and also

$$\left|f(z) - f(a) - (z - a)f'(a)\right| < \varepsilon\,|z - a|\,, \quad \forall z \in D(a;\delta),\ z \neq a.$$

(Note that by hypothesis f is differentiable on a neighborhood of each point of R.) Take now n sufficiently large so that $D_n = 2^{-n} D < \delta$, and thus $R_n \subset D(a;\delta)$. On the other hand, by the fundamental theorem of calculus,

$$\int_{\partial R_n} \mathrm{d}z = \int_{\partial R_n} z\,\mathrm{d}z = 0.$$

From these considerations it follows that

$$\left|\int_{\partial R} f\right| \leqslant 4^n \left|\int_{\partial R_n} f\right| = 4^n \left|\int_{\partial R_n} \big[f(z) - f(a) - f'(a)(z-a)\big]\,\mathrm{d}z\right|$$
$$< 4^n \int_{\partial R_n} \varepsilon\,|z-a|\;|\mathrm{d}z| \leqslant 4^n \cdot 2^{-n} D\varepsilon \cdot P_n = 4^n \cdot 2^{-n} D\varepsilon \cdot 2^{-n} P = PD\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and PD is constant, the theorem is proved. □

Generalized Cauchy–Goursat theorem. *Let a be an interior point in R, and assume that the function $f : \mathbb{C} \to \mathbb{C}$ is analytic on $R \setminus \{a\}$ and $\lim_{z\to a}[(z-a)f(z)] = 0$. Then $\int_{\partial R} f = 0$.*

Proof. Let $Q \subset R$ be a square centered at a with sides parallel to the axes of length $l > 0$ sufficiently small so that $|(z-a)f(z)| < \varepsilon$ if $z \in Q \setminus \{a\}$. By prolonging the sides of Q we can subdivide the rectangle R into 9 subrectangles $Q, R_1, \ldots, R_8$. Thus

$$\int_{\partial R} f = \int_{\partial Q} f + \sum_{i=1}^{8} \int_{\partial R_i} f\,.$$

The function f is analytic on each of the rectangles R_i, since $a \notin R_i \subset R$. By the Cauchy–Goursat theorem $\int_{\partial R_i} f = 0$ for $i = 1, \ldots, 8$, and hence

$$\left|\int_{\partial R} f\right| = \left|\int_{\partial Q} f\right| < \varepsilon \int_{\partial Q} \frac{|\mathrm{d}z|}{|z-a|} \leqslant \varepsilon \cdot \frac{2}{l} \cdot 4l = 8\varepsilon\,,$$

and the theorem follows. □

5.2.2 Homotopy. Cauchy's theorem. Deformation theorem

- Let $A \subset \mathbb{C}$ be a region, and let γ_1 and γ_2 be two continuous curves in A with the same endpoints $z_1, z_2 \in A$ ($z_1 \neq z_2$), or two closed continuous curves in A. We shall say that γ_1 is **homotopic** to γ_2 in A if it can be deformed *in a continuous way* into γ_2 *without leaving* A. In the first case (*homotopy of open curves with fixed endpoints*), the endpoints of all deformed curves must remain equal to z_1 and z_2, whereas in the second one (*homotopy of closed curves*) all deformed curves must be closed.

- It is important to note that the concept of homotopy A *depends on the region* A considered. In other words, two homotopic curves in a certain region A may not be homotopic in another region A'.

- Note that a point $z_0 \in A$ is a constant closed curve: $\gamma(t) = z_0$, $\forall t \in [a,b]$. In particular, $\int_{z_0} f = 0$ for any function f.

Cauchy's theorem. *Let γ be a closed arc homotopic to a point in a region A. If $f : A \to \mathbb{C}$ is analytic on A then*

$$\int_\gamma f = 0. \tag{5.3}$$

Sketch of the proof. Let us assume, for simplicity, that $\gamma \subset A$ is a simple closed arc. Since γ is homotopic to a point in A, it is intuitively clear (and can be rigorously proved) that the interior D of γ is contained in A. Given $\varepsilon > 0$, we cover the plane with a lattice of closed squares Q_j with sides parallel to the axes of length $\delta > 0$. Let us denote by J the collection of indices such that $Q_j \subset D$ if $j \in J$. (The set J is finite, for D is bounded since the image of the arc γ is a compact set.) It can be shown that if δ is chosen sufficiently small then

$$\left|\int_\gamma f - \int_{\partial Q} f\right| < \varepsilon\,, \tag{5.4}$$

where ∂Q is the boundary of $Q \equiv \bigcup_{j\in J} Q_j$ traversed counterclockwise[1]. On the other hand, it is clear that

$$\int_{\partial Q} f = \sum_{j\in J}\int_{\partial Q_j} f \tag{5.5}$$

(where ∂Q_j is the boundary of the square Q_j traversed counterclockwise), since the integrals along the sides of the squares Q_j not belonging to ∂Q cancel in pairs. Since $Q_j \subset D \subset A$ for all $j \in J$ and f is analytic on A, the Cauchy–Goursat theorem implies that

$$\int_{\partial Q_j} f = 0\,, \qquad \forall j \in J,$$

and from (5.4) and (5.5) it follows that

$$\left|\int_\gamma f\right| < \varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, the theorem is proved. □

Definition 5.4. A *region* $A \subset \mathbb{C}$ is **simply connected** if every continuous closed curve γ contained in A is homotopic to a point in A.

Note: Intuitively, a simply connected region is an open set that "consists of only one only one piece" and "has no holes". For example, $\mathbb{C}$ or an open disk are simply connected regions, but an open disk without one of its points is not. However, the complex plane minus a ray is simply connected.

Applying Cauchy's theorem to a simply connected region, we deduce the following two corollaries:

Corollary 5.5. *If $A \subset \mathbb{C}$ is a* simply connected *region and $f : A \to \mathbb{C}$ is analytic on A then*

$$\int_\gamma f = 0,$$

for any closed arc contained in A.

Proof. Indeed, if A is a simply connected region any closed arc γ contained in A is homotopic to a point in that set, so $\int_\gamma f = 0$ by virtue of Cauchy's theorem. □

Corollary 5.6. *If $f : A \to \mathbb{C}$ is analytic on a* simply connected *region A then f admits a primitive function on A.*

Proof. By the previous corollary, $\int_\gamma f = 0$ for every closed arc γ contained in A. This implies that f possesses a primitive function on A, by virtue of the equivalence ii) $\Longleftrightarrow$ iii) of the path independence theorem on page 73. □

Using Cauchy's theorem one can prove the following general result, which plays a fundamental role in complex analysis.

[1] Without loss of generality, it may be assumed that γ is positively oriented, since $\int_\gamma f = 0 \Longleftrightarrow \int_{-\gamma} f = 0$.

Deformation theorem. *Let γ_1 and γ_2 be two homotopic arcs in a region A, and let $f : A \to \mathbb{C}$ be analytic on A. Then*

$$\int_{\gamma_1} f = \int_{\gamma_2} f.$$

Sketch of the proof. Assume, to begin with, that $\gamma_1, \gamma_2 \subset A$ are two homotopic arcs in A with the same endpoints $z_1 \neq z_2$. It is intuitively clear that $\gamma_1 - \gamma_2$ is a closed arc homotopic to a point in A. By Cauchy's theorem,

$$0 = \int_{\gamma_1 - \gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f,$$

which establishes the theorem in this case. Consider next two closed arcs $\gamma_1, \gamma_2 \subset A$ homotopic in A. Suppose, for the sake of concreteness, that γ_1 and γ_2 are both simple and (for instance) γ_1 is interior to γ_2 (cf. Fig. 5.1 left). Since γ_1 and γ_2 are homotopic in A, it is intuitively clear that the set D bounded by both arcs is entirely contained in A. Let $z_1 \in \gamma_1$ and $z_2 \in \gamma_2$, and call L the segment joining z_1 with z_2. The arc $L + \gamma_2 - L - \gamma_1$ is closed, is contained in A (since $L \subset D \subset A$) and is homotopic to a point in that region (cf. Fig. 5.1 right). By Cauchy's theorem,

$$0 = \int_{L+\gamma_2-L-\gamma_1} f = \int_L f + \int_{\gamma_2} f - \int_L f - \int_{\gamma_1} f = \int_{\gamma_2} f - \int_{\gamma_1} f,$$

which proves the theorem also in this case. □

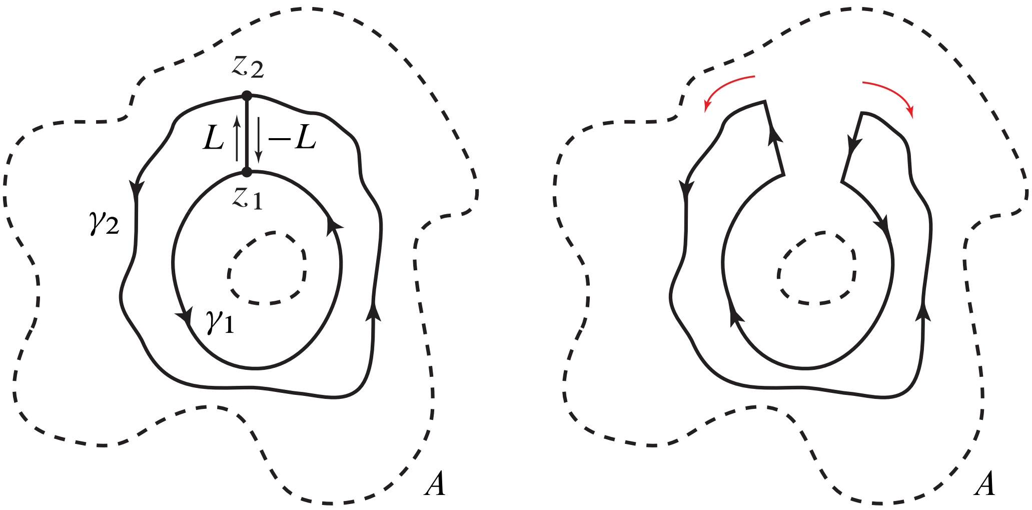

Figure 5.1: Homotopic arcs γ_1 and γ_2 (left) and intermediate curve of the deformation of the closed arc $L + \gamma_2 - L - \gamma_1$ to a point (right).

We will also need the following generalization of Cauchy's theorem, which is proved using the deformation theorem together with the generalized Cauchy–Goursat theorem:

Generalized Cauchy's theorem. *Let $\gamma : [a,b] \to \mathbb{C}$ be a closed arc homotopic to a point in a region A, and let $z_0 \in A \setminus \gamma([a,b])$. If f is analytic on $A \setminus \{z_0\}$ and $\lim_{z\to z_0} [(z - z_0) f(z)] = 0$ then $\int_\gamma f = 0$.*

Sketch of the proof. If z_0 is in the exterior of the curve, then γ is homotopic to a point in the region $A \setminus \{z_0\}$, and $\int_\gamma f = 0$ by Cauchy's theorem applied to that region. Alternatively, if z_0 lies in the interior of γ, since this curve is homotopic to a point in A it can be shown that there is a sufficiently small square $Q \subset A$ centered at z_0 such that γ is homotopic to ∂Q in $A \setminus \{z_0\}$. By the deformation theorem applied to f in the region $A \setminus \{z_0\}$,

$$\int_\gamma f = \int_{\partial Q} f = 0\,,$$

where the last equality follows from the generalized Cauchy–Goursat theorem. $\square$

5.3 Cauchy's integral formula and its consequences

5.3.1 Index

- If γ is a closed arc and $a \notin \gamma$, we define the **index** of a with respect to γ as

$$\boxed{n(\gamma, a) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\mathrm{d}z}{z-a}}\,.$$

- If γ is the circle with center a and radius $r > 0$ traversed m times counterclockwise ($\gamma(t) = a + r\,\mathrm{e}^{it}$, with $t \in [0, 2m\pi]$) then

$$n(\gamma, a) = \frac{1}{2\pi\mathrm{i}} \int_0^{2m\pi} \frac{ir\mathrm{e}^{it}}{r\mathrm{e}^{it}}\,\mathrm{d}t = m\,.$$

  Likewise, if γ is the circle with center a and radius $r > 0$ traversed m times clockwise,

$$n(\gamma, a) = -m\,.$$

  By virtue of the deformation theorem, this suggests that $n(\gamma, a)$ provides the *number of turns that the curve makes around* a, counting as positive the turns made counterclockwise.

Example: If z_0 is *exterior* to a circle (or any other simple closed curve) γ, then $(z - z_0)^{-1}$ is analytic on $A = \mathbb{C} \setminus \{z_0\}$ and γ is homotopic to a point in $A \Longrightarrow n(\gamma, z_0) = 0$, by Cauchy's theorem.

Proposition 5.7. *$n(\gamma, z_0)$ is an integer.*

Proof. Assume, for simplicity, that $\gamma : [a, b] \to \mathbb{C}$ is C^1 on $[a, b]$. Let

$$h(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0}\,\mathrm{d}s\,,$$

so that $n(\gamma, z_0) = h(b)/(2\pi\mathrm{i})$. On the other hand, h is differentiable on $[a, b]$ (the integrand is continuous, since the denominator does not vanish), and

$$h'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0} \quad\Longrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-h(t)}\,[\gamma(t) - z_0]\right) = 0, \quad \forall t \in [a, b].$$

Thus $\mathrm{e}^{-h(t)}\big(\gamma(t) - z_0\big)$ is constant on $[a, b]$, and hence (since $h(a) = 0$)

$$\gamma(a) - z_0 = \mathrm{e}^{-h(b)}(\gamma(b) - z_0) = \mathrm{e}^{-h(b)}(\gamma(a) - z_0) \overset{z_0 \notin \gamma}{\Longrightarrow} \mathrm{e}^{-h(b)} = 1 \Longrightarrow h(b) = 2n\pi\mathrm{i},\ \ n \in \mathbb{Z}.$$

$\square$

5.3.2 Cauchy's integral formula

The following result, which can be easily proved using the generalized Cauchy theorem, is one of the cornerstones of complex analysis.

