Bethe–Salpeter studies of mesons beyond rainbow-ladder

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C. S. Fischer and RW, Phys. Rev. D 78 2008 074006,
C. S. Fischer and RW, Phys. Rev. Lett. 103 2009 122001,
C. S. Fischer and RW,

[arXiv:0808.3372]
[arXiv:0905.2291]
in preparation

4. November 2009
1. Introduction
2. The Old
3. The New
4. Results
5. Next steps and Conclusions
Quantum Chromodynamics

\[ Z[J, \eta, \bar{\eta}] = \int \mathcal{D}[A, \psi, \bar{\psi}] \exp \left\{ - \int_x \bar{\psi}(\not{D} + m)\psi + \frac{1}{4} F_{\mu\nu}^2 + \int_x A_\mu^a J_\mu^a + \bar{\eta}\psi + \eta\bar{\psi} \right\} \]

**Strong interaction described by QCD**

- Gauge theory with gauge group $SU(3)$
- non-abelian – self-interactions between gauge fields.
- Degrees of freedom are quarks and gluons

**Exhibits asymptotic freedom**

- weak-coupling at high momenta
- \( \rightarrow \) perturbation theory (PT) applicable

**Confinement**

- Strong coupling for small momenta (no PT)
- Physical observables colourless bound-states
- mesons, baryons, ... glueballs, exotics.
Quantum Chromodynamics

Need non-perturbative tools

- lattice QCD
- effective theories (NJL, χPT)
- functional renormalisation group
- Dyson-Schwinger equations

Why use Dyson-Schwinger equations??

- *ab initio*
- continuum approach (IR accessible)
- (necessarily) gauge-fixed → explore confinement mechanisms
- Bound-states via Bethe-Salpeter/Faddeev equations
  - complementary to other approaches
- DSE requires truncation – errors difficult to quantify and control
Dyson-Schwinger equations

\[ \int \mathcal{D}\varphi \left[ -\frac{\delta S}{\delta \varphi} + J \right] \exp \left( -S + \int J\varphi \right) = 0 \]

- Derive from observation that integral of total derivative vanishes.
- Quantum analogue of classical equations of motion.

\[ \Phi_i = \left\{ \hat{A}_\mu, \hat{\Psi}, \hat{\bar{\Psi}}, \ldots \right\} \text{ superfield} \quad \Gamma[\Phi_i] \text{ effective action} \]

\[ S[\Phi_i] \text{ classical action} \]

Provides (infinite) set of coupled integral equations for the Euclidean Green’s functions.

\[ \left\{ \frac{\delta \Gamma[\Phi_i]}{\delta \Phi_i} - \frac{\delta S[\Phi_i]}{\delta \Phi_i} \right\} \left[ \Phi_i \rightarrow \Phi_i + \left( \frac{\delta^2 \Gamma}{\delta \Phi_i \delta \Phi_j} \right)^{-1} \frac{\delta}{\delta \Phi_j} \right] = 0 . \]

[ F. J. Dyson, Phys. Rev. 75 (1949) 1736. ]

[ J. S. Schwinger, Proc. Nat. Acad. Sci. 37 (1951) 452. ]


Dyson-Schwinger equations – quark propagator

\[ S^{-1} (p; \mu) = Z_2 (\mu^2, \Lambda^2) S_0^{-1} (p; \mu) + \Sigma (p, \mu, \Lambda) \]

\[ \Sigma (p, \mu, \Lambda) = -C_F g^2 Z_1 F (\mu^2, \Lambda^2) \int_q^\Lambda i \gamma^\mu S (q; \mu) D^{\mu \nu} (k; \mu) \Gamma^\nu (p, k; \mu) \]

- \( S(p; \mu) \) quark propagator
- \( \Sigma(p, \mu, \Lambda) \) quark self-energy
- \( D^{\mu \nu} (p; \mu) \) gluon propagator
- \( \Gamma^\mu (k, p; \mu) \) quark-gluon vertex.
- \( Z_1 F, Z_2 \) renormalisation constants.

\[ S^{-1}(p; \mu) = -i \not{p} A (p^2; \mu^2) + B (p^2; \mu^2) \]

\[ D^{\mu \nu} (p; \mu) = \left( \delta^{\mu \nu} - \frac{p^\mu p^\nu}{p^2} \right) \frac{Z (p^2; \mu^2)}{p^2} \]
Dyson-Schwinger equations – quark propagator

Truncation necessary – gluon, quark-gluon vertex.

