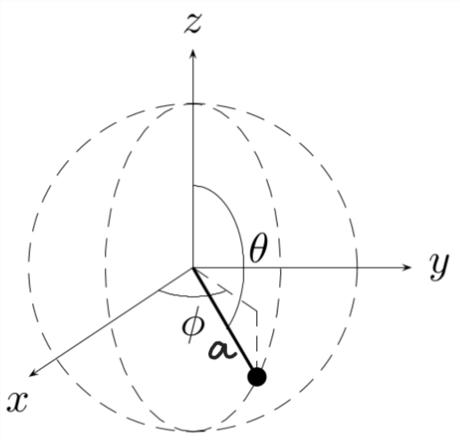
**1 [2.5 points].** A spherical pendulum consists of a mass $m$ attached to a fixed point by an inextensible string of length $a$ subject to a uniform gravitational field, so that it moves on the sphere. Write the Lagrangian. Derive the Euler-Lagrange equations. Find the equilibrium points and study if oscillations about them are stable. Write the Hamiltonian. Find two conserved quantities.




The mass coordinates are

$$x = a \sin\theta \cos\phi$$
$$y = a \sin\theta \sin\phi$$
$$z = a \cos\theta$$

The Lagrangian is then

$$L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz = \frac{1}{2} m a^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) - mga\cos\theta$$

The Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \quad , \quad q^i = \theta, \phi$$

take in our case the form

- $p_\theta := \frac{\partial L}{\partial \dot\theta} = ma^2\dot{\theta}$ , $\quad \frac{\partial L}{\partial \theta} = ma^2 \sin\theta\cos\theta\,\dot{\phi}^2 + mga\sin\theta \implies$

$$\implies ma^2\ddot{\theta} - ma^2\sin\theta\cos\theta\,\dot{\phi}^2 - mga\sin\theta = 0$$

$$\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 - \frac{g}{a}\sin\theta = 0 \qquad (1)$$

- $p_\phi := \frac{\partial L}{\partial \dot\phi} = ma^2\sin^2\theta\,\dot{\phi}$ , $\quad \frac{\partial L}{\partial \phi} = 0 \implies \frac{d}{dt}(ma^2\sin^2\theta\,\dot{\phi}) = 0 \implies$

$$\implies p_\phi \text{ is conserved}, \quad J = \sin^2\theta\,\dot{\phi} = \text{const.} \qquad (2)$$

In terms of $J$, eq. (1) can be cast as

$$\ddot{\theta} - \frac{\cos\theta}{\sin^3\theta} J^2 - \frac{g}{a}\sin\theta = 0 \qquad (3)$$

The Hamiltonian is given by

$$H = \dot{q}^i p_i - L = \dot{\theta} p_\theta + \dot{\phi} p_\phi - L$$

$$= \frac{p_\theta}{ma^2} p_\theta + \frac{p_\phi}{ma^2 \sin^2\theta} p_\phi - \left\{ \frac{1}{2} ma^2 \left[ \left( \frac{p_\theta}{ma^2} \right)^2 + \sin^2\theta \left( \frac{p_\phi}{ma^2 \sin^2\theta} \right)^2 \right] - mga \cos\theta \right\}$$

$$H = \frac{1}{2ma^2} \left( p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} \right) + mga \cos\theta$$

The canonical momentum $p_\phi$ is conserved. Since the gravitational force is conservative the Hamiltonian, which is equal to the total energy, is also conserved.

Let us now discuss the equilibrium points and their stability.

Equilibrium points have $\theta = \theta_0 = \text{const}$. Since $J = \sin^2\theta\, \dot{\phi} = \text{const}$, they have $\dot{\phi} = \text{const}$. We distinguish two cases

<u>Case $\dot{\phi} = 0$.</u> Then $J = 0$ and eq. (3) reduces to

$$\ddot{\theta} - \frac{g}{a} \sin\theta = 0 \qquad (4)$$

This is the equation of the simple pendulum. Its equilibrium points are

$$\sin\theta_0 = 0 \quad \Rightarrow \quad \theta_0 = 0, \pi$$

To study their stability, set $\theta = \theta_0 + \delta\theta$, substitute in eq. (4), expand in powers of $\delta\theta$ and keep terms up to order one. This gives

$$\delta\ddot{\theta} - \underbrace{\frac{g}{a} \cancel{\sin\theta_0}}_{0} - \frac{g}{a} \cos\theta_0\, \delta\theta = 0 \quad \Rightarrow \quad \delta\ddot{\theta} = \frac{g}{a} \cos\theta_0\, \delta\theta$$

It follows that

a) $\frac{g}{a} \cos\theta_0 = 1$ for $\theta_0 = 0$ (north pole) $\Rightarrow$ $\theta_0 = 0$ is unstable

b) $\frac{g}{a} \cos\theta_0 = -1$ for $\theta_0 = \pi$ (south pole) $\Rightarrow$ $\theta_0 = \pi$ is stable