Cauchy's integral formula. *Let $f : A \to \mathbb{C}$ be analytic on a region A, let γ be a closed arc homotopic to a point in A, and let $a \in A$ be a point not on γ. Then*

$$n(\gamma, a) \cdot f(a) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(z)}{z-a}\, \mathrm{d}z\,.$$

Proof. The function

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z-a}, & z \neq a \\ f'(a), & z = a \end{cases}$$

is analytic on $A \setminus \{a\}$ and $\lim_{z\to a}[(z-a)g(z)] = \lim_{z \to a}[f(z) - f(a)] = 0$ (since f is continuous on a for it is differentiable at that point). From the generalized Cauchy theorem it follows that

$$0 = \int_\gamma g = \int_\gamma \frac{f(z)-f(a)}{z-a}\,\mathrm{d}z = \int_\gamma \frac{f(z)}{z-a}\,\mathrm{d}z - f(a)\int_\gamma \frac{\mathrm{d}z}{z-a} = \int_\gamma \frac{f(z)}{z-a}\,\mathrm{d}z - f(a)\cdot 2\pi \mathrm{i} n(\gamma, a).$$

□

If $f : \mathbb{C} \to \mathbb{C}$ is analytic on an open set A and $z \in A$, we can apply Cauchy's integral formula to the circle γ centered at z with radius r sufficiently small so that $D(z; 2r) \subset A$, since γ is homotopic to a point on $D(z; 2r)$ and hence on A. If the circle γ is positively oriented then $n(\gamma, z) = 1$, and thus Cauchy's integral formula makes it possible to express $f(z)$ as

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w.$$

Taking *formally* the derivative with respect to z under the integral sign we obtain

$$f^{(k)}(z) = \frac{k!}{2\pi \mathrm{i}} \int_\gamma \frac{f(w)}{(w-z)^{k+1}}\,\mathrm{d}w, \quad \forall k \in \mathbb{N}\,. \tag{5.6}$$

In particular, if (5.6) holds, f *is differentiable infinitely many times at any point* $z \in A$. Let us see next how the differentiation under the integral sign leading to equation (5.6) can be rigorously justified:

Lemma 5.8. *Consider the* Cauchy-type integral

$$G(z) = \int_\gamma \frac{g(w)}{w-z}\,\mathrm{d}w,$$

where $g : \mathbb{C} \to \mathbb{C}$ is a continuous *function on an arc γ (not necessarily closed) and $z \notin \gamma([a, b])$. Then G is differentiable infinitely many times at any point $z_0 \notin \gamma([a, b])$, with*

$$G^{(k)}(z_0) = k! \int_\gamma \frac{g(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w. \tag{5.7}$$

Proof. The proof proceeds by induction on k.
i) Assume, to begin with, that $k = 1$. Let $z_0 \notin \gamma([a, b])$, and define

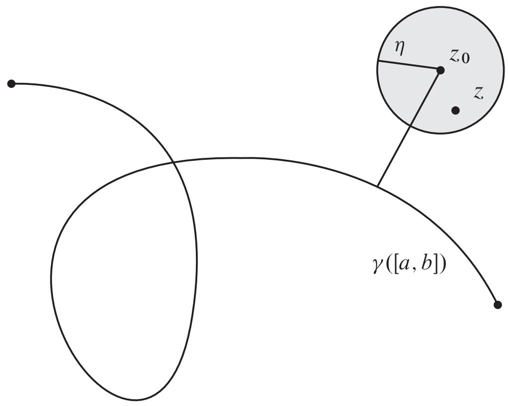

Figure 5.2: Cauchy-type integral

$$2\eta = \min_{t\in[a,b]} |\gamma(t) - z_0| > 0, \quad M = \max_{t\in[a,b]} \left|g\big(\gamma(t)\big)\right| .$$

Note that $\eta > 0$ and $M < \infty$ by the continuity of γ and $g \circ \gamma$ on the compact interval $[a, b]$. If $z \in D(z_0; \eta)$ with $z \neq z_0$ we have

$$\frac{G(z) - G(z_0)}{z - z_0} = \int_\gamma \frac{1}{z - z_0}\left[\frac{1}{w - z} - \frac{1}{w - z_0}\right] g(w)\, \mathrm{d}w.$$

But

$$\frac{1}{z - z_0}\left[\frac{1}{w - z} - \frac{1}{w - z_0}\right] = \frac{1}{(w - z)(w - z_0)} = \frac{1}{(w - z_0)^2}\,\frac{w - z_0}{w - z} = \frac{1}{(w - z_0)^2}\left(1 + \frac{z - z_0}{w - z}\right). \tag{5.8}$$

If w lies on the curve γ, from the definition of M and η it follows that

$$|g(w)| \leqslant M , \qquad |w - z_0| \geqslant 2\eta , \qquad |w - z| \geqslant \eta .$$

Thus

$$\begin{aligned}\left|\frac{G(z) - G(z_0)}{z - z_0} - \int_\gamma \frac{g(w)}{(w - z_0)^2}\, \mathrm{d}w\right| &= |z - z_0| \left|\int_\gamma \frac{g(w)}{(w - z_0)^2(w - z)}\, \mathrm{d}w\right| \\ &\leqslant |z - z_0| \cdot \frac{M\, l(\gamma)}{4\eta^2 \cdot \eta} \xrightarrow[z \to z_0]{} 0.\end{aligned}$$

ii) Assume now that the lemma holds for $k = 1, \dots, n - 1$, and let us prove it for $k = n$. Let us first show that $G^{(n-1)}$ is continuous at $z_0 \in \mathbb{C} \setminus \gamma([a, b])$. Indeed, by the induction hypothesis we have

$$G^{(n-1)}(z) = (n - 1)! \int_\gamma \frac{g(w)}{(w - z)^n}\, \mathrm{d}w .$$

Multiplying the first equality in (5.8) by $1/(w - z)^{n-1}$ we obtain

$$\frac{1}{(w - z)^n} = \frac{1}{(w - z)^{n-1}(w - z_0)} + \frac{z - z_0}{(w - z)^n(w - z_0)} ,$$

and thus

$$\begin{aligned}G^{(n-1)}(z) - G^{(n-1)}(z_0) = (n - 1)! &\left[\int_\gamma \frac{g(w)}{(w - z)^{n-1}(w - z_0)}\, \mathrm{d}w - \int_\gamma \frac{g(w)}{(w - z_0)^n}\, \mathrm{d}w\right] \\ &+ (n - 1)!\, (z - z_0) \int_\gamma \frac{g(w)}{(w - z)^n(w - z_0)}\, \mathrm{d}w .\end{aligned} \tag{5.9}$$

By the induction hypothesis applied to $g(w)/(w - z_0)$ (which is also continuous on γ, since $z_0 \notin \gamma([a,b])$), the function

$$\int_\gamma \frac{g(w)}{(w-z)^{n-1}(w-z_0)}\,\mathrm{d}w$$

is differentiable, and hence continuous, if $z \in \mathbb{C} \setminus \gamma([a,b])$. This implies that the term in brackets in the right-hand side of (5.9) tends to 0 as $z \to z_0$. As to the integral in the second term of the RHS of the latter equation, proceeding as in the case $k = 1$ it can be shown that

$$\left|\int_\gamma \frac{g(w)}{(w-z)^n(w-z_0)}\,\mathrm{d}w\right| \leqslant \frac{M\,l(\gamma)}{2\eta^{n+1}}\,, \qquad \forall z \in D(z_0;\eta)\,,$$

which implies that the second term of the RHS of (5.9) also tends to 0 as $z \to z_0$.

Let us see, finally, that $G^{(n-1)}$ is differentiable at z_0. Indeed, if $z \in D(z_0;\eta) \setminus \{z_0\}$, dividing (5.9) by $z - z_0$ we obtain

$$\frac{G^{(n-1)}(z) - G^{(n-1)}(z_0)}{z-z_0} = \frac{(n-1)!}{z-z_0}\left[\int_\gamma \frac{g(w)}{(w-z)^{n-1}(w-z_0)}\,\mathrm{d}w - \int_\gamma \frac{g(w)}{(w-z_0)^n}\,\mathrm{d}w\right] + (n-1)!\int_\gamma \frac{g(w)}{(w-z)^n(w-z_0)}\,\mathrm{d}w\,. \tag{5.10}$$

Again by the induction hypothesis applied to $g(w)/(w-z_0)$, when $z \to z_0$ the first term of the RHS of this identity tends to

$$\begin{aligned}(n-1)!\left.\frac{\mathrm{d}}{\mathrm{d}z}\right|_{z=z_0}\int_\gamma \frac{g(w)}{(w-z)^{n-1}(w-z_0)}\,\mathrm{d}w &= (n-1)\left.\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\right|_{z=z_0}\int_\gamma \frac{g(w)}{(w-z)(w-z_0)}\,\mathrm{d}w\\ &= (n-1)(n-1)!\int_\gamma \frac{g(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w\,.\end{aligned}$$

As to the second term of (5.10), from what was proved above about the continuity of $G^{(n-1)}(z_0)$ applied now to the Cauchy-type integral

$$\int_\gamma \frac{g(w)}{(w-z)^n(w-z_0)}\,\mathrm{d}w$$

it follows that the latter integral is a continuous function at z_0, and thus it tends to

$$\int_\gamma \frac{g(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w\,.$$

when $z \to z_0$. From these assertions it follows from (5.10) that $G^{(n-1)}$ is differentiable at z_0, with

$$\begin{aligned}G^{(n)}(z_0) &= (n-1)(n-1)!\int_\gamma \frac{g(w)}{(w-z_0)^{n+1}} + (n-1)!\int_\gamma \frac{g(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w\\ &= n!\int_\gamma \frac{g(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w\,.\end{aligned}$$

□

5.3.3 Cauchy's integral formula for the derivatives

From the previous lemma it is easy to prove the following theorem, which generalizes Cauchy's integral formula to the derivatives of arbitrary order of an analytic function:

Cauchy's integral formula for the derivatives. *Let $f : A \to \mathbb{C}$ be an analytic function on a region A. Then f is* infinitely differentiable *at any point of A. Moreover, if $\gamma : [a, b] \to A$ is a closed arc homotopic to a point in A and $z_0 \in A \setminus \gamma([a, b])$ then*

$$n(\gamma, z_0) \cdot f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}} \, dw, \quad k \in \mathbb{N}. \tag{5.11}$$

Proof. Let $z_0 \in A \setminus \gamma([a, b])$, and let D be a neighborhood of z_0 contained in A not intersecting γ (for instance, the disc $D(z_0; \eta)$ in the proof of the previous lemma). Since $n(\gamma, z)$ is a Cauchy-type integral (with $g = 1/2\pi i$), it is a continuous function of z for all $z \in D$, and since it must be an *integer* number it is necessarily *constant* on the latter disc. Thus, if

$$F(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} \, dw \,, \qquad z \in D,$$

from the Cauchy integral formula it follows that

$$F(z) = n(\gamma, z) f(z) = n(\gamma, z_0) f(z) \,, \qquad \forall z \in D \,. \tag{5.12}$$

Since F is also of Cauchy type, equations (5.7) and (5.12) imply that

$$F^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)^{k+1}} \, dw = n(\gamma, z_0) f^{(k)}(z) \,, \qquad \forall z \in D \,.$$

Equation (5.11) then follows by setting $z = z_0$. □

From the previous theorem one can easily prove the following fundamental property of analytic functions:

Theorem 5.9. *If $f : \mathbb{C} \to \mathbb{C}$ is analytic on an arbitrary set C, then f is* differentiable infinitely many times *at every point of C.*

Proof. Given a point $z \in C$, it suffices to apply the previous theorem to any neighborhood A of z on which f be analytic. □

The previous result makes it possible to easily prove the following theorem, a (partial) converse of Cauchy's theorem:

Morera's theoerm. *If $f : \mathbb{C} \to \mathbb{C}$ is continuous on a region A and $\int_\gamma f = 0$ for any closed arc γ contained in A, then f is analytic on A.*

Proof. The path independence theorem implies that there is a function $F : \mathbb{C} \to \mathbb{C}$ analytic on A such that $f = F'$ on A. Since F is analytic on A, it is infinitely differentiable on this region. In particular, $f' = F''$ exists at every point of A. □

Exercise. Does the above result hold if we just assume that $\int_\gamma f = 0$ for any closed arc $\gamma \subset A$ homotopic to a point in A?

5.3.4 Liouville's theorem

The following inequalities for the modulus of the derivatives of an analytic function can be easily proved using Cauchy's integral formula for the derivatives:

Cauchy's inequalities. *Let f be analytic on a region A, let $a \in A$, and assume that $\overline{D}(a; R) \subset A$. If $M(R) = \max_{|z-a|=R} |f(z)|$ then*

$$\left|f^{(k)}(a)\right| \leqslant \frac{k!}{R^k}\, M(R), \qquad \forall k = 0, 1, 2, \ldots .$$

Proof. Note, to begin with, that the existence of the *maximum* of $|f|$ on the circle is guaranteed, since this set is compact and f continuous (being analytic on A). If γ is the positively oriented circle of radius R centered in a, then γ is homotopic to a point in A and $n(\gamma, a) = 1$. Cauchy's integral formula for the k-th derivative thus yields

$$\left|f^{(k)}(a)\right| = \frac{k!}{2\pi}\left|\int_\gamma \frac{f(z)}{(z-a)^{k+1}}\,\mathrm{d}z\right| \leqslant \frac{k!}{2\pi}\int_\gamma \frac{|f(z)|}{|z-a|^{k+1}}\,|\mathrm{d}z| \leqslant \frac{k!}{2\pi}\frac{M(R)}{R^{k+1}}\,2\pi R = \frac{k!}{R^k}\,M(R).$$

□

From the previous result one can prove the following theorem, which is a key result in the study of global properties of analytic functions:

Liouville's theorem. *If $f : \mathbb{C} \to \mathbb{C}$ is* **entire** *(that is, analytic on the whole complex plane) and $|f|$ is* bounded *on $\mathbb{C}$, then f is* constant.

Proof. The hypothesis implies that there is $K > 0$ such that $|f(z)| < K$ for all $z \in \mathbb{C}$. If $a \in \mathbb{C}$, since f is analytic on $\mathbb{C}$ we can apply Cauchy's inequality for the first derivative to *any* closed disc $\overline{D}(a; R)$, obtaining

$$\left|f'(a)\right| < \frac{M(R)}{R} \leqslant \frac{K}{R}, \quad \forall R > 0\,.$$

Since the latter inequality is valid for *all* $R > 0$, it follows that $|f'(a)| = 0$ at all $a \in \mathbb{C}$. Since $\mathbb{C}$ is a connected set, f must be constant on $\mathbb{C}$. □

Liouville's theorem provides one of the simplest proofs of the fundamental theorem of Algebra:

Fundamental theorem of algebra. *A polynomial of degree $n \geqslant 1$ has at least one root in $\mathbb{C}$.*

Proof. Let $p(z) = \sum_{i=0}^n a_i z^i$, with $a_n \neq 0$ and $n \geqslant 1$. If p had no roots, the function $f = 1/p$ would be entire. We shall prove that this is impossible by showing that in such a case the function f would be also bounded and not constant, in contradiction with Liouville's theorem.

