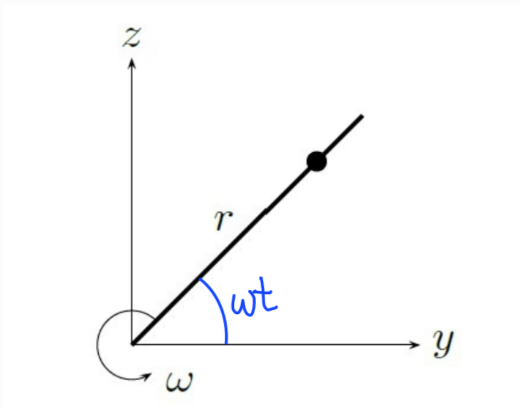
3.5. A particle of mass m slides over a rigid straight wire. The wire has one end fixed at the origin of the yz plane and rotates counterclockwise about the x axis. The particle is initially at a distance a from the origin and at rest with respect to the wire. Write the Lagrangian, find the equations of motion. Determine the equilibrium points and discuss their stability. Take as y-axis the axis initally containing the wire. Assume that the Earth's gravitational force is equal to $-mg\mathbf{e}_3$.



$x = 0$

$y = r\cos\omega t$

$z = r\sin\omega t$

The Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

$$= \frac{1}{2}m\left[(\dot{r}\cos\omega t - r\omega\sin\omega t)^2 + (\dot{r}\sin\omega t + r\omega\cos\omega t)^2\right] - mgr\sin\omega t$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\omega^2) - mgr\sin\omega t$$

This gives the Euler-Lagrange equation

$$m\ddot{r} - (mr\omega^2 - mg\sin\omega t) = 0 \iff \ddot{r} - r\omega^2 = -g\sin\omega t$$

Its solution is

$$r = Ae^{\omega t} + Be^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t$$

At $t = 0$

$$\left.\begin{array}{l} r(0) = a \Rightarrow a = A + B \\ \dot{r}(0) = 0 \Rightarrow \omega(A - B) - \frac{g}{2\omega} = 0 \end{array}\right\} \Rightarrow A = \frac{1}{2}\left(a + \frac{g}{2\omega^2}\right), \quad B = \frac{1}{2}\left(a - \frac{g}{2\omega^2}\right) \Rightarrow$$

$$\Rightarrow r(t) = \underbrace{\frac{1}{2}\left(a + \frac{g}{2\omega^2}\right)}_{>0} e^{\omega t} + \frac{1}{2}\left(a - \frac{g}{2\omega^2}\right)e^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t$$

For large enough t $r(t) > a$, so the ball abandons the bar