Case $\dot{\phi} = \text{const} =: \omega \neq 0$. In this case the mass rotates about the z-axis with constant angular velocity $\omega$, tracing a circle contained in the $\theta = \theta_0$ plane. The value of $\omega$ can be obtained from eq. (4) as

$$\sin\theta_0 \cos\theta_0\, \omega^2 + \frac{g}{a}\sin\theta_0 = 0 \quad \Rightarrow \quad \omega^2 = -\frac{g}{a\cos\theta_0} > 0 \quad \Rightarrow$$

$$\omega \text{ real} \Rightarrow \omega^2 > 0 \Rightarrow \theta_0 \in \left[\frac{\pi}{2}, \pi\right]$$

To study the stability of this orbit, set $\theta = \theta_0 + \delta\theta$ in eq. (4) and proceed as above:

$$\ddot{\delta\theta} - \frac{\cos(\theta_0+\delta\theta)}{\sin^3(\theta_0+\delta\theta)} J^2 - \frac{g}{a}\sin(\theta_0+\delta\theta) = 0$$

$$\ddot{\delta\theta} - \left[\cancel{\frac{\cos\theta_0}{\sin^3\theta_0}} - \left(\frac{1}{\sin^2\theta_0} + \frac{3\cos^2\theta_0}{\sin^4\theta_0}\right)\delta\theta\right] J^2 - \frac{g}{a}\left(\cancel{\sin\theta_0} + \cos\theta_0\, \delta\theta\right) = 0$$

Use $J = \sin^2\theta_0\, \omega$ and $-\frac{g}{a} = \omega^2 \cos\theta_0$,

$$\ddot{\delta\theta} + \delta\theta\left[(\sin^2\theta_0 + 3\cos^2\theta_0)\omega^2 + \omega^2\cos^2\theta_0\right] = 0$$

$$\ddot{\delta\theta} = -\omega^2(1+3\cos^2\theta_0)\,\delta\theta \quad \Longleftrightarrow \quad \ddot{\delta\theta} = -\Omega^2\,\delta\theta$$

$$\Omega^2 = \omega^2(1+3\cos^2\theta_0)$$

Since $\Omega^2 > 0$, the orbit $\theta = \theta_0$ with $\frac{J^2}{\sin^4\theta_0} = -\frac{g}{a\cos\theta_0}$ is stable, with oscillations

$$\delta\theta = A\cos(\Omega t + \varphi)$$

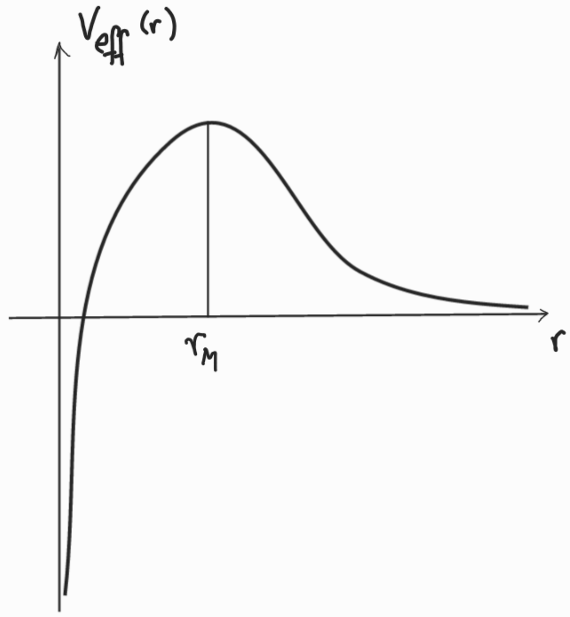
of frequency $\Omega$

**2 [2.5 points].** A particle of mass $\mu$ moves in a central force field with potential energy $V = -k/r^3$, $k > 0$. If the particle approaches from infinity with impact parameter $b$ and velocity $v_0$, find the condition that $b$ and $v_0$ must satisfy for the particle to fall into the origin.

The particle's total energy is

$$E = \frac{1}{2}\mu \dot{r}^2 + V_{eff}(r) \quad , \quad V_{eff}(r) = V(r) + \frac{\ell^2}{2\mu r^2} = -\frac{k}{r^3} + \frac{\ell^2}{2\mu r^2}$$

The effective potential $V_{eff}(r)$ has the plot in the figure




The maximum is located at $r_M$

$$\left.\frac{dV_{eff}}{dr}(r)\right|_{r=r_M} = 0 \quad \Leftrightarrow \quad \frac{3k}{r_M^4} - \frac{\ell^2}{\mu r_M^3} = 0 \Leftrightarrow \frac{1}{r_M^3}\left(\frac{3k}{r_M} - \frac{\ell^2}{\mu}\right) = 0 \Rightarrow$$

$$\Rightarrow \begin{cases} r_M \to \infty . & \text{Discard it} \\ r_M = \dfrac{3k\mu}{\ell^2} . & \text{Maximum, with} \end{cases}$$