In order to prove that f is bounded, notice that if $z \neq 0$

$$|p(z)| \geqslant |z|^n \left(|a_n| - \frac{|a_{n-1}|}{|z|} - \cdots - \frac{|a_0|}{|z|^n}\right).$$

Since $\dfrac{|a_k|}{|z|^{n-k}} \xrightarrow[|z|\to\infty]{} 0$ $\;(k = 0, \ldots, n-1)$, there is $M > 1$ such that

$$|z| > M \implies \frac{|a_k|}{|z|^{n-k}} < \frac{|a_n|}{2n}, \quad k = 0, \ldots, n-1.$$

Thus

$$|z| > M \implies |p(z)| > |z|^n \left(|a_n| - n \cdot \frac{|a_n|}{2n} \right) = \frac{|a_n|}{2} |z|^n > \frac{|a_n|}{2} > 0\,,$$

and hence

$$|z| > M \implies |f(z)| < \frac{2}{|a_n|}\,.$$

On the other hand, since by hypothesis f is analytic (and thus continuous) on the closed disc of radius M centered at 0, there is $c > 0$ such that $|f(z)| \leqslant c$ if $|z| \leqslant M$. (Recall that a function $g : \mathbb{R}^n \to \mathbb{R}$ continuous on a compact set $K \subset \mathbb{R}^n$ is bounded on K.) Therefore, we have shown that

$$|f(z)| = \frac{1}{|p(z)|} \leqslant \max\left(\frac{2}{|a_n|}, c \right), \qquad \forall z \in \mathbb{C}.$$

This contradicts Liouville's theorem, since $f \equiv 1/p$ is also entire but not constant (p is not constant, for $a_n \neq 0$ and $n \geqslant 1$). $\square$

Chapter 6

Series representation of analytic functions

6.1 Power series. Taylor's theorem

6.1.1 Sequences and series of complex numbers

Definition 6.1. A **sequence** of complex numbers $\{z_n\}_{n=1}^{\infty}$ **converges** to $z \in \mathbb{C}$ ($\iff \lim_{n\to\infty} z_n = z$) if

$$\boxed{\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \quad \text{such that} \quad n \geqslant N \Longrightarrow |z_n - z| < \varepsilon}\,.$$

- Note that the definition is identical to that of the real case, replacing the absolute value (or the norm) by the modulus.

Properties:

i) $\lim_{n\to\infty} z_n$, if it exists, is unique.

ii) $\lim_{n\to\infty} z_n = z \equiv x + \mathrm{i}y \iff \lim_{n\to\infty} \mathrm{Re}(z_n) = x,\ \lim_{n\to\infty} \mathrm{Im}(z_n) = y.$

iii) $\lim_{n\to\infty} z_n = z,\ \lim_{n\to\infty} w_n = w \quad \Longrightarrow \quad \lim_{n\to\infty} (z_n + w_n) = z + w\,,\ \lim_{n\to\infty} z_n w_n = zw\,.$

iv) If, in addition, $w_n \neq 0$ for all $n \in \mathbb{N}$ and $w \neq 0$, then $\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{z}{w}\,.$

- **The Cauchy criterion:**

$$\boxed{\exists \lim_{n\to\infty} z_n \iff \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \quad \text{such that} \quad n, m \geqslant N \Longrightarrow |z_n - z_m| < \varepsilon}\,.$$

Proof.

$\Longrightarrow$) $|z_n - z_m| \leqslant |z_n - z| + |z_m - z|$

$\Longleftarrow$) $z_n = x_n + \mathrm{i}y_n \Longrightarrow |x_n - x_m| \leqslant |z_n - z_m|\,,\ |y_n - y_m| \leqslant |z_n - z_m| \Longrightarrow \{x_n\}_{n=1}^{\infty}$ y $\{y_n\}_{n=1}^{\infty}$ convergent (Cauchy real sequences) $\Longrightarrow \{z_n\}_{n=1}^{\infty}$ convergent. ☐

Definition 6.2. The series $\sum_{k=1}^{\infty} z_k$ **converges** to $s \in \mathbb{C}$ $\left(\iff \sum_{k=1}^{\infty} z_k = s \right)$ if the sequence of **partial sums** $\{\sum_{k=1}^{n} z_k\}_{n=1}^{\infty}$ converges to s, that is

$$\sum_{k=1}^{\infty} z_k = s \iff \lim_{n\to\infty} \sum_{k=1}^{n} z_k = s.$$

- $\sum_{k=1}^{\infty} z_k$ convergent $\Longrightarrow \lim_{n\to\infty} z_n = 0$. Indeed,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\left(\sum_{k=1}^{n} z_k - \sum_{k=1}^{n-1} z_k\right) = \lim_{n\to\infty}\sum_{k=1}^{n} z_k - \lim_{n\to\infty}\sum_{k=1}^{n-1} z_k = s - s = 0\,.$$

Definition 6.3. The series $\sum_{k=1}^{\infty} z_k$ is **absolutely convergent** if $\sum_{k=1}^{\infty} |z_k|$ is convergent.

Proposition 6.4. *An absolutely convergent series $\sum_{k=1}^{\infty} z_k$ is convergent.*

Proof. This is a consequence of the Cauchy criterion, since if (for instance) $m > n$ we have

$$\left|\sum_{k=1}^{m} z_k - \sum_{k=1}^{n} z_k\right| = \left|\sum_{k=n+1}^{m} z_k\right| \leqslant \sum_{k=n+1}^{m} |z_k| = \left|\sum_{k=1}^{m} |z_k| - \sum_{k=1}^{n} |z_k|\right|.$$

□

6.1.2 Sequences and series of functions. Uniform convergence

Definition 6.5. A **sequence of functions** $f_n : A \to \mathbb{C}$ defined on a set $A \subset \mathbb{C}$ $(n \in \mathbb{N})$ **converges pointwise** to a function f on A if for any $z \in A$ we have $\lim_{n\to\infty} f_n(z) = f(z)$. Likewise, the **series of functions** $\sum_{k=1}^{\infty} f_k$ converges pointwise to the function g on A if for any $z \in A$ we have $\sum_{k=1}^{\infty} f_k(z) = g(z)$.

Definition 6.6. The sequence of functions $\{f_n\}_{n=1}^{\infty}$ defined on A **converges uniformly to** f **on** A if

$$\boxed{\forall\varepsilon > 0,\ \exists N \in \mathbb{N} \quad \text{such that} \quad n \geqslant N \Longrightarrow |f_n(z) - f(z)| < \varepsilon,\ \textbf{for all}\ z \in A}\,.$$

Likewise, $\sum_{k=1}^{\infty} f_k$ converges uniformly to g on A if the sequence of functions $\{\sum_{k=1}^{n} f_k\}_{n=1}^{\infty}$ converges uniformly to g on A, that is, if

$$\boxed{\forall\varepsilon > 0,\ \exists N \in \mathbb{N} \quad \text{such that} \quad n \geqslant N \Longrightarrow \left|\sum_{k=1}^{n} f_k(z) - g(z)\right| < \varepsilon,\ \textbf{for all}\ z \in A}\,.$$

- Obviously, if a sequence or series of functions converges uniformly to a function f on A, then such a sequence or series converges pointwise to this function. However, *the pointwise convergence of a sequence or series of functions does not imply, in general, its uniform convergence*.

- **Cauchy criterion for the uniform convergence:** $\{f_n\}_{n=1}^{\infty}$ converges uniformly on A if and only if

$$\boxed{\forall\varepsilon > 0,\ \exists N \in \mathbb{N} \quad \text{such that} \quad n, m \geqslant N \Longrightarrow |f_n(z) - f_m(z)| < \varepsilon,\ \textbf{for all}\ z \in A}\,.$$

Proof. First of all, clearly the uniform convergence of f_n to f on A implies the Cauchy criterion. Conversely, the Cauchy criterion for numerical sequences implies that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to a function f on A. Taking the limit $m \to \infty$ in the uniform Cauchy condition it follows that if $n \geqslant N$ then $|f_n(z) - f(z)| \leqslant \varepsilon$ for all $z \in A$. □

Similarly, the series $\sum\limits_{k=1}^{\infty} f_k$ converges uniformly to a function g on A if and only if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \quad \text{such that} \quad m > n \geqslant N \Longrightarrow \left|\sum_{k=n+1}^{m} f_k(z)\right| < \varepsilon, \ \textbf{for all} \ z \in A.$$

Weierstrass M-test. *Consider the sequence of functions $f_k : A \subset \mathbb{C} \to \mathbb{C}$ $(k \in \mathbb{N})$, and assume that $|f_k(z)| \leqslant M_k$ for all $z \in A$ and all $k \in \mathbb{N}$. If the numerical series $\sum_{k=1}^{\infty} M_k$ converges, then $\sum_{k=1}^{\infty} f_k$ converges absolutely and uniformly on A.*

Proof. According to the Cauchy criterion for numerical series, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$m > n \geqslant N \quad \Longrightarrow \quad \left|\sum_{k=n+1}^{m} M_k\right| < \varepsilon.$$

But then

$$m > n \geqslant N \quad \Longrightarrow \quad \left|\sum_{k=n+1}^{m} f_k(z)\right| \leqslant \sum_{k=n+1}^{m} |f_k(z)| \leqslant \sum_{k=n+1}^{m} M_k = \left|\sum_{k=n+1}^{m} M_k\right| < \varepsilon, \quad \forall z \in A\,.$$

By the Cauchy criterion for uniform convergence, $\sum_{k=1}^{\infty} f_k$ converges absolutely and uniformly on A.
□

• If $\{f_n\}_{n=1}^{\infty}$ converges *uniformly* to f on A and $f_n : A \to \mathbb{C}$ is *continuous* on A for all $n \in \mathbb{N}$, then f is *continuous* on A. Similarly, if f_n is continuous on A for all $n \in \mathbb{N}$ and $\sum_{k=1}^{\infty} f_k$ converges uniformly to g on A, then g is continuous on A.

The proof of this result is identical to that of the real case, simply replacing the absolute value by the modulus.

Lemma 6.7. *Let f_n be continuous on A for all $n \in \mathbb{N}$. If $\{f_n\}_{n=1}^{\infty}$ converges* uniformly *to f over an arc γ contained in A then*

$$\boxed{\lim_{n\to\infty} \int_\gamma f_n = \int_\gamma f \equiv \int_\gamma \lim_{n\to\infty} f_n}\,.$$

In particular, if f_k is continuous on A for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} f_k$ converges uniformly over γ we have

$$\boxed{\int_\gamma \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int_\gamma f_k}\,.$$

Proof. Note to begin with that f is continuous on $\gamma \subset A$ due to the uniform convergence of f_n to f on γ, and hence the integral $\int_\gamma f$ exists. Given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \varepsilon$ for all $z \in \gamma$ and $n \geqslant N$. We then have:

$$n \geqslant N \quad \Longrightarrow \quad \left|\int_\gamma f_n - \int_\gamma f\right| = \left|\int_\gamma (f_n - f)\right| \leqslant \int_\gamma |f_n(z) - f(z)|\,|\mathrm{d}z| \leqslant \varepsilon\, l(\gamma)\,.$$

□

Definition 6.8. We say that a sequence of functions $f_n : \mathbb{C} \to \mathbb{C}$ **converges normally** to a function f on an open set $A \subset \mathbb{C}$ if $f_n \to f$ uniformly on any closed disc contained in A. Likewise, the series of functions $\sum_{k=1}^{\infty} f_n$ converges normally to g on A if the sequence of partial sums $\sum_{k=1}^{n} f_n$ converges uniformly to g on any closed disc contained in A.

Clearly, if $A \subset \mathbb{C}$ is an open set we have

$$f_n \to f \ \text{ uniformly on } A \implies f_n \to f \ \text{ normally on } A \implies f_n \to f \ \text{ pointwise on } A$$

and similarly

$$\sum_{n=1}^{\infty} f_n \to g \ \text{ uniformly on } A \implies \sum_{n=1}^{\infty} f_n \to g \ \text{ normally on } A \implies \sum_{n=1}^{\infty} f_n \to g \ \text{ pointwise on } A\,.$$

Analytic convergence theorem. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of analytic functions on an open set A such that $f_n \to f$ normally over A. Then f is analytic on A, and $f_n' \to f'$ normally over A.*

Proof. In the first place, by the uniform convergence of f_n to f on closed discs contained in A, f is continuous on each closed disc contained in A, and is thus continuous on A. Let $\overline{D}(a;r) \subset A$. If γ is a closed arc contained in $D(a;r)$ then

$$\int_\gamma f = \lim_{n\to\infty} \int_\gamma f_n = 0,$$

by Lemma 6.7 and Cauchy's theorem (f_n is analytic on $D(a;r) \subset A$ and $D(a;r)$ is simply connected). By Morera's theorem, f is analytic on $D(a;r)$, and thus on A.

In order to show that $f_n' \to f'$ uniformly on $\overline{D}(a;r)$, note that there is $R > r$ such that $\overline{D}(a;R) \subset A$. Given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|f_n(w) - f(w)| < \varepsilon$ for all $w \in \overline{D}(a;R)$ and $n \geqslant N$. If $z \in \overline{D}(a;r)$ and γ is the (positively oriented) circle of radius R centered at a, from Cauchy's integral formula for the first derivative it follows that

$$\left|f_n'(z) - f'(z)\right| = \frac{1}{2\pi}\left|\int_\gamma \frac{f_n(w) - f(w)}{(w-z)^2}\,\mathrm{d}w\right| \leqslant \frac{1}{2\pi}\frac{\varepsilon}{(R-r)^2}\,2\pi R = \frac{\varepsilon R}{(R-r)^2}, \qquad \forall n \geqslant N.$$

□

Corollary 6.9. *Let $\sum_{k=1}^{\infty} g_k$ be a series of analytic functions on an open set A converging normally to a function g over A. Then g is analytic on A, and $\sum_{k=1}^{\infty} g_k'$ converges normally to g' on A.*

In particular, notice that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}z}\sum_{k=1}^{\infty} g_k = \sum_{k=1}^{\infty} g_k'(z) \quad \text{in } A};$$

in other words, if the hypothesis of the latter corollary hold *the series can be differentiated term by term on A.*

6.1.3 Power series

A **power series** centered at $z_0 \in \mathbb{C}$ is a series of the form

$$\sum_{k=0}^{\infty} a_k (z-z_0)^k, \qquad a_k \in \mathbb{C} \ \ (k = 0, 1, \dots). \tag{6.1}$$

Abel's theorem. *For any power series (6.1) there is a unique* R *with* $0 \leqslant R \leqslant \infty$, *called the* **radius of convergence** *of the series, satisfying:*

i) The series converges absolutely *and* normally *if* $|z - z_0| < R$.

ii) The series diverges if $|z - z_0| > R$.

iii) If $R > 0$, *the sum of the series is an* analytic *function on the* **convergence disk** $D(z_0; R)$, *whose derivative is obtained by differentiating the series term by term.*

Proof. Clearly, from i) and ii) it follows that R is unique if it exists. We shall prove that

$$R = \sup I\,, \qquad I \equiv \left\{r \geqslant 0 : \{|a_n|\, r^n\}_{n=0}^{\infty} \text{ bounded}\right\}.$$

Note that if $\{|a_n|\, r^n\}_{n=0}^{\infty}$ is bounded, so is $\{|a_n|\, \rho^n\}_{n=0}^{\infty}$ for all $\rho \leqslant r$, so that the set I is an *interval* with lower endpoint 0. In particular, $R = \infty$ if $\{|a_n|\, r^n\}_{n=0}^{\infty}$ is bounded for all $r \geqslant 0$. With this definition, the assertion ii) is trivial: indeed, if $|z - z_0| > R$ the sequence

$$\left\{|a_n|\, |z - z_0|^n\right\}_{n=0}^{\infty} = \{|a_n (z - z_0)^n|\}_{n=0}^{\infty}$$

is not bounded (since $|z - z_0| \notin I$), and thus the general term of the series does not tend to zero as $n \to \infty$.

