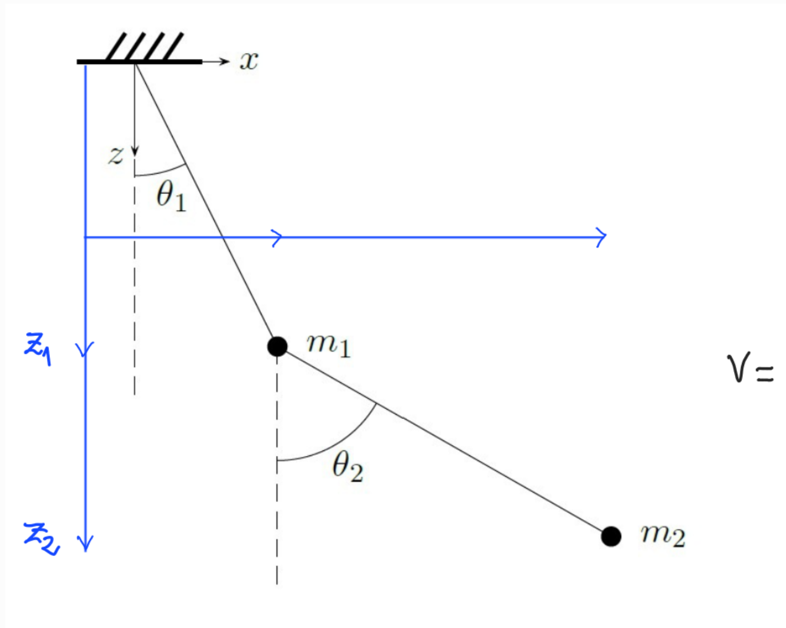
3.3. Consider the double pendulum with masses m_1 and m_2 in the figure. Find the costraints. Make a sensible choice of generalized coordinates. Write the system's Lagrangian and the equations of motion.



$V = -mgz$

The x and z-coordinates of the masses are given by

$$\left.\begin{aligned} x_1 &= l_1 \sin\theta_1 \\ z_1 &= l_1 \cos\theta_1 \end{aligned}\right\}$$

$$\left.\begin{aligned} x_2 &= x_1 + l_2 \sin\theta_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \\ z_2 &= z_1 + l_2 \cos\theta_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2 \end{aligned}\right\}$$

It follows that

$$\left.\begin{aligned} \dot{x}_1 &= l_1 \cos\theta_1 \, \dot{\theta}_1 \\ \dot{z}_1 &= -l_1 \sin\theta_1 \, \dot{\theta}_1 \end{aligned}\right\} \qquad \left.\begin{aligned} \dot{x}_2 &= l_1 \cos\theta_1 \, \dot{\theta}_1 + l_2 \cos\theta_2 \, \dot{\theta}_2 \\ \dot{z}_2 &= -l_1 \sin\theta_1 \, \dot{\theta}_1 - l_2 \sin\theta_2 \, \dot{\theta}_2 \end{aligned}\right\}$$

The Lagrangian reads

$$L = T - V = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{z}_1)^2 + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{z}_2^2) + m_1 g z_1 + m_2 g z_2$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \theta_2) \dot{\theta}_1 \dot{\theta}_2 \right]$$

$$+ m_1 g \, l_1 \cos\theta_1 + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

$$L = \frac{1}{2}(m_1+m_2)\,l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2\cos(\theta_1-\theta_2)\,\dot{\theta}_1\dot{\theta}_2$$
$$+ (m_1+m_2)\,g l_1\cos\theta_1 + m_2 g l_2\cos\theta_2$$

The Euler–Lagrange equation for θ_1 is

$$\left.\begin{aligned}
\frac{\partial L}{\partial\dot{\theta}_1} &= (m_1+m_2)\,l_1^2\dot{\theta}_1 + m_2 l_1 l_2\cos(\theta_1-\theta_2)\,\dot{\theta}_2\\
\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) &= (m_1+m_2)\,l_1^2\ddot{\theta}_1 - m_2 l_1 l_2\sin(\theta_1-\theta_2)\,(\cancel{\dot{\theta}_1}-\dot{\theta}_2)\,\dot{\theta}_2\\
&\quad + m_2 l_1 l_2\cos(\theta_1-\theta_2)\,\ddot{\theta}_2\\
\frac{\partial L}{\partial\theta_1} &= -m_2 l_1 l_2 \cancel{\sin(\theta_1-\theta_2)}\,\dot{\theta}_1\dot{\theta}_2 - (m_1+m_2)\,g l_1\sin\theta_1
\end{aligned}\right\} \Rightarrow$$

$$\Rightarrow (m_1+m_2)\,l_1^2\ddot{\theta}_1 + m_2 l_1 l_2\cos(\theta_1-\theta_2)\,\ddot{\theta}_2 + m_2 l_1 l_2\sin(\theta_1-\theta_2)\,\dot{\theta}_2^2 + (m_1+m_2)\,g l_1\sin\theta_1 = 0 \qquad (3.3.1)$$

Similarly for the EL equation for θ_2 we have

$$m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\cos(\theta_1-\theta_2)\,\ddot{\theta}_1 - m_2 l_1 l_2\sin(\theta_1-\theta_2)\,\dot{\theta}_1^2 + m_2 g l_2\sin\theta_2 = 0 \qquad (3.3.2)$$