- Weak-coupling expansion reproduces all diagrams of perturbation theory
  \[
  m(\mu^2) = 0 \implies B(p^2; \mu^2) / A(p^2; \mu^2) = 0
  \]

- Dynamical mass generation is non-perturbative.
Dyson-Schwinger equations – gluon propagator

\[
\begin{align*}
\begin{array}{c}
\text{Quark loops neglected} \\
\text{Faddeev-Popov → ghosts (and ghost DSE)} \\
\text{Landau gauge – truncating is simpler} \\
\text{Solvable and good agreement with Lattice!}
\end{array}
\end{align*}
\]

IR scaling/decoupling a moot point – irrelevant for mesons.
Dyson-Schwinger equations – quark-gluon vertex

\[ \Gamma^\mu(k, p; \mu) = \sum_{i=1}^{4} \left( f_i^{(1)} \gamma^\mu + f_i^{(2)} k^\mu + f_i^{(3)} p^\mu \right) \tau_i(k, p) \]

\[ \tau_i(k, p) = \{ \mathbb{1}, \phi , k , [k , \phi ] \} \]

- \( f_i^{(j)}(k^2, p^2, k \cdot p; \mu^2) \) scalar dressing functions
- Twelve covariants in total.
- Non-trivial momentum dependence

Tree level, \( \Gamma^\mu(k, p, \mu) = \gamma^\mu \) or:

\[ f_i^{(j)} = \begin{cases} 
1 & \text{if } i = j = 1 \\
0 & \text{otherwise}
\end{cases} \]
Bethe-Salpeter equations

- connect gauge-dependent Green’s functions to physical observables
- describe colourless bound-states in terms of quarks and gluons

appear in quark 4-pt function, $G$ or in amputated connected part $T$

\[ G = G_0 + G_0 T G_0 \]

where $G_0$ is a product of 2 dressed quark propagators.

Relate $T$ matrix to 2-quark scattering kernel $K$ by Dyson sum

\[ T = K + KG_0K + KG_0KG_0K + \cdots \]

In integral notation (Dyson equation):

\[ T = K (1 + G_0 T) \]
Bethe-Salpeter equations

Introduce bound state amplitude $\Psi$ as residues of scattering matrix:

$$T \leftarrow M^2 \rightarrow P^2 \frac{\Psi \overline{\Psi}}{P^2 + M^2}$$

- insert into Dyson eq and compare residues
- gives bethe-salpeter equation at pole $P^2 = -M^2$.
- normalisation from $T' = -T \left( T^{-1} \right)' T$

$$\overline{\Psi} \left[ \mathcal{N} \frac{d}{dP^2} T^{-1} \right] \Psi = 1$$
\[ \Gamma^{(\mu)}(q, P) = \sum_i \tau_i(q, P) T_i^{(\mu)}(q, P) \]

Relativistic covariants

\[
\begin{array}{c|cc}
\text{Component} & 0^- & 0^{++} \\
T_1 & \mathbb{1} & \circ & \circ \\
T_2 & -iP & \circ & \bullet \\
T_3 & -i\phi & \bullet & \circ \\
T_4 & \phi, P & \circ & \circ \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\text{Component} & 1^{--} & 1^{++} & 1^{+-} \\
T_1^\mu & i\gamma^\mu_T & \circ & \circ \\
T_2^\mu & \gamma^\mu_T P & \circ & \bullet \\
T_3^\mu & -\gamma^\mu_T \phi + p^\mu_T P_T & \bullet & \circ \\
T_4^\mu & i\gamma^\mu_T [P_T, \phi] + 2ip_T^\mu P_T & \circ & \circ \\
T_5^\mu & p^\mu_T \mathbb{1} & \circ & \circ \\
T_6^\mu & ip_T^\mu P_T & \bullet & \circ \\
T_7^\mu & -ip_T^\mu \phi & \circ & \bullet \\
T_8^\mu & p_T^\mu [P_T, \phi] & \circ & \circ \\
\end{array}
\]

J^{PC} = 0^{--}, 1^{++}, 1^{+-}

prefixied by \(\gamma_5\).

So far: said nothing about the truncation.
Axial-vector Ward-Takahashi identity

\[ \{ \gamma^5 \Sigma (-p_-) + \Sigma (p_+) \gamma^5 \}_{\alpha\beta} = - \int K_{\alpha\gamma, \delta\beta} (p, q, P) \{ \gamma^5 S (-q_-) + S (q_+) \gamma^5 \}_{\gamma\delta} \]

axWTI connects quark self-energy and the bethe-salpeter kernel

- truncation must consider this
- important for chiral properties of pseudoscalars
  - Gell-Mann–Oakes-Renner / massless pion in the chiral limit
  - difficult and intricate to implement in practice.

\[ m_{\pi}^2 f_\pi \simeq 2m_q \langle \bar{q}q \rangle \]
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Rainbow-ladder truncation

\[ \Gamma^\mu (k, p; \mu) = iZ_2 \gamma^\mu \]

\[ K_{\alpha \gamma, \beta \delta} (p, q, P) = Z_2^2 g^2 \left( \frac{\lambda^i}{2} \right)_{AC} \left( \frac{\lambda^i}{2} \right)_{BD} (i\gamma^\mu)_{\alpha \gamma} D_{\mu \nu} (q) (i\gamma^\nu)_{\beta \delta} \]