In order to prove i), note to begin with that if $R = 0$ the series diverges for all $z \neq z_0$, and there is nothing to prove. Otherwise, assume that $R > 0$, and let $0 < r < R$. Then $r \in I$ (by definition of supremum), and (again by definition of supremum) there are $r < \rho < R$ and $M > 0$ such that $|a_n|\, \rho^n < M$ for all n. If $|z - z_0| \leqslant r$ we have:

$$|a_n (z - z_0)^n| = |a_n|\, \rho^n \left(\frac{|z - z_0|}{\rho}\right)^n \leqslant M \left(\frac{r}{\rho}\right)^n .$$

By the Weierstrass M-test, the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely and uniformly on $\overline{D}(z_0; r)$; in particular, it converges absolutely on $D(z_0; R)$. This also implies that the series converges uniformly on any closed disc contained in $D(z_0; R)$, since a closed disc contained in $D(z_0; R)$ is also contained in some closed disc centered at z_0 with radius smaller than R. This establishes assertion i). Assertion iii) then follows from the analytic converge theorem. □

• *The radius of convergence of the derivative of a power series is equal to the radius of convergence of the power series.*

In fact, by Abel's theorem, it suffices to see that the series $\sum\limits_{k=1}^{\infty} k a_k (z - z_0)^{k-1}$ diverges if $|z - z_0| > R$, where R is the radius of convergence of the original series $\sum\limits_{k=0}^{\infty} a_k (z - z_0)^k$. And indeed

$$k\, |a_k|\, |z - z_0|^{k-1} = |z - z_0|^{-1} \cdot k\, |a_k|\, |z - z_0|^k \geqslant |z - z_0|^{-1} \cdot |a_k|\, |z - z_0|^k .$$

By definition of R, the last term is *not* bounded when $|z - z_0| > R$. Then the general term of the series $\sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}$ does not tend to zero if $|z - z_0| > R$, so that the latter series diverges if $|z - z_0| > R$.

Repeated application of Abel's theorem and the previous result yield the following

Theorem 6.10. *Let* $0 < R \leqslant \infty$ *be the radius of convergence of the series* $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$. *Then* f *is* infinitely differentiable *on* $D(z_0; R)$, *with*

$$f^{(n)}(z) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1) a_k (z - z_0)^{k-n}$$
$$= \sum_{k=0}^{\infty} (k+n)(k+n-1)\cdots(k+1) a_{k+n} (z - z_0)^k, \quad \forall n \in \mathbb{N}, \quad \forall z \in D(z_0; R).$$

The radius of convergence of the above series is again R, *and the coefficients* a_n *are given by*

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \qquad \forall n = 0, 1, 2, \ldots .$$

Corollary 6.11 (Uniqueness of power series)**.** *If there is* $r > 0$ *such that*

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = \sum_{k=0}^{\infty} b_k (z - z_0)^k, \qquad \forall z \in D(z_0, r),$$

then $a_k = b_k$ *for all* $k = 0, 1, 2, \ldots$.

Proof. $a_k = b_k = f^{(k)}(z_0)/k!$, where $f(z)$ is the sum of either series. □

• *The ratio and root tests are valid in the complex case.* Indeed, let us consider the series $\sum_{k=0}^{\infty} z_k$, and assume that $\lim_{n\to\infty} |z_{n+1}| \,/\, |z_n| = c$, with $c \in \mathbb{R}_+$ or $c = +\infty$. If $c < 1$, the series of nonnegative real numbers $\sum_{k=0}^{\infty} |z_k|$ is convergent, so that the series $\sum_{k=0}^{\infty} z_k$ is absolutely convergent. If, on the other hand, $c > 1$ (o $c = +\infty$) then $|z_n|$ is not bounded as $n \to \infty$, so that the series $\sum_{k=0}^{\infty} z_k$ diverges (since its general term does not tend to zero as $n \to \infty$). The root test is established using a similar argument.

• If there exists (*or is* $+\infty$) $\lim_{n\to\infty} \dfrac{|a_n|}{|a_{n+1}|}$, *then* $\boxed{R = \lim_{n\to\infty} \dfrac{|a_n|}{|a_{n+1}|}}$.

Likewise, if there exists (*or is* $+\infty$) $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ *then* $\boxed{R = 1/\lim_{n\to\infty} \sqrt[n]{|a_n|}}$.

Let us prove e.g. the first formula. By the root test, if $z \neq z_0$ the series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ converges if

$$\lim_{n\to\infty} \frac{|a_{n+1}|\,|z - z_0|^{n+1}}{|a_n|\,|z - z_0|^n} = |z - z_0| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1,$$

and diverges if

$$|z - z_0| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1.$$

The second formula follows by a similar argument using the root test.

• *The radius of convergence* R *of the power series* $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ *can be computed using* Hadamard's formula

$$\boxed{R = 1/\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Note: if $x_n \geqslant 0$ for all $n \in \mathbb{N}$,

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geqslant n\}.$$

The limit superior always exist, is equal to infinity if and only if $\{x_n\}_{n=1}^{\infty}$ is not bounded above, and coincides with the ordinary limit when this limit exists.

Example 6.12. Let us consider the **geometric series** with ratio $z \in \mathbb{C}$, given by $\sum_{k=0}^{\infty} z^k$. This is a power series centered at $z_0 = 0$, with unit radius of convergence (since $a_n = 1$ for all n). Thus the geometric series converges absolutely if $|z| < 1$ and diverges if $|z| > 1$. This result can be proved in a more elementary way noting that if $|z| > 1$ the general terms of the series is not bounded (its modulus tends to infinity as $n \to \infty$), so that the series is divergent. On the contrary, if $|z| < 1$ the n-th partial sum of the series is given by

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z} \xrightarrow[n\to\infty]{} \frac{1}{1 - z},$$

since $|z|^{n+1} \xrightarrow[n\to\infty]{} 0$ for $|z| < 1$. Note that the geometric series is divergent at all points of the boundary of the convergence disc, since $|z| = 1$ implies that the general term of the series has unit modulus, and thus cannot tend to zero as $k \to \infty$. In summary,

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}, \qquad |z| < 1.$$

6.1.4 Taylor's theorem

In the previous subsection we have shown that the sum of a power series is an analytic function in its convergence disk. We will now prove that, conversely, an analytic function on an open disk can be represented by a convergent power series on that disk:

Taylor's theorem. *If f is analytic on the disc $D(z_0; r)$ (with $r > 0$), it admits the* **Taylor series** *expansion*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k, \qquad \text{for all } z \in D(z_0; r). \tag{6.2}$$

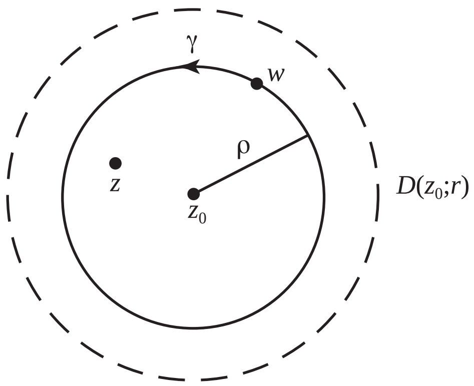

Figure 6.1: Taylor's theorem

Proof. Let z be a fixed point in $D(z_0; r)$, and let $\rho > 0$ such that $|z - z_0| < \rho < r$. If γ is the (positively oriented) circle with radius ρ and center z_0, from Cauchy's integral formula it follows that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw.$$

On the other hand,

$$\frac{1}{w-z}=\frac{1}{w-z_0+z_0-z}=\frac{1}{w-z_0}\frac{1}{1-\dfrac{z-z_0}{w-z_0}}=\frac{1}{w-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k .$$

Note that the geometric series in the right-hand side is convergent, for $w \in \gamma \implies |z-z_0| < \rho = |w-z_0|$. Since $f(w)$ is analytic on γ, it is bounded γ (which is a compact set), so that

$$\left|\frac{f(w)}{w-z_0}\right|\left|\frac{z-z_0}{w-z_0}\right|^k < \frac{M}{\rho}\left|\frac{z-z_0}{\rho}\right|^k , \qquad \forall w \in \gamma .$$

The *numerical* series (that is, independent from w) $\frac{M}{\rho}\sum_{k=0}^{\infty}\left|\frac{z-z_0}{\rho}\right|^k$ is convergent (geometric series with ration smaller than 1). By the Weierstrass M-test, the series

$$g(w) \equiv \sum_{k=0}^{\infty}\frac{f(w)}{w-z_0}\left(\frac{z-z_0}{w-z_0}\right)^k$$

converges uniformly and absolutely over γ. Integrating term by term (cf. Lemma 6.7) we obtain

$$\begin{aligned} f(z)=\frac{1}{2\pi \mathrm{i}}\int_\gamma g(w)\,\mathrm{d}w &= \frac{1}{2\pi \mathrm{i}}\sum_{k=0}^{\infty}\int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}(z-z_0)^k\,\mathrm{d}w \\ &= \sum_{k=0}^{\infty}(z-z_0)^k\frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w = \sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k , \end{aligned}$$

where the last equality follows from Cauchy's integral formula for the k-th derivative. □

Let f be analytic on a nonempty open set $A \subset \mathbb{C}$. If $z_0 \in A \neq \mathbb{C}$, we define the **distance of z_0 to the boundary ∂A of A** as

$$d(z_0;\partial A)=\inf\{|z-z_0| : z \in \partial A\} .$$

It is easy to show that $d(z_0;\partial A) \in (0,\infty)$. If $A=\mathbb{C}$, which has no boundary, by definition we shall say that the latter distance is infinite. Clearly, the open disc centered at z_0 with radius $d(z_0;\partial A)$ is contained in A. Applying Taylor's theorem to the latter disc we obtain the following

Corollary 6.13. *The radius of convergence of the Taylor series centered at $z_0 \in A$ of a function f analytic on A is* greater than or equal to *the distance of z_0 to the boundary of A.*

- The radius of convergence of the Taylor series of f (6.2) may be *greater* than $d(z_0;\partial A)$. This is for example the case if $f(z)=\operatorname{Log} z$ and $z_0=-1+\mathrm{i}$ (see the following exercise).

Exercise. Show that the Taylor series of $\operatorname{Log} z$ centered at $-1+\mathrm{i}$ has radius of convergence $\sqrt{2}$, while the distance from $-1+\mathrm{i}$ to the boundary of the analyticity region of Log is equal to 1. To which function does the above Taylor series converge on $D(-1+\mathrm{i};\sqrt{2})$?

Solution. The function $f(z)=\operatorname{Log} z$ is analytic on $A=\mathbb{C}\setminus\big(\mathbb{R}^-\cup\{0\}\big)$, so that $\partial A=\mathbb{R}^-\cup\{0\}$. If $z_0=-1+\mathrm{i}$ then

$$d(z_0;\partial A)=1\equiv d .$$

On the other hand,

$$f'(z)=\frac{1}{z} \implies f^{(k)}(z)=(-1)^{k-1}\frac{(k-1)!}{z^k}, \qquad k=1,2,\dots .$$

Therefore, the Taylor series of f centered at z_0 is

$$f(z) = \mathrm{Log}(z) = \mathrm{Log}\, z_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k\, z_0^k} (z - z_0)^k \tag{6.3}$$

(notice that $\mathrm{Log}(z_0) = \frac{1}{2}\log 2 + \frac{3\pi i}{4}$, although this fact is not important). By the root test, the radius of convergence of this series is

$$R = \lim_{k\to\infty} \sqrt[k]{k|z_0|^k} = |z_0| \lim_{k\to\infty} \sqrt[k]{k} = |z_0| = \sqrt{2} > d = 1 .$$

The Taylor series of $\mathrm{Log}\, z$ centered at $z_0 \equiv -1 + i$ converges to $F(z) = \log_{[0,2\pi)}$ on $D(z_0; \sqrt{2})$. Indeed, f and F clearly coincide on $D(z_0; 1)$, so that

$$F^{(k)}(z_0) = f^{(k)}(z_0) , \qquad k = 0, 1, \ldots .$$

Thus the Taylor series of F centered at z_0 coincides with that of f. Since F is analytic on the disc $D(z_0; \sqrt{2})$, $F(z)$ is equal on the latter disc to the sum of its Taylor series centered at z_0, that is, to the sum of the series (6.3).

Proposition 6.14. *Let f be analytic on A, let $z_0 \in A$, and assume that f* is not bounded *on the disc with center z_0 and radius $d(z_0; \partial A)$. Then the radius of convergence of the Taylor series of f centered at z_0 is* exactly equal *to $d(z_0; \partial A)$.*

Proof. Let R be the radius of convergence of the Taylor series of f centered at z_0, and assume that $R > d(z_0; \partial A) \equiv d$. If

$$F(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k , \qquad |z - z_0| < R ,$$

then $F = f$ on $D(z_0; d)$. On the other hand, F is bounded on $\overline{D}(z_0; d)$, for this closed disc is a compact set entirely contained in $D(z_0; R)$, and F is continuous (analytic) on the latter disc. In particular, F is bounded on the open disc $D(z_0; d)$. But this contradicts the hypothesis, since $F = f$ on $D(z_0; d)$. □

6.1.5 Zeros of analytic functions

In this subsection we shall summarize some fundamental properties of the set of zeros of an analytic function.