The effective potential at the maximum takes the value

$$V_{eff}(r_M) = -k\left(\frac{\ell^2}{3k\mu}\right)^3 + \frac{\ell^2}{2\mu}\left(\frac{\ell^2}{3k\mu}\right)^2 = -\frac{k}{2}\left(\frac{\ell^2}{3k\mu}\right)^3 = \frac{\ell^6}{54k^2\mu^3}$$

For the particle to fall into the origin, the total energy must be larger than $V_{eff}(r_M)$

$$E \geq V_{eff}(r_M) = \frac{\ell^6}{54 k^2 \mu^3} \tag{5}$$

Since energy is conserved and the incoming velocity is $v_0$, one has

$$E = \text{const} = E(r \to \infty) = \frac{1}{2}\mu v_0^2$$

This and the angular momentum of the particle being

$$\ell = \mu b v_0$$

gives for inequality (5)

$$\frac{1}{2}\mu v_0^2 \geq \frac{\mu^6 b^6 v_0^6}{54 k^2 \mu^3} \iff \left(\frac{\mu v_0^2 b^3}{3k}\right)^2 \leq 3$$

**3 [2.5 points].** Light is emitted at time $ct_0 = 0$ from position $\mathbf{x}_0 = (2, 0, 0)$. At what points $\mathbf{x}_1$ can it be observed at time $ct_1 = 3$? An observer is in an intertial frame $S'$ that moves with respect to the source of emission with velocity $v = c/2$ along the positive $x'$ axis. What is the trajectory for the light rays that he will observe?

The light wave front equation is

$$|\vec{x}-\vec{x}_0| = c(t-t_0)$$

At $ct_1=3$, the light wave front will be at

$$|\vec{x}_1 - 2\hat{e}_1| = 3 \iff (x_1-2)^2 + y_1^2 + z_1^2 = 9 \quad \ldots$$ sphere of radius 3 with center at $\vec{x}_0 = (2,0,0)$

Since the velocity of light is the same in all inertial frames the light wave front equation in frame $S'$ is

$$|\vec{x}' - \vec{x}_0'| = c(t' - t_0')$$

We must find $\vec{x}_0'$ and $ct_0'$. Using

$$x' = \gamma(x - vt), \quad ct' = \gamma(ct - \frac{v}{c}x), \quad y'=y, \quad z'=z$$

$$\gamma = \sqrt{\frac{1}{1-(v^2/c^2)}}$$

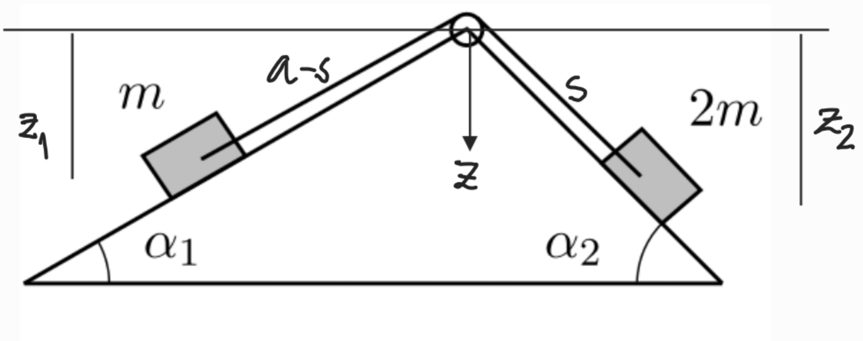
for $v = c/2$ and $\vec{x}_0 = (2,0,0)$ we have $\gamma = \frac{2}{\sqrt{3}}$ and

$$x_0' = \frac{4}{\sqrt{3}}, \quad ct_0' = -\frac{2}{\sqrt{3}}, \quad y' = z' = 0$$

The wave front observed from $S'$ is

$$|\vec{x}' - \frac{4}{\sqrt{3}}\hat{e}_1| = c(t' - \frac{2}{\sqrt{3}})$$

**4 [2.5 points].** Two bodies of masses $m$ and $2m$ are attached to the ends of an inextensible massless string of length $a$. The masses slide without friction over two slopes. See the figure. Write the Lagrangian of the system and derive the Euler-Lagrange equations. Find the linear acceleration of the masses.




$V = -mgz$

The z-coordinates of the masses are

$$z_1 = (a-s)\sin\alpha_1 \qquad z_2 = s\sin\alpha_2$$

The Lagrangian is then

$$L = T - V = \frac{1}{2} m\dot{s}^2 + \frac{1}{2} 2m\dot{s}^2 + mgz_1 + 2mgz_2$$

$$= \frac{3}{2} m\dot{s}^2 + mg\left[(a-s)\sin\alpha_1 + 2\sin\alpha_2\right]$$

Using that

$$\frac{\partial L}{\partial \dot{s}} = 3m\dot{s} \qquad \frac{\partial L}{\partial s} = mg(-\sin\alpha_1 + 2\sin\alpha_2)$$

the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0$$

becomes

$$3\ddot{s} - g(\sin\alpha_1 - 2\sin\alpha_2) = 0$$

The acceleration with which the masses move is

$$\ddot{s} = \frac{1}{3} g(\sin\alpha_1 - 2\sin\alpha_2)$$