- One-gluon exchange
Rainbow-ladder truncation

\[ \lambda_1 \left( q^2 \right) \]

\[ \Gamma^\mu \left( k, p; \mu \right) = iZ_2 \gamma^\mu \]

\[ K_{\alpha \gamma, \beta \delta} \left( p, q, P \right) = Z_2^2 g^2 \left( \frac{\lambda^i}{2} \right)_{\alpha \gamma} \left( \frac{\lambda^i}{2} \right)_{\beta \delta} \left( i\gamma^\mu \right)_{\alpha \gamma} D^{\mu \nu} \left( q \right) \left( i\gamma^\nu \right)_{\beta \delta} \]

- One-gluon exchange
- Also: dress \( \gamma^\mu \) vertex with function \( \lambda_1 \left( q^2 \right) \)
- Reduces twelve components of \( k^2, p^2, q^2 \) to one with \( q^2 \) dependence.

\[ \lambda_1 \left( q^2 \right) \text{ allows part of vertex-dressing to be included.} \]
Rainbow-ladder truncation – Practical application

How do we apply this truncation

Phenomenology dictates:
- feature $D_\chi$SB – chiral condensate

Other considerations:
- perturbative limit, . . . other inputs calculable from QCD

Interaction must provide enough interaction strength for $D_\chi$SB
- Munczek–Nemirowksy
- Maris–Tandy effective interaction
- “Watson” interaction
- Soft-Divergent model

Whole host of things are calculable

- Bound-state mass
- Electroweak decay constants
- Electromagnetic form factors
  - pion charge radius
  - magnetic moments
- transition form factors (hadronic decays)

After fixing phenomenology in Meson sector, apply to three-body systems:

- Nucleon/Delta from quark-diquark model
- full-three body equation

[ P. Maris, P. C. Tandy, PRC 60 (1999) 055214 ]
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Bethe–Salpeter studies of mesons beyond rainbow-ladder
Symmetry preserving truncation: obtained by cutting internal quark-lines in the quark-Dyson-Schwinger equation

- Perform dressed skeleton expansion to perform truncation
- IR power counting, $1/N_c$ expansion yields dominant contributions.
Beyond Rainbow-Ladder: what has been done?

\[
\begin{align*}
\text{Generally, extreme approximations have been made} \\
\text{Trivialization of gluon momentum} \\
\quad \text{in both the quark DSE and vertex DSE} \\
\quad \text{in either the quark DSE or vertex DSE} \\
\quad \text{Dominant vertex contribution neglected/untreatable} \\
\quad \text{in (1) meson amplitude has } p = 0 \text{ fixed.} \\
\quad \text{Meson states unbound (e.g. scalar, axial-vector).}
\end{align*}
\]

\[
D_{\mu\nu}(q) \propto \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \delta^{(4)}(q)
\]
Beyond Rainbow-Ladder: what has been done?

Comparing model (1) to (3) we see a sign change. Consequence of zero momentum exchange?

– qualitative statements from (1,2) dangerous
Beyond Rainbow-Ladder: what must be done?

To investigate dominant contributions we need:

- a gluon with non-trivial momentum dependence
- appropriate internal vertex dressings

Tackle two-loop integrals over NP propagators and vertices. **Challenging.**

\[
\alpha\lambda_1(q^2)Z(q^2) = \frac{\pi D}{\omega^2} q^4 e^{-q^2/\omega^2}
\]

First study: compare to existing literature

employ **model gluon:**

- provides $D_{\chi}$ SB
- UV suppressed (don't worry about renormalisation)
- no trivialization of the gluon momentum

→ study qualitative impact of vertex corrections
→ framework generalises to realistic input.
Beyond Rainbow-Ladder: what must be done?

What do we gain?

- **Truncation does not restrict what mesons we can study:**

  \[
  \begin{array}{cccc}
  \pi & \sigma & \rho & a_1 \\
  0^{--} & 0^{++} & 1^{--} & 1^{++} \\
  \end{array}
  \]

- **Quark-gluon vertex:**
  - all twelve components
  - full momentum dependence \((k^2, p^2, k \cdot p)\)
  - imparts quark-mass dependence on interaction.

- **Different interactions:**
  - vector\(\times\)vector
  - vector\(\times\)scalar
  - different degrees of attraction/repulsion in different meson channels.
Beyond Rainbow-Ladder: what must be done?

Technical challenges?