Proposition 6.15. *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic on a point $a \in \mathbb{C}$, and assume that $f(a) = 0$. Then either f vanishes identically on a neighborhood of a, or f does not vanish on a punctured neighborhood of the latter point.*

Proof. If f is analytic at a then f is differentiable on some neighborhood $D(a; r) \equiv D$ of a. By Taylor's theorem,

$$f(z) = \sum_{k=1}^{\infty} c_k (z - a)^k , \qquad |z - a| < r .$$

If the coefficients c_k are all zero, then $f = 0$ on D. On the contrary, if there is $n \in \mathbb{N}$ such that

$$c_0 = \cdots = c_{n-1} = 0, \quad c_n \neq 0,$$

and thus

$$f(z) = \sum_{k=n}^{\infty} c_k (z - a)^k = (z - a)^n \sum_{k=0}^{\infty} c_{k+n} (z - a)^k \equiv (z - a)^n g(z) , \qquad |z - a| < r .$$

The function $g(z)$ is analytic on D (since it is the sum of a convergent power series on D), and $g(a) = c_n \neq 0$. Since g is continuous at a, there is $0 < \delta < r$ such that $g(z) \neq 0$ for all $z \in D(a;\delta)$. In particular, $f(z) = (z-a)^n g(z)$ does not vanish on the punctured neighborhood $D(a;\delta) - \{a\}$ of a. $\square$

• As before, let f be analytic at $a \in \mathbb{C}$ and not identically zero on a neighborhood of a. If $f(a) = 0$, by the previous lemma there is $n \in \mathbb{N}$ such that

$$f(z) = (z-a)^n g(z)\,,$$

with g analytic an nonzero on a neighborhood of a. Moreover, in this case we have

$$f(a) = \cdots = f^{(n-1)}(a) = 0, \quad f^{(n)}(a) \neq 0\,,$$

since $f^{(k)}(a) = k!c_k$. We then say that f has a **zero of order** n at a.

• With the help of the previous proposition one can prove the following fundamental property of analytic functions, which is the basis of the **principle of analytic continuation**:

Theorem 6.16. *If $f : \mathbb{C} \to \mathbb{C}$ is analytic on a* region *A and vanishes on a neighborhood of a point $z_0 \in A$, then f is identically zero on all of A.*

6.2 Laurent series. Laurent's theorem

6.2.1 Laurent series

A series of the form

$$f(z) = \sum_{k=1}^{\infty} \frac{b_k}{(z-z_0)^k} \tag{6.4}$$

is a power series in the variable $w = (z-z_0)^{-1}$. Thus, if ρ is the radius of convergence of this power series and $R = 1/\rho$ (with $0 \leqslant R \leqslant \infty$), the series (6.4) converges if $|z-z_0| > R$ and diverges if $|z-z_0| < R$, the convergence being absolute and uniform in the complement of any disc $D(z_0;r)$ with $r > R$. Moreover, the function f is analytic on the region of convergence $|z-z_0| > R$, since it is the composition of the power series $g(w) \equiv \sum_{k=1}^{\infty} b_k\, w^k$, analytic for $|w| < \rho \equiv 1/R$, with the function $h(z) = (z-z_0)^{-1}$.

Consider next the more general series

$$f(z) = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k \equiv \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k. \tag{6.5}$$

The first series converges absolutely to an analytic function for $|z-z_0| > R_1$, while the second one does so if $|z-z_0| < R_2$. Thus, f is well-defined and analytic on the open **annulus**

$$C(z_0;R_1,R_2) = \{z : R_1 < |z-z_0| < R_2\}\,,$$

called the **annulus of convergence**, whenever $0 \leqslant R_1 < R_2 \leqslant \infty$. In addition (by the results on power series) the convergence of both series (6.5) is absolute and uniform on any closed subannulus contained in $C(z_0;R_1,R_2)$. A series of the form (6.5) is known as a **Laurent series** centered at z_0.

Proposition 6.17. *If the Laurent series*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

converges on the annulus $C(z_0; R_1, R_2)$ (*with* $0 \leqslant R_1 < R_2 \leqslant \infty$) *then*

$$\boxed{a_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz\,, \qquad \forall n \in \mathbb{Z}}\,,$$

where γ_r *is the positively oriented circle with center* z_0 *and radius* r (*with* $R_1 < r < R_2$).

Proof. Indeed, by definition of f we have:

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{1}{2\pi i} \int_{\gamma_r} \sum_{k=-\infty}^{\infty} a_k (z-z_0)^{k-n-1}\, dz\,.$$

The series under the integral sign is a Laurent series converging on the annulus $C(z_0; R_1, R_2)$, and thus converging *uniformly* on the circle γ_r (by the properties of Laurent series). Applying Lemma 6.7 we obtain:

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \sum_{k=-\infty}^{\infty} a_k \cdot \frac{1}{2\pi i} \int_{\gamma_r} (z-z_0)^{k-n-1}\, dz\,.$$

By the fundamental theorem of calculus, for any integer $j \neq -1$ we have

$$\int_{\gamma_r} (z-z_0)^j\, dz = \int_{\gamma_r} \frac{d}{dz}\left[\frac{(z-z_0)^{j+1}}{j+1}\right] dz = 0\,,$$

and thus

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz = a_n \cdot \frac{1}{2\pi i} \int_{\gamma_r} \frac{dz}{z-z_0} = a_n \cdot n(\gamma_r, z_0) = a_n\,.$$

□

Corollary 6.18 (uniqueness of Laurent series)**.** *If*

$$\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k\,, \qquad R_1 < |z-z_0| < R_2\,, \tag{6.6}$$

then $a_k = c_k$ *for all* $k \in \mathbb{Z}$.

6.2.2 Laurent's theorem

As we have seen above, Laurent series are analytic functions in their annuli of convergence. Conversely, we shall prove below that an analytic function on an open annulus is the sum of a Laurent series:

Laurent's theorem. *Let* f *be an analytic function on the annulus* $C(z_0; R_1, R_2)$, *with* $0 \leqslant R_1 < R_2 \leqslant \infty$. *If* $R_1 < r < R_2$, *let* γ_r *be the positively oriented circle with center* z_0 *and radius* r, *and define*

$$a_k = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{(z-z_0)^{k+1}}\, dz, \qquad \forall k \in \mathbb{Z}. \tag{6.7}$$

Then f *admits the* ***Laurent series expansion***

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k\,, \qquad R_1 < |z-z_0| < R_2\,, \tag{6.8}$$

where the series on the right-hand side converges absolutely and uniformly on each closed subannulus contained in $C(z_0; R_1, R_2)$.

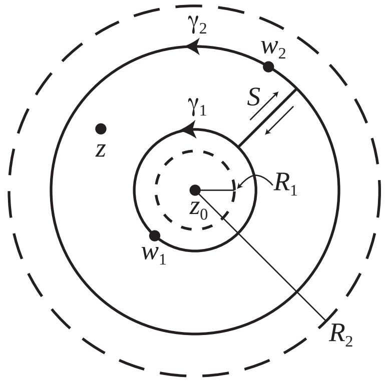

Figure 6.2: Laurent's theorem

Proof. Let $z \in C(z_0; R_1, R_2)$ and take r_1 y r_2 such that $R_1 < r_1 < |z - z_0| < r_2 < R_2$, so that the closed annulus $\overline{A} = \overline{C}(z_0; r_1, r_2)$ is contained in $C(z_0; R_1, R_2)$. Let us denote $\gamma_{r_1} \equiv \gamma_1$, $\gamma_{r_2} \equiv \gamma_2$. The closed curve $S + \gamma_2 - S - \gamma_1$ is homotopic to a point in $C(z_0; R_1, R_2)$ (see fig. 6.2). By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{S+\gamma_2-S-\gamma_1} \frac{f(w)}{w - z}\, dw = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w - z}\, dw \equiv f_2(z) - f_1(z)\,.$$

The proof of Laurent's theorem basically consists in expanding f_1 y f_2 as power series in $(z - z_0)^{-1}$ and $z - z_0$, respectively. For f_2, repeating the argument used in the proof of Taylor's theorem one obtains

$$f_2(z) = \frac{1}{2\pi i} \int_{\gamma_2} f(w) \frac{1}{w - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^k dw = \sum_{k=0}^{\infty} (z - z_0)^k \cdot \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w - z_0)^{k+1}}\, dw,$$

where the last step is justified by the uniform convergence of the series over γ_2 ($w \in \gamma_2 \implies |z - z_0| / |w - z_0| = |z - z_0| / r_2 < 1$). As to f_1, it suffices to observe that if $r_1 < |z - z_0|$ then

$$\frac{1}{w - z} = -\frac{1}{z - z_0} \frac{1}{1 - \dfrac{w - z_0}{z - z_0}} = -\frac{1}{z - z_0} \sum_{k=0}^{\infty} \left(\frac{w - z_0}{z - z_0} \right)^k .$$

Again, the convergence of the geometric series of the right-hand side is uniform over γ_1, since

$$w \in \gamma_1 \implies \frac{|w - z_0|}{|z - z_0|} = \frac{r_1}{|z - z_0|} < 1\,,$$

Applying Lemma 6.7 we obtain

$$\begin{aligned} -f_1(z) &= \frac{1}{2\pi i} \int_{\gamma_1} f(w) \sum_{k=0}^{\infty} \frac{(w - z_0)^k}{(z - z_0)^{k+1}}\, dw = \sum_{k=0}^{\infty} (z - z_0)^{-k-1} \cdot \frac{1}{2\pi i} \int_{\gamma_1} f(w)(w - z_0)^k\, dw \\ &= \sum_{n=-\infty}^{-1} (z - z_0)^n \cdot \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(w - z_0)^{n+1}}\, dw. \end{aligned}$$

By the deformation theorem, $\int_{\gamma_r} f(w)(w - z_0)^{-n-1}\, dw$ is independent of r if $R_1 < r < R_2$, which proves (6.7)–(6.8). The annulus of convergence of the Laurent series (6.7)–(6.8) is at least $C(z_0; R_1, R_2)$; hence, from the properties of Laurent series it follows that the latter series converges absolutely and uniformly on any closed subannulus centered at z_0 and contained in $C(z_0; R_1, R_2)$. □

6.3 Classification of isolated singularities

Definition 6.19. A function $f : \mathbb{C} \to \mathbb{C}$ has an **isolated singularity** at $z_0 \in \mathbb{C}$ if f is not differentiable at z_0, but is analytic in some punctured neighborhood $C(z_0; 0, r)$ (with $r > 0$) of z_0.

By Laurent's theorem, if f has an isolated singularity at z_0 there is $r > 0$ such that f admits a Laurent expansion (6.5) on $C(z_0; 0, r)$:

$$\boxed{f(z) = \sum_{k=1}^{\infty} \frac{b_k}{(z - z_0)^k} + \sum_{k=0}^{\infty} a_k (z - z_0)^k, \qquad \text{if } 0 < |z - z_0| < r}\,.$$

i) If $b_k = 0$ for all $k \in \mathbb{N}$, we say that z_0 is a **removable singularity** of f.

ii) If $b_p \neq 0$ and $b_k = 0$ for all $k > p$, the point z_0 is a **pole of order** p of f.

iii) Finally, if there are infinitely many coefficients $b_k \neq 0$ we say that f has an **essential singularity** at z_0.

Definition 6.20. The series $\displaystyle\sum_{k=1}^{\infty} b_k (z - z_0)^{-k}$ is called the **principal part** of the Laurent series of f at z_0. The coefficient b_1 of the Laurent series is called the **residue** of f at z_0:

$$\boxed{\operatorname{Res}(f; z_0) = b_1}\,.$$

- If f has a removable singularity at z_0, there exists the limit

$$\lim_{z \to z_0} f(z) = a_0\,. \tag{6.9}$$

If we define $f(z_0) = a_0$, the function f is analytic on $D(z_0; r)$ (since the power series representing f on $0 < |z - z_0| < r$ converges on the latter disc). Conversely, if f is analytic on a punctured neighborhood of z_0 and equation (6.9) is satisfied, then f has a removable singularity at z_0. Indeed, (6.9) implies that

$$\boxed{\lim_{z \to z_0} \big[(z - z_0) f(z)\big] = 0}\,. \tag{6.10}$$

From the generalized Cauchy theorem it follows that

$$a_{-m} = \frac{1}{2\pi \mathrm{i}} \int_\gamma f(z)(z - z_0)^{m-1}\, \mathrm{d}z = 0\,, \qquad \forall m \in \mathbb{N}\,.$$

Thus f *has a removable singularity at* z_0 *if and only if* $f(z)$ *has a limit when* z *tends to* z_0. In fact, one can prove the slightly more general result:

Proposition 6.21. *Let* $z_0 \in \mathbb{C}$ *be an isolated singularity of* $f : \mathbb{C} \to \mathbb{C}$*. Then* f *has a removable singularity at* z_0 *if and only if the condition* (6.10) *is satisfied.*

Proof. Indeed, if f has a removable singularity at z_0 it has a limit at z_0, which implies (6.10). Conversely, if this condition is satisfied then the generalized Cauchy theorem implies that all the coefficients a_k with $k < 0$ of the Laurent expansion of f centered at z_0 are zero. $\square$

Example 6.22. The function $f(z) = \sin z / z$, defined for all $z \neq 0$, has a removable singularity at the origin. Indeed, although formally f is not defined (and thus is not analytic) at $z = 0$, the condition (6.10) is satisfied, since $\lim_{z \to 0} [z f(z)] = \lim_{z \to 0} \sin z = 0$. The Laurent series of f on the annulus of analyticity $C(0; 0, \infty)$ can be easily computed from the Taylor series of $\sin z$:

$$f(z) = \frac{1}{z} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k+1)!}\,, \qquad z \neq 0\,.$$

If we define $f(0) = \lim_{z \to 0} f(z) = 1$, the function f coincides with the sum of the previous series for all $z \in \mathbb{C}$, and is thus an entire function.