The equilibrium points are the solutions $\theta_1(t) = \theta_{10}$, $\theta_2(t) = \theta_{20}$ for all t. Upon substitution in eqs. (3.3.1) and (3.3.2) one has

$$\sin\theta_{10} = 0\ ,\quad \sin\theta_{20} = 0 \quad\Rightarrow\quad \theta_{10} = 0,\pi;\quad \theta_{20} = 0,\pi$$

Discard $\theta_{10} = \pi$, $\theta_{20} = \pi$, since they are clearly unstable. Oscillations about θ_{10} and θ_{20} are $\theta_1 = \theta_{10} + \delta\theta_1 = \delta\theta_1$, $\theta_2 = \theta_{20} + \delta\theta_2 = \delta\theta_2$. Substituting in eqs. (3.3.1) and (3.3.2) and keeping terms of order 1 in $\delta\theta_1$, $\delta\theta_2$, one has

$$\left.\begin{aligned}
&(m_1+m_2)\,l_1^2\ddot{\theta}_1 + m_2 l_1 l_2\ddot{\theta}_2 + (m_1+m_2)\,g l_1\theta_1 + O(\theta_1^2, \theta_2^2, \theta_1\theta_2) = 0\\
&m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\ddot{\theta}_1 + m_2 g l_2\theta_2 = 0
\end{aligned}\right\}$$

Divide the first equation by $(m_1+m_2)\,l_1^2$ and the second one by $m_2 l_1 l_2$,

$$\left.\begin{aligned}
&\ddot{\theta}_1 + \frac{m_2}{m_1+m_2}\frac{l_2}{l_1}\ddot{\theta}_2 + \frac{g}{l_1}\theta_1 = 0\\
&\ddot{\theta}_1 + \frac{l_2}{l_1}\ddot{\theta}_2 + \frac{g}{l_1}\theta_2 = 0
\end{aligned}\right\} \qquad \mu := \frac{m_2}{m_1+m_2}\,,\quad \lambda := \frac{l_2}{l_1} \qquad (3.3.3)$$

Make the ansatz

$$\theta_1 = A_1 \cos(\omega_1 t + \varphi_1), \quad \theta_2 = A_2 \cos(\omega_2 t + \varphi_2)$$

Eqs. (3.3.3) become

$$\left.\begin{aligned} -\omega_1^2 \theta_1 - \mu\lambda\omega_2^2 \theta_2 + \frac{g}{\ell_1}\theta_1 = 0 \\ -\omega_1^2 \theta_1 - \lambda\omega_2^2\theta_2 + \frac{g}{\ell_1}\theta_2 = 0 \end{aligned}\right\} \qquad \left.\begin{aligned} \theta_1\left(\frac{g}{\ell_1} - \omega_1^2\right) - \theta_2 \mu\lambda\omega_2^2 = 0 \\ \theta_1\omega_1^2 + \theta_2\left(\lambda\omega_2^2 - \frac{g}{\ell_1}\right) = 0 \end{aligned}\right\}$$

To have non-zero θ_1 and θ_2 we need

$$\det\begin{pmatrix} \frac{g}{\ell_1} - \omega_1^2 & -\mu\lambda\omega_2^2 \\ \omega_1^2 & \lambda\omega_2^2 - \frac{g}{\ell_1} \end{pmatrix} = 0 \iff -\lambda\omega_1^2\omega_2^2 + \frac{g}{\ell_1}\omega_1^2 + \lambda\frac{g}{\ell_1}\omega_2^2 - \left(\frac{g}{\ell_1}\right)^2 + \mu\lambda\omega_1^2\omega_2^2 = 0$$

$$-\lambda\omega_2^2\left(\omega_1^2 - \frac{g}{\ell_1} - \mu\omega_1^2\right) = \frac{g}{\ell_1}\left(\frac{g}{\ell_1} - \omega_1^2\right)$$

$$\frac{\ell_1\lambda}{g}\omega_2^2 = \frac{\omega_1^2 - \frac{g}{\ell_1}}{\omega_1^2 - \frac{g}{\ell_1} - \mu\omega_1^2}$$

Case $m_2 \ll m_1$. $\mu = \frac{m_2}{m_1 + m_2} = \frac{m_2}{m_1}\frac{1}{1 + \frac{m_2}{m_1}} = \frac{m_2}{m_1}\left[1 + O\left(\frac{m_2}{m_1}\right)\right]$

$$\frac{\ell_1\lambda}{g}\omega_2^2 = \frac{\omega_1^2 - \frac{g}{\ell_1}}{\omega_1^2\left(1 - \frac{m_2}{m_1}\left[1 + O(m_2/m_1)\right]\right) - \frac{g}{\ell_1}} = 1 + \frac{m_2}{m_1}\frac{\omega_1^2}{\omega_1^2 - \frac{g}{\ell_1}} + O\left(\left(\frac{m_2}{m_1}\right)^2\right)$$

Case $m_1 \ll m_2$. $\mu = \frac{1}{1 + \frac{m_1}{m_2}} = 1 - \frac{m_1}{m_2} + O\left(\left(\frac{m_1}{m_2}\right)^2\right)$

$$\frac{\ell_1\lambda}{g}\omega_2^2 = \frac{\omega_1^2 - \frac{g}{\ell_1}}{\omega_1^2\left[1 - 1 + \frac{m_1}{m_2} + O\left(\left(\frac{m_1}{m_2}\right)^2\right)\right] - \frac{g}{\ell_1}} =$$

$$= \left(1 - \frac{\ell_1\omega_1^2}{g}\right)\left[1 + \frac{m_1}{m_2}\frac{\ell_1\omega_1^2}{g} + O\left(\left(\frac{m_1}{m_2}\right)^2\right)\right]$$