Auxiliary normalisation condition: (Leon–Cutkosky)

- momentum dependent kernel complicates evaluation

Bound state in Eucl. space, $P^0 = iM$
quark and vertex momenta $(q \pm P/2)^2$:

- must be evaluated for complex momenta
- region bounded by parabola dependent upon meson mass.
Normalisation: Leon–Cutkosky

\[ \delta^{ij} = \frac{\partial}{\partial P^2} \left( \text{tr} \left[ \left( \Gamma_{\pi}^i S \Gamma_{\pi}^j S \right) + \int_q \left[ \left[ \bar{\chi}_{\pi}^i \right]_{sr} K_{tu;rs} \left[ \chi_{\pi}^j \right]_{ut} \right] \right] \right) \]

usual approach to determining normalisation condition

- lines: quark propagator.
- dashed lines: quark propagator fixed wrt. derivative.
- wiggles: gluon propagator.

Normalisation: Nakanishi

\[
\left( \frac{d \ln(\lambda)}{dP^2} \right)^{-1} = \text{tr} \int_k \bar{\Gamma} S \Gamma S
\]

Where \( \lambda \) is the eigenvalue obtained via \( \Gamma = \lambda K \Gamma \).

alternative hidden in the literature

- considerably simpler
- valid for all truncations
- first time applied beyond rainbow-ladder

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Results: Rainbow-Ladder benchmark

Calculate bound-state masses with bare vertex:
Results: Rainbow-Ladder benchmark

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Bethe–Salpeter studies of mesons beyond rainbow-ladder
Consider dominant non-Abelian corrections to vertex

\[
\begin{align*}
\text{diagram 1} & = \text{diagram 2} + \text{diagram 3} \\
\text{diagram 4} & = \text{diagram 5} \\
\text{diagram 6} & = \text{diagram 7} + \text{diagram 8} + \text{diagram 9}
\end{align*}
\]
Results: non-Abelian corrections

\begin{tabular}{c|c|c|c|c|c|c|c|c|c}
  \hline
  Mass [GeV] & 0.0 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 1.0 & 1.1 & 1.2 & 1.3 \\
  \hline
  \(\pi\) & 0.142 & & & & & & & & & & & & & \\
  \(\sigma\) & 0.884 & 0.881 & 0.973 & & & & & & & & & & & \\
  \(\rho\) & & & & & & & & & & & & & & \\
  \(a_1\) & 0.881 & & & & & & & & & & & & & \\
  \(b_1\) & & & & & & & & & & & & & & \\
  \hline
\end{tabular}

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Results: Abelian corrections

Consider sub-leading Abelian corrections to vertex

\[ \text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} \]

\[ \text{Diagram 4} = \frac{-1}{2} + \text{Diagram 5} \]

\[ \text{Diagram 6} = \text{Diagram 7} + \text{Diagram 8} \]
Results: Abelian corrections

<table>
<thead>
<tr>
<th>Mass [GeV]</th>
<th>0.137</th>
<th>0.602</th>
<th>0.734</th>
<th>0.889</th>
<th>0.915</th>
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<tr>
<td>π</td>
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<tr>
<td>(a_1)</td>
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<tr>
<td>(b_1)</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Attractive

Watson/Cassing/Tandy

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Results: non-Abelian and Abelian corrections

Put them all-together!

\[
\begin{align*}
\text{non-Abelian corrections:} & & \text{Abelian corrections:} \\
\end{align*}
\]
Results: non-Abelian and Abelian corrections

Mass [GeV]

- $\pi$: (essentially the same as RL + NA)
- $\sigma$
- $\rho$
- $a_1$
- $b_1$

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In addition to non-resonant diagrams

- resonant contributions from, e.g. pion exchange

→ *approximate and include*

(Ignore contribution from sub-leading Abelian diagrams)
Results: non-Abelian and unquenching effects

- Pion corrections attractive
- Abelian diagram irrelevant

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Results: non-Abelian and unquenching effects

- Implicit mass-dependence of vertex corrections
- Unquenching: shifts towards corrected lattice results.

Next steps

**Pion electromagnetic form-factor**

![Diagram of pion electromagnetic form-factor](image)

**Beyond Impulse-Approximation**

- Consistent with BSE truncation – must satisfy current conservation
- Dressed quark-photon vertex
- Explicitly three-loop - (implicitly eight)

"include inputs from gluon DSE, internal quark-gluon vertex, dressed three-gluon vertex. → examine meson spectrum"
Conclusions

Summary

- Reached an era where rainbow-ladder is no longer the only feasible truncation

- First study no trivialization of momenta:
  - Leading non-Abelian corrections
  - Sub-leading Abelian corrections
  - Also hadronic resonance contributions

- Full calculation:
  - Solve the full system of equations without overt simplifications

- Not just a toy-model
  - Foundation on which to build future *ab initio* bound-state studies
  - Just change the inputs to something realistic!

- Found that sub-leading Abelian corrections are further suppressed dynamically
  - Dominant non-Abelian corrections are generally repulsive

- Take everything with a pinch of salt: wait until realistic study completed