- By definition, f has a pole of order p at z_0 if and only if there is $r > 0$ such that

$$0 < |z - z_0| < r \Longrightarrow f(z) = \frac{1}{(z - z_0)^p} \bigg(b_p + b_{p-1}(z - z_0) + \cdots + b_1(z - z_0)^{p-1} + \sum_{k=0}^{\infty} a_k (z - z_0)^{k+p} \bigg) \equiv \frac{F(z)}{(z - z_0)^p},$$

where F is analytic on $D(z_0; r)$ and $F(z_0) = b_p \neq 0$. Thus *f has a pole of order p at z_0 if and only if $(z - z_0)^p f(z)$ has a removable singularity at z_0, and*

$$\boxed{\exists \lim_{z \to z_0} \big[(z - z_0)^p f(z)\big] \neq 0}\,. \tag{6.11}$$

In fact, one can prove a slightly more general result:

Proposition 6.23. *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic on a punctured neighborhood of z_0. Then f has a pole of order p at z_0 if and only if the condition (6.11) is satisfied.*

Proof. Indeed, as we have seen above the condition (6.11) certainly holds when f has a pole of order p at z_0. Conversely, if such a condition is satisfied then z_0 is an isolated singularity of f, since (6.11) clearly implies that f does not have a limit when $z \to z_0$, and thus it is not differentiable at this point. Moreover, applying Proposition 6.21 to the function $(z - z_0)^p f(z)$ it follows that $(z - z_0)^p f(z)$ has a removable singularity at z_0. The remark just before this proposition then implies that f has a pole of order p at z_0. ☐

- If f has a pole of order p at z_0, then

$$f(z) = \frac{F(z)}{(z - z_0)^p}, \qquad 0 < |z - z_0| < r,$$

where F is analytic on $D(z_0; r)$ and $F(z_0) = b_p \neq 0$. By continuity, there is $0 < \delta \leqslant r$ such that $F(z) \neq 0$ if $|z - z_0| < \delta$, so that

$$\frac{1}{f(z)} = (z - z_0)^p \frac{1}{F(z)}, \qquad 0 < |z - z_0| < \delta,$$

with $1/F$ analytic and nonvanishing on $D(z_0; \delta)$. Thus $1/f$ has a removable singularity (zero of order p) at z_0. Conversely (see Section 6.1.5) if f has a zero of order $p > 0$ at z_0 then $1/f$ has a pole of order p at z_0. Hence:

Proposition 6.24. *f has a pole of order p at z_0 if and only if the function*

$$g(z) = \begin{cases} \frac{1}{f(z)}, & z \neq z_0 \\ 0, & z = z_0 \end{cases}$$

has a zero of order p at z_0.

Example 6.25. Consider the function $f(z) = \csc^2 z$, analytic at $z \neq k\pi$, with $k \in \mathbb{Z}$. In this case the function $g(z) = \sin^2 z$ has a double zero at each of the (obviously isolated) singularities $z = k\pi$ of f, since $\sin z$ has a simple zero[1] at these points (for $\sin(k\pi) = 0$, $\cos(k\pi) = (-1)^k \neq 0$). Thus f has a double pole at each of the points $z = k\pi$, with $k \in \mathbb{Z}$.

[1] It is immediate to show that when $h(z)$ has a simple zero at a point z_0, then for any $n \in \mathbb{N}$ the function $h(z)^n$ has a zero of order n at this point.

• Assume that $f = g/h$, where g and h are analytic functions at z_0 and with zeros of order $n \geqslant 0$ and $m \geqslant 1$, respectively, at that point. If g and h do not vanish identically on a neighborhood of z_0, as we have seen in Section 6.1.5 there exists $r > 0$ such that

$$|z - z_0| < r \quad \Longrightarrow \quad g(z) = (z - z_0)^n G(z)\,, \quad h(z) = (z - z_0)^m H(z)\,,$$

with G and H analytic and non-vanishing on $D(z_0; r)$. In particular, since $h(z) \neq 0$ on a punctured neighborhood of z_0, this point is an isolated singularity of f. Using the previous expressions for g and h we obtain

$$0 < |z - z_0| < r \Longrightarrow f(z) = \frac{g(z)}{h(z)} = \frac{(z - z_0)^n G(z)}{(z - z_0)^m H(z)} \equiv (z - z_0)^{n-m} R(z),$$

with $R \equiv G/H$ analytic (quotient of analytic functions with $H(z) \neq 0$) and nonzero at z_0 (since $G(z_0) \neq 0$). Then:

i) If $n \geqslant m$, f has a removable singularity (zero of order $n - m$) at z_0.

ii) If $n < m$, f has a pole of order $m - n$ at z_0.

From the above discussion it follows that the *singularities of the quotient of two analytic functions which do not vanish identically must be either poles or removable singularities*

Proposition 6.26. *Let $f = g \cdot h$, with g analytic and nonzero at z_0, and assume that z_0 is an isolated singularity of h. Then z_0 is an isolated singularity of f, of the same type as is for h.*

Proof. Clearly f has an isolated singularity at z_0, since h analytic on $C(z_0; 0, r)$ $(r > 0)$ and g analytic on $D(z_0; r)$ imply that $f \equiv g \cdot h$ is analytic on $C(z_0; 0, r)$. Moreover, f *cannot* be differentiable at z_0, since in such a case $h = f/g$ would be differentiable at the latter point (quotient of differentiable functions at z_0 with a nonvanishing denominator at z_0).

If z_0 is a removable singularity of h then h coincides with an analytic function on a punctured neighborhood of z_0, and thus the same is true for f. Hence in this case f also has a removable singularity at z_0. On the other hand, if h has a pole of order p at z_0, then $h(z) = (z - z_0)^{-p} H(z)$, with H analytic on a neighborhood of z_0 and $H(z_0) \neq 0 \Longrightarrow f(z) = (z - z_0)^{-p} \cdot g(z)\, H(z) \equiv (z - z_0)^{-p} F(z)$, with $F = gH$ analytic on a neighborhood of z_0 and $F(z_0) = g(z_0) H(z_0) \neq 0 \Longrightarrow f$ has a pole of order p at z_0. Finally, if h has an essential singularity at z_0 then the same is true for f, since otherwise $h = \frac{1}{g} \cdot f$ would have a removable singularity or a pole at z_0 (notice that $1/g$ is analytic on a neighborhood of z_0, since $g(z_0) \neq 0$). □

Example 6.27. The function $f(z) = \mathrm{e}^{1/z}$ is analytic on $\mathbb{C} \setminus \{0\}$, and has an essential singularity at the origin. Indeed,

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{z^k}\,, \qquad \forall z \neq 0\,.$$

By the uniqueness of Laurent series, this is the Laurent expansion of f on the annulus $C(0; 0, \infty)$. In particular, since

$$b_k = \frac{1}{k!} \neq 0\,, \qquad \forall k = 1, 2, \dots\,,$$

$z = 0$ is an essential singularity of f. Consider next the function $f(z) = \mathrm{e}^{\cot z}$, analytic on $\mathbb{C}$ excepting the points $z = k\pi$ with $k \in \mathbb{Z}$. The latter points are simple poles of $\cot z = \cos z / \sin z$ (simple zeros of $\sin z$, while $\cos z$ does not vanish). Thus, on a punctured neighborhood of $k\pi$ we have

$$\cot z = \frac{c_k}{z - k\pi} + g_k(z)\,,$$

with $c_k \neq 0$ and g_k analytic on the latter neighborhood[2]. Consequently

$$e^{\cot z} = e^{\frac{c_k}{z-k\pi}}\, e^{g_k(z)}\,,$$

where e^{g_k} is analytic at $k\pi$ (composition of analytic functions) and nonzero at this point. From Proposition 6.26 it follows that f has an essential singularity at $k\pi$ for all $k \in \mathbb{Z}$.

- *If f has a pole at z_0 then*

$$\lim_{z\to z_0} |f(z)| = \infty; \tag{6.12}$$

in particular, $|f|$ is not bounded on a punctured neighborhood of z_0. However, (6.12) does *not* hold if f has an *essenctal* singularity at z_0. For instance, $f(z) = e^{1/z}$ has an essential singularity at the origin, and $f(z_n) = 1$ if $z_n = 1/(2n\pi i) \xrightarrow[n\to\infty]{} 0$, for all $n \in \mathbb{N}$. In fact, the following theorem implies that (6.12) *only holds if f has a pole* at z_0:

Casorati–Weierstrass theorem. *If f has an* essential singularity *at z_0 and $a \in \mathbb{C}$, there is a sequence $\{z_n\}_{n=1}^{\infty}$ such that $z_n \to z_0$ and $f(z_n) \to a$.*

Note: As a matter of fact, it can be proved (**Picard's big theorem**) that for any complex number a, with at most one exception, there is sequence $\{z_n\}_{n=1}^{\infty}$ such that $z_n \to z_0$ and $f(z_n) = a$ for all $n \in \mathbb{N}$ (cf. $f(z) = e^{1/z}$).

Exercise. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function satisfying $\lim_{|z|\to\infty} |f(z)| = \infty$. Show that f is a polynomial.

[2]By Laurent's theorem, the previous formula is valid if $0 < |z - k\pi| < \pi$.

Chapter 7

Evaluation of integrals using residues

7.1 Residue theorem

The following theorem is the basis for the application of the results of the previous two chapters to the computation of definite integrals on the real line:

Residue theorem. *Let $z_1, \ldots, z_n$ be n distinct points in a region A, and let γ be an arc homotopic to a point in A and such that no z_i lies on γ. If f is analytic on $A \setminus \{z_1, \ldots z_n\}$ then*

$$\int_\gamma f = 2\pi \mathrm{i} \sum_{k=1}^{n} n(\gamma, z_k)\ \mathrm{Res}(f; z_k).$$

Proof. By Laurent's theorem, for each $i = 1, \ldots, n$ there is a punctured neighborhood $C(z_i; 0, \varepsilon_i)$ of z_i on which f is represented by its Laurent series expansion

$$f(z) = \sum_{k=1}^{\infty} b_{ik}(z - z_i)^{-k} + \sum_{k=0}^{\infty} a_{ik}(z - z_i)^k \equiv S_i(z) + f_i(z), \qquad 0 < |z - z_i| < \varepsilon_i\,,$$

with f_i analytic on the disc $D(z_i; \varepsilon_i)$. Moreover, by the properties of the Laurent series the series defining the principal part $S_i(z)$ converges absolutely to an analytic function on $\mathbb{C} \setminus \{z_i\}$, the convergence being also *uniform* on the exterior of any open disc centered on z_i.

Let us next show that the function

$$g(z) = f(z) - \sum_{k=1}^{n} S_k(z)\,,$$

which clearly is analytic on $A \setminus \{z_1, \ldots, z_n\}$, has a removable singularity on the points z_i ($i = 1, \ldots, n$). Indeed, for each $i = 1, \ldots, n$ we have

$$g(z) = f_i(z) + S_i(z) - \sum_{k=1}^{n} S_k(z) = f_i(z) - \sum_{1 \leqslant k \neq i \leqslant n} S_k(z)\,,$$

on a sufficiently small punctured neighborhood of z_i. Defining $g(z_i) = \lim_{z \to z_i} g(z)$, the function g is thus *analytic* on all A, so that by Cauchy's theorem we have

$$\int_\gamma g = 0 \implies \int_\gamma f = \sum_{k=1}^{n} \int_\gamma S_k.$$

Consider now the integral $\int_\gamma S_k$. Since $\mathbb{C} \setminus \gamma$ is open (indeed, γ is compact, and thus closed, for it is the image of the compact interval $[a, b]$ under the continuous mapping which parametrizes the arc), there is $\delta_k > 0$ such that $D(z_k;\delta_k) \cap \gamma = \emptyset$. Therefore, the Laurent series defining S_k converges uniformly on γ, which by virtue of Lemma 6.7 yields

$$\int_\gamma S_k \equiv \int_\gamma \sum_{j=1}^\infty b_{kj}(z-z_k)^{-j}\,\mathrm{d}z = \sum_{j=1}^\infty \int_\gamma b_{kj}(z-z_k)^{-j}\,\mathrm{d}z = b_{k1}\cdot 2\pi\mathrm{i}n(\gamma,z_k)$$
$$\equiv 2\pi\mathrm{i}\cdot n(\gamma,z_k)\ \mathrm{Res}(f;z_k)\,,$$

since $\int_\gamma (z-z_k)^{-j} = 0$ for any integer $j \neq 1$ on account of the fundamental theorem of calculus. This completes the proof. □

7.2 Methods for calculating residues

• Let $f(z) = g(z)/h(z)$, with g, h analytic on a neighborhood of z_0, $g(z_0) \neq 0$, $h(z_0) = 0$ and $h'(z_0) \neq 0$. Then f has a *simple pole* at z_0 (simple zero of the denominator and nonvanishing numerator), with residue

$$\boxed{\mathrm{Res}(f;z_0) = \frac{g(z_0)}{h'(z_0)}}\,.$$

Indeed, by Taylor's theorem, on a neighborhood of z_0 we have

$$h(z) = \sum_{k=1}^\infty \frac{h^{(k)}(z_0)}{k!}(z-z_0)^k = (z-z_0)\sum_{k=1}^\infty \frac{h^{(k)}(z_0)}{k!}(z-z_0)^{k-1} \equiv (z-z_0)\,H(z),$$

with H analytic at z_0 (power series convergent on a neighborhood of z_0) and $H(z_0) = h'(z_0)$. Thus

$$f(z) = \frac{1}{(z-z_0)}\cdot\frac{g(z)}{H(z)}.$$

Since g/H is analytic at z_0 (for $H(z_0) = h'(z_0) \neq 0$), using again Taylor's theorem we obtain the expansion

$$\frac{g(z)}{H(z)} = \frac{g(z_0)}{H(z_0)} + \sum_{k=1}^\infty a_k(z-z_0)^k = \frac{g(z_0)}{h'(z_0)} + \sum_{k=1}^\infty a_k(z-z_0)^k\,,$$

so that

$$f(z) = \frac{g(z_0)}{h'(z_0)}\,\frac{1}{z-z_0} + \sum_{k=1}^\infty a_k\,(z-z_0)^{k-1}\,,$$

from where it follows that the residue of f at z_0 is given by $\dfrac{g(z_0)}{h'(z_0)}$, as claimed.

Example 7.1. Let us compute the integral

$$I = \int_{|z|=8} \tan z\ \mathrm{d}z\,.$$

The function $f(z) = \tan z$ is singular at the points $z_k = (2k+1)\frac{\pi}{2}$, with $k \in \mathbb{Z}$, none of which lies on the circle $|z| = 8$. Besides, in the interior of any open disc there are obviously a finite number of singularities of f, so that we can apply the residue theorem taking as the region A any open disc centered at the origin with radius greater than 8. In this way we obtain

$$I = 2\pi\mathrm{i}\sum_{|z_k|<8} \mathrm{Res}(\tan;z_k) = 2\pi\mathrm{i}\sum_{k=-3}^{2} \mathrm{Res}(\tan;z_k)\,,$$

since $\frac{5\pi}{2} < 8 < \frac{7\pi}{2}$. In order to compute the residue of tan at the singularity z_k, it suffices to note that z_k is a simple pole (for $\sin z_k \neq 0$ and $\cos'(z_k) = -\sin z_k \neq 0$), so that

$$\operatorname{Res}(\tan; z_k) = \frac{\sin z_k}{-\sin z_k} = -1\,.$$

Thus $I = 2\pi i \cdot (-6) = -12\pi i\,.$

- If f has a *pole of order* n at z_0 then

$$\boxed{\operatorname{Res}(f; z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right]}\,. \tag{7.1}$$

Indeed, on a punctured neighborhood of z_0 the function f is represented by the Laurent expansion

$$f(z) = \frac{b_n}{(z-z_0)^n} + \frac{b_{n-1}}{(z-z_0)^{n-1}} + \cdots + \frac{b_1}{z - z_0} + g(z),$$

where g is analytic at z_0 (convergent power series). Thus, on a punctured neighborhood of z_0 we have

$$(z-z_0)^n f(z) = b_n + b_{n-1}(z-z_0) + \cdots + b_1(z-z_0)^{n-1} + G(z) \equiv F(z), \tag{7.2}$$

where $G(z) = (z-z_0)^n g(z)$ is analytic at z_0 with a zero of order $\geqslant n$ at this point, and F is analytic at z_0. By Taylor's theorem,

$$\begin{aligned} \operatorname{Res}(f; z_0) = b_1 = \frac{F^{(n-1)}(z_0)}{(n-1)!} &= \frac{1}{(n-1)!} \lim_{z \to z_0} F^{(n-1)}(z) \\ &= \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right]. \end{aligned}$$

Note: from equation (7.2) it follows that $(z - z_0)^n f(z)$ has a removable singularity at z_0, so that the formula (7.1) is often written (with a slight abuse of notation) in the simpler form

$$\operatorname{Res}(f; z_0) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right] \Big|_{z=z_0}.$$

Exercise. If $f = g/h$ with g and h analytic at z_0, $h(z_0) = h'(z_0) = 0$ and $g(z_0) \neq 0$, $h''(z_0) \neq 0$, show that f has a *pole of order* 2 at z_0, with residue

$$\operatorname{Res}(f; z_0) = 2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3} \frac{g(z_0)\, h'''(z_0)}{[h''(z_0)]^2}.$$

Solution. The function f clearly has a double pole at z_0, so that we can apply the formula (7.1). By Taylor's theorem, on a punctured neighborhood of z_0 we have

$$\frac{h(z)}{(z-z_0)^2} = h_2 + h_3(z-z_0) + H(z)\,,$$

with

$$h_2 = \frac{1}{2} h''(z_0)\,, \qquad h_3 = \frac{1}{6} h'''(z_0) \tag{7.3}$$

and H analytic at z_0 with a zero of order at least 2 at this point. Applying (7.1) one thus obtains

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} \frac{d}{dz} \left[\frac{g(z)}{h_2 + h_3(z-z_0) + H(z)} \right] = \frac{h_2\, g'(z_0) - h_3\, g(z_0)}{h_2^2}\,.$$

Substituting in the latter equation h_2 and h_3 by the expressions (7.3) yields the proposed formula.

7.3 Evaluation of definite integrals

In this section we shall use the notation

$$\boxed{H = \{z \in \mathbb{C} : \operatorname{Im} z \geqslant 0\}, \qquad L = \{z \in \mathbb{C} : \operatorname{Im} z \leqslant 0\}}$$

to respectively denote the (closed) upper and lower half-planes.

7.3.1 $\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x$

- **Conditions:**

  i) f analytic on $H \setminus \{z_1, \dots, z_n\}$, with $z_k \in H \setminus \mathbb{R}$ (i.e., f has at most a *finite* number of singularities in H, none of which can lie on the *real axis*)

  ii) $\exists p > 1$, $R > 0$ y $M > 0$ such that

$$|f(z)| < \frac{M}{|z|^p}, \qquad \forall z \in H, \ \ |z| > R$$

- **Result:** $\boxed{2\pi \mathrm{i} \sum_{k=1}^{n} \operatorname{Res}(f; z_k)}$.

(Note that the sum is extended to the singularities of f in the upper half-plane H.)

Proof. Let γ_r be the positively oriented half-circle of radius r, with $r > R$ large enough so that all the singularities of f in H are in the interior of γ_r (see fig. 7.1).

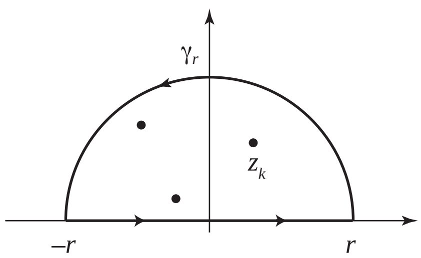

Figure 7.1: half-circle γ_r

Since $n(\gamma_r, z_k) = 1$ for $k = 1, \dots, n$, by the residue theorem it follows that

$$\int_{\gamma_r} f = 2\pi \mathrm{i} \sum_{k=1}^{n} \operatorname{Res}(f; z_k) = \int_{-r}^{r} f(x)\ \mathrm{d}x + \int_0^{\pi} f(r\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{i} r \mathrm{e}^{\mathrm{i}\theta}\ \mathrm{d}\theta. \tag{7.4}$$

Since $|f(x)| < M\,|x|^{-p}$ with $p > 1$ for $|x| > R$, the first integral on the right-hand side converges to $\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x$ when $r \to \infty$ (comparison test). As to the second one, its modulus is bounded by $M\pi r^{1-p}$, which tends to 0 as $r \to \infty$. Taking the $r \to \infty$ limit in (7.4) we obtain the above mentioned result.

□

- **Notes.:**

  i) If f is analytic on $L \setminus \{z_1, \ldots, z_n\}$, with $z_k \in L \setminus \mathbb{R}$, and condition ii) in the previous page holds on the lower half-plane L, then

$$\boxed{\int_{-\infty}^{\infty} f(x)\, \mathrm{d}x = -2\pi\mathrm{i} \sum_{k=1}^{n} \mathrm{Res}(f; z_k)}\,.$$

  The minus sign is due to the fact that in this case we have to integrate f along the semicircle of center 0 and radius r in the lower half-plane traversed in the *clockwise* direction, and thus $n(\gamma_r, z_k) = -1$.

  ii) If $f = P/Q$, with $P \neq 0$ and Q polynomials such that $Q(x) \neq 0$ for all $x \in \mathbb{R}$, then f satisfies the previous conditions (both on H and L) if and only if $\deg Q \geqslant \deg P + 2$.

- **Example:** $\displaystyle\int_{-\infty}^{\infty} \frac{x\, \mathrm{d}x}{(x^2 + 4x + 13)^2}$.

In this example

$$f(z) = \frac{z}{(z^2 + 4z + 13)^2} \equiv \frac{P(z)}{Q(z)}\,,$$

with singularities (double poles) at the zeros $z = -2 \pm 3\mathrm{i} \notin \mathbb{R}$ of the denominator Q. Since $\deg Q = 4 \geqslant \deg P + 2 = 3$, we have

$$I \equiv \int_{-\infty}^{\infty} \frac{x\, \mathrm{d}x}{(x^2 + 4x + 13)^2} = 2\pi\mathrm{i}\, \mathrm{Res}(f; -2 + 3\mathrm{i}).$$

The function f has a double pole at $z_0 \equiv -2 + 3\mathrm{i}$, with residue (cf. (7.1))

$$\frac{\mathrm{d}}{\mathrm{d}z}\Big[(z - z_0)^2 f(z)\Big]\Big|_{z=z_0} = \frac{\mathrm{d}}{\mathrm{d}z} \frac{z}{(z + 2 + 3\mathrm{i})^2}\Big|_{z=-2+3\mathrm{i}} = \frac{1}{(6\mathrm{i})^2} - \frac{2(-2 + 3\mathrm{i})}{(6\mathrm{i})^3} = \frac{4}{(6\mathrm{i})^3} = \frac{4\mathrm{i}}{6^3}\,.$$

Thus $I = -\dfrac{8\pi}{6^3} = -\dfrac{\pi}{27}$.

7.3.2 Trigonometric integrals: $\displaystyle\int_0^{2\pi} R(\cos\theta, \sin\theta)\, \mathrm{d}\theta$

- **Conditions:** $R(x, y)$ rational function of two variables with a nonvanishing denominator on the unit circle $x^2 + y^2 = 1$.

- **Result:** $\boxed{2\pi\mathrm{i} \displaystyle\sum_{|z_k|<1} \mathrm{Res}(f; z_k)}$, where

$$f(z) = \frac{1}{\mathrm{i}z} R\left(\frac{1}{2}\left(z + z^{-1}\right), \frac{1}{2\mathrm{i}}\left(z - z^{-1}\right)\right)$$

and z_k are the singularities of f (necessarily in finite number, since f is a rational function).

Proof. The function $f(z)$ has no singularities on the unit circle γ, since $f(\mathrm{e}^{\mathrm{i}\theta}) = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta} R(\cos\theta, \sin\theta)$ for $\theta \in [0, 2\pi)$. If we parametrize $\int_\gamma f$ in the usual way as $z = \mathrm{e}^{\mathrm{i}\theta}$, we obtain

$$\int_\gamma f = \int_0^{2\pi} R(\cos\theta, \sin\theta)\, \mathrm{d}\theta\,.$$

The result then follows from the residue theorem, since f (being a rational function of z) has a finite number of singularities inside the unit circle. □

- **Example:** $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{(5-3\sin\theta)^2}$.

In this case

$$f(z) = \frac{1}{\mathrm{i}z\left[5-\frac{3}{2\mathrm{i}}\left(z-\frac{1}{z}\right)\right]^2} = \frac{4\mathrm{i}z}{(3z^2-10\mathrm{i}z-3)^2} = \frac{4\mathrm{i}z}{9\left(z-\frac{\mathrm{i}}{3}\right)^2(z-3\mathrm{i})^2} .$$

Thus the value of the integral is given by

$$I = -\frac{8\pi}{9}\operatorname{Res}\left(g;\frac{\mathrm{i}}{3}\right), \quad \text{with} \quad g(z) = \frac{z}{(z-3\mathrm{i})^2}\cdot\frac{1}{\left(z-\frac{\mathrm{i}}{3}\right)^2} \equiv \frac{h(z)}{\left(z-\frac{\mathrm{i}}{3}\right)^2} ,$$

so that

$$\operatorname{Res}\left(g;\frac{\mathrm{i}}{3}\right) = h'(\mathrm{i}/3) = \frac{1}{\left(\frac{\mathrm{i}}{3}-3\mathrm{i}\right)^2} - \frac{\frac{2\mathrm{i}}{3}}{\left(\frac{\mathrm{i}}{3}-3\mathrm{i}\right)^3} = \frac{\frac{\mathrm{i}}{3}-3\mathrm{i}-\frac{2\mathrm{i}}{3}}{\left(\frac{\mathrm{i}}{3}-3\mathrm{i}\right)^3} = \frac{-\frac{10\mathrm{i}}{3}}{-\frac{8^3}{3^3}\mathrm{i}^3} = -\frac{10\cdot 3^2}{8^3} .$$

Hence $I = \dfrac{10\pi}{8^2} = \dfrac{5\pi}{32}$.

7.3.3 Fourier transforms: $\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega x} f(x)\, \mathrm{d}x$

- **Conditions:**

  i) $\omega > 0$

  ii) f analytic on $H \setminus \{z_1,\ldots,z_n\}$, with $z_k \in H \setminus \mathbb{R}$ (that is, f has at most a *finite* number of singularities on the upper half-plane, none of which lying on the real axis)

  iii) $|f(z)| \to 0$ as $|z| \to \infty$ on H, that is

$$\forall\varepsilon > 0,\ \exists R > 0 \ \text{ such that } \ |z| > R,\ z \in H \Longrightarrow |f(z)| < \varepsilon$$

- **Result:** $\boxed{2\pi\mathrm{i}\sum_{k=1}^{n}\operatorname{Res}\left(\mathrm{e}^{\mathrm{i}\omega z}f(z);z_k\right)}$.

(Again, the sum runs over the singularities of f on the upper half-plane H.)

Proof. Given $\varepsilon > 0$, let γ be the (positively oriented) rectangle with vertices at $-x_1, x_2, x_2+\mathrm{i}y_1, -x_1+\mathrm{i}y_1$, with x_1, x_2, y_1 greater than R and large enough so that all the singularities of f on H lie inside γ (cf. fig. 7.2).

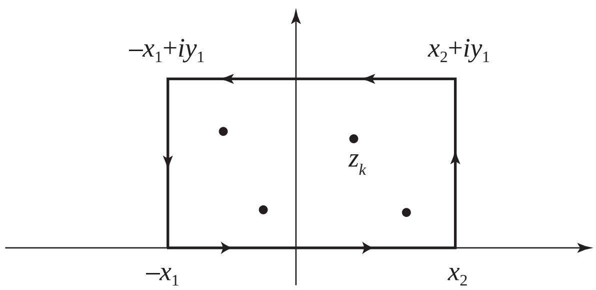

Figure 7.2: rectangle γ

Then

$$\begin{aligned}
\int_\gamma e^{i\omega z} f(z)\,dz &= 2\pi i \sum_{k=1}^{n} \operatorname{Res}\left(e^{i\omega z} f(z); z_k\right) \\
&= \int_{-x_1}^{x_2} e^{i\omega x} f(x)\,dx + i\int_0^{y_1} e^{i\omega(x_2+iy)} f(x_2+iy)\,dy \\
&\quad - \int_{-x_1}^{x_2} e^{i\omega(x+iy_1)} f(x+iy_1)\,dx - i\int_0^{y_1} e^{i\omega(-x_1+iy)} f(-x_1+iy)\,dy \\
&\equiv I_1 + I_2 - I_3 - I_4 .
\end{aligned}$$

If y_1 is chosen large enough so that $(x_1+x_2)e^{-\omega y_1} < 1/\omega$ we have

$$\begin{aligned}
|I_2| &\leqslant \varepsilon \int_0^{y_1} e^{-\omega y}\,dy = \frac{\varepsilon}{\omega}(1-e^{-\omega y_1}) < \frac{\varepsilon}{\omega}, \\
|I_3| &\leqslant \varepsilon\,(x_1+x_2)\,e^{-\omega y_1} < \frac{\varepsilon}{\omega}, \\
|I_4| &\leqslant \frac{\varepsilon}{\omega}(1-e^{-\omega y_1}) < \frac{\varepsilon}{\omega} .
\end{aligned}$$

Thus

$$\left| \int_{-x_1}^{x_2} e^{i\omega x} f(x)\,dx - 2\pi i \sum_{k=1}^{n} \operatorname{Res}\left(e^{i\omega z} f(z); z_k\right) \right| < \frac{3\varepsilon}{\omega} .$$

Since $\varepsilon > 0$ is arbitrary, if x_1 and x_2 tend separately to infinity it is shown that the integral converges to the above stated result. □

- **Notes:**

  i) If $\omega < 0$ and f satisfies conditions analogous to ii)–iii) on the *lower* half-plane L, it can likewise be shown that

  $$\boxed{\int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx = -2\pi i \sum_{k=1}^{n} \operatorname{Res}\left(e^{i\omega z} f(z); z_k\right)} ,$$

  where $z_1, \ldots, z_n$ are the singularities of f on L.

  ii) If $f = P/Q$, with $P \neq 0$ and Q polynomials with $Q(x) \neq 0$ for all $x \in \mathbb{R}$, the previous conditions on f are satisfied (both on H and L) if and only if $\deg Q \geqslant \deg P + 1$.

- **Example:** $\displaystyle\int_0^\infty \frac{\cos(\omega x)}{x^4+x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(\omega x)}{x^4+x^2+1}\,dx \equiv I(\omega),\ \ \omega > 0 .$

The integral is the real part of

$$J(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x^4+x^2+1}\,dx .$$

In fact, since $\sin(\omega x)$ is an odd function $\operatorname{Im} J(\omega) = 0$, and thus $I(\omega) = J(\omega)$. We can apply the previous result to the rational function $f(z) = \frac{1}{2}(z^4+z^2+1)^{-1}$ on the upper half-plane (f has no singularities on the real axis). The singularities (poles) of f are the zeros of the quartic equation

$$z^4 + z^2 + 1 = 0 \iff z^2 = \frac{1}{2}(-1 \pm i\sqrt{3}) = e^{\pm\frac{2\pi i}{3}} \iff z = \pm e^{\pm\frac{\pi i}{3}} .$$

The only singularities on the upper half-plane are

$$z_1 = e^{\frac{\pi i}{3}} = \frac{1}{2}(1+i\sqrt{3}), \qquad z_2 = -e^{-\frac{\pi i}{3}} = \frac{1}{2}(-1+i\sqrt{3}) = -\bar{z}_1 .$$

The residue of $e^{i\omega z} f(z)$ in any of these singularities z_k can be easily computed, since $e^{i\omega z} f(z) = g(z)/h(z)$ with $g(z_k) \neq 0$, $h(z_k) = 0$ and $h'(z_k) \neq 0$:

$$\operatorname{Res}(f; z_k) = \frac{1}{4}\,\frac{e^{i\omega z_k}}{z_k(2z_k^2 + 1)}\,.$$

Thus

$$I = \frac{\pi i}{2}\left[\frac{e^{i\omega z_1}}{z_1(2z_1^2+1)} - \frac{e^{-i\omega \bar{z}_1}}{\bar{z}_1(2\bar{z}_1^2+1)}\right] = -\pi \operatorname{Im}\left[\frac{e^{i\omega z_1}}{z_1(2z_1^2+1)}\right] \equiv -\pi \operatorname{Im} A.$$

Since

$$A = \frac{2e^{\frac{i\omega}{2}(1+i\sqrt{3})}}{(1+i\sqrt{3})\,i\sqrt{3}} = \frac{2e^{\frac{\omega}{2}(i-\sqrt{3})}}{\sqrt{3}\,(i-\sqrt{3})} = -\frac{i+\sqrt{3}}{2\sqrt{3}}\,e^{\frac{\omega}{2}(i-\sqrt{3})},$$

we finally obtain

$$I = -\pi \operatorname{Im} A = \frac{\pi}{2\sqrt{3}} e^{-\frac{\sqrt{3}}{2}\omega}\left(\cos\frac{\omega}{2} + \sqrt{3}\,\sin\frac{\omega}{2}\right), \qquad \omega > 0\,.$$

7.3.4 Cauchy principal value

Assume that $f : \mathbb{R} \to \mathbb{R}$ is not bounded on a neighborhood of $x_0 \in \mathbb{R}$, and that the improper integrals $\int_{-\infty}^{b} f(x)\,dx$ and $\int_{c}^{\infty} f(x)\,dx$ are convergent for all $b < x_0 < c$. In this case we define the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ as

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\varepsilon\to 0+}\int_{-\infty}^{x_0-\varepsilon} f(x)\,dx + \lim_{\delta\to 0+}\int_{x_0+\delta}^{\infty} f(x)\,dx\,.$$

Clearly, if this improper integral exists then

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\varepsilon\to 0+}\left[\int_{-\infty}^{x_0-\varepsilon} f(x)\,dx + \int_{x_0+\varepsilon}^{\infty} f(x)\,dx\right].$$

The right-hand side of the latter expression is called the **Cauchy principal value** of the improper integral, and shall be denoted as

$$\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\varepsilon\to 0+}\left[\int_{-\infty}^{x_0-\varepsilon} f(x)\,dx + \int_{x_0+\varepsilon}^{\infty} f(x)\,dx\right].$$

(This definition can be generalized in an obvious way to the case where f has a finite number of singularities on the real axis). Therefore, *if* the improper integral exists then its principal value also exists, and the equality

$$\int_{-\infty}^{\infty} f(x)\,dx = \mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx$$

is then satisfied. Note, however, that the Cauchy principal value may exist when the improper integral does not. For instance, if f is an *odd* function singular at $x_0 = 0$ but otherwise integrable at $\pm\infty$ then $\mathrm{PV}\int_{-\infty}^{\infty} f(x)\,dx = 0$.

Lemma 7.2. *Assume that f is an analytic function with a* simple pole *at $z_0 \in \mathbb{C}$, and let γ_ε be the arc of a circle $\gamma_\varepsilon(t) = z_0 + \varepsilon\,e^{it}$, with $t \in [t_0, t_0 + \alpha]$ (fig. 7.3). Then*

$$\lim_{\varepsilon\to 0+}\int_{\gamma_\varepsilon} f = i\alpha\,\operatorname{Res}(f; z_0).$$

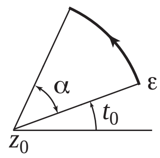

Figure 7.3: curve γ_ε

Proof. From Laurent's theorem, it follows that

$$f(z) = \frac{b_1}{z - z_0} + g(z), \qquad 0 < |z - z_0| < r,$$

with g analytic on $D(z_0; r)$, so that $|g(z)| < M$ for $|z - z_0| \leqslant r/2$. If $0 < \varepsilon \leqslant r/2$ we have

$$\int_{\gamma_\varepsilon} f = b_1 \int_{\gamma_\varepsilon} \frac{dz}{z - z_0} + \int_{\gamma_\varepsilon} g\,.$$

But

$$b_1 \int_{\gamma_\varepsilon} \frac{dz}{z - z_0} = b_1 \int_{t_0}^{t_0+\alpha} \frac{i\varepsilon e^{it}}{\varepsilon e^{it}}\, dt = i b_1 \alpha = i\alpha \operatorname{Res}(f; z_0),$$

whereas

$$\left| \int_{\gamma_\varepsilon} g \right| \leqslant M\varepsilon\alpha \xrightarrow[\varepsilon \to 0+]{} 0\,.$$

□

- Assume that $f : \mathbb{C} \to \mathbb{C}$ satisfies:

  i) f is analytic on $H \setminus \{z_1, \dots, z_n\}$, with $z_k \in H$ (i.e., f has at most a *finite* number of singularities on the upper half-plane), and the possible singularities of f on the real axis are all *simple poles.*

In addition, one of the following two conditions holds:

ii) $\exists p > 1,\ R > 0,\ M > 0$ such that $|f(z)| < \dfrac{M}{|z|^p}$ if $|z| > R$ and $z \in H$;

ii′) $f(z) = e^{i\omega z} g(z)$, with $\omega > 0$ and $|g(z)| \to 0$ when $|z| \to \infty$ on H.

Then PV $\int_{-\infty}^{\infty} f(x)\, dx$ exists and is given by

$$\boxed{\operatorname{PV} \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}(f; z_k) + \pi i \sum_{z_k \in \mathbb{R}} \operatorname{Res}(f; z_k)}\,.$$

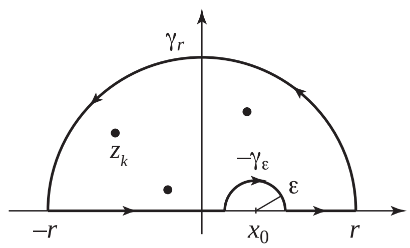

Figure 7.4: curve γ

Proof. Assume, for instance, that f satisfies conditions i) y ii). For simplicity, we shall restrict ourselves to the case in which f has a single singularity x_0 on the real axis. If $r > \max(|x_0|, R)$ is sufficiently large and $\varepsilon > 0$ is small enough so that all the singularities of f on $H - \{x_0\}$ lie on the interior of the curve γ in fig. 7.4, integrating f along the latter curve we have

$$\int_\gamma f = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}(f; z_k) = \int_{-r}^{x_0-\varepsilon} f(x)\, dx - \int_{\gamma_\varepsilon} f + \int_{x_0+\varepsilon}^{r} f(x)\, dx + \int_{\gamma_r} f\,.$$

By the discussion in Section 7.3.1, the integrals $\int_{-\infty}^{x_0-\varepsilon} f(x)\, dx$ and $\int_{x_0+\varepsilon}^{\infty} f(x)\, dx$ are convergent, and

$$\lim_{r\to\infty} \int_{\gamma_r} f = 0.$$

Taking the limit $r \to \infty$ one thus obtains

$$2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}(f; z_k) = \int_{-\infty}^{x_0-\varepsilon} f(x)\, dx + \int_{x_0+\varepsilon}^{\infty} f(x)\, dx - \int_{\gamma_\varepsilon} f\,.$$

The result then follows by making $\varepsilon \to 0+$ and using the previous lemma with $\alpha = \pi$. □

- **Note:** If we replace H by L and $\omega > 0$ by $\omega < 0$ in the previous conditions, then

$$\boxed{\operatorname{PV}\int_{-\infty}^{\infty} f(x)\, dx = -2\pi i \sum_{\operatorname{Im} z_k < 0} \operatorname{Res}(f; z_k) - \pi i \sum_{z_k \in \mathbb{R}} \operatorname{Res}(f; z_k)}\,.$$

- **Example:** $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx \equiv I\,.$

If we define $f(x) = \sin x / x$ for $x \neq 0$ and $f(0) = 1$ then f is continuous at 0 and even, so that

$$I = \frac{1}{2} \int_{-\infty}^{\infty} f(x)\, dx\,.$$

This integral is *not* of the type studied in Section 7.3.1, since $|\sin z| = \left(\cosh^2 y - \cos^2 x\right)^{1/2} \to \infty$ faster than any power of $|z|$ when $|y| \to \infty$. Neither the relation

$$I = \frac{1}{2} \operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx,$$

is satisfied, since the real part of the integral on the right-hand side is clearly divergent at the origin (the integrand behaves at this point as $1/x$). However,

$$\operatorname{PV}\int_{-\infty}^{\infty} \frac{\cos x}{x}\, dx = 0$$

for $\cos x$ is even, while

$$\operatorname{PV}\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx\,,$$

due to the convergence of the integral on the right-hand side. Thus

$$I = \frac{1}{2i} \operatorname{PV}\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx\,.$$

The function $g(z) = e^{iz}/z$ has a simple pole at the origin and satisfies condition ii′) above (with $\omega = 1 > 0$), so that

$$I = \frac{\pi}{2} \operatorname{Res}\left(\frac{e^{iz}}{z}; 0\right) = \frac{\pi}{2}\,.$$

• **Example:** $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\, \mathrm{d}x \equiv I$.

In this case

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, \mathrm{d}x = \frac{1}{4}\int_{-\infty}^{\infty} \frac{1-\cos(2x)}{x^2}\, \mathrm{d}x = \frac{1}{4}\,\mathrm{PV}\int_{-\infty}^{\infty} \frac{1-\mathrm{e}^{2\mathrm{i}x}}{x^2}\, \mathrm{d}x\,,$$

since

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^2}\, \mathrm{d}x = 0$$

since the integrand is an odd function. If $g(z) = (1-\mathrm{e}^{2\mathrm{i}z})/z^2$ then

$$|g(z)| \leqslant \frac{1+\left|\mathrm{e}^{2\mathrm{i}z}\right|}{|z|^2} = \frac{1+\mathrm{e}^{-2\,\mathrm{Im}\,z}}{|z|^2} \leqslant \frac{2}{|z|^2}, \qquad \mathrm{Im}\,z \geqslant 0,\ \ z \neq 0,$$

and thus condition ii) is satisfied on the upper half-plane. Moreover, $z = 0$ is a simple pole of g (the numerator has a simple zero and the denominator a double one at the origin), with residue

$$\mathrm{Res}(g;0) = \frac{\mathrm{d}}{\mathrm{d}z}\big(1-2\mathrm{e}^{\mathrm{i}z}\big)\Big|_{z=0} = -2\mathrm{i}\mathrm{e}^{\mathrm{i}z}\big|_{z=0} = -2\mathrm{i}.$$

Therefore,

$$I = \frac{1}{4}\cdot \pi\mathrm{i}\cdot(-2\mathrm{i}) = \frac{\pi}{2}\,.$$

• **Example:** $\displaystyle\mathrm{PV}\int_{-\infty}^{\infty} \frac{\sin x\ \mathrm{d}x}{(x-1)(x^2+4)} \equiv I$.

Here

$$I = \mathrm{Im}\,J, \qquad J \equiv \mathrm{PV}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}\ \mathrm{d}x}{(x-1)(x^2+4)}\,.$$

The function

$$f(z) = \frac{\mathrm{e}^{\mathrm{i}z}}{(z-1)(z^2+4)}$$

is analytic on $\mathbb{C}\setminus\{1,\pm 2\mathrm{i}\}$, and the singularity at $z = 1$ is clearly a simple pole. Besides, condition ii′) is clearly satisfied on the upper half-plane ($\omega = 1 > 0$), so that

$$\begin{aligned} J &= \pi\mathrm{i}\,[\mathrm{Res}(f;1) + 2\,\mathrm{Res}(f;2\mathrm{i})] = \pi\mathrm{i}\left[\frac{\mathrm{e}^{\mathrm{i}}}{5} + \frac{2\mathrm{e}^{-2}}{(2\mathrm{i}-1)\cdot 4\mathrm{i}}\right] = \pi\mathrm{i}\left[\frac{\mathrm{e}^{\mathrm{i}}}{5} - \frac{\mathrm{e}^{-2}}{2(2+\mathrm{i})}\right] \\ &= \pi\mathrm{i}\left[\frac{\mathrm{e}^{\mathrm{i}}}{5} - \frac{(2-\mathrm{i})\mathrm{e}^{-2}}{10}\right]. \end{aligned}$$

Thus

$$I = \frac{\pi}{5}\left(\cos 1 - \frac{1}{\mathrm{e}^2}\right).$$

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