Classical Mechanics - Final exam - January 17th, 2024

(Time: 3 hours)

1 [3 points]. A particle of mass m and charge q moves in an EM field described by the electric and magnetic potentials

$$\Phi = 0\,, \qquad \mathbf{A} = \frac{B}{2}\,(y\mathbf{e}_1 - x\mathbf{e}_2)\,.$$

Write the Lagrangian. Derive and solve the Euler-Lagrange equations assuming that the particle is initially at the origin with velocity v_0 along the x-direction. Consider the change of generalized coordinates

$$x \to x' = x + y\,\delta\theta\,, \qquad y \to y' = y - x\,\delta\theta\,, \qquad z \to z' = z \tag{2}$$

with $\delta\theta \ll 1$. Is it a symmetry of the system? If so, identify the associated conserved quantity.

(See problem 3.11 solved in the lectures) The Lagrangian of a particle in an EM $(\Phi,\, \mathbf{A})$ is

$$L = \frac{1}{2}\,\dot{\mathbf{x}}^2 - q\Phi + q\,\dot{\mathbf{x}}\cdot\mathbf{A}\,.$$

In the case at hand, this takes the form (0.5)

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{qB}{2}\left(\dot{x}y - \dot{y}x\right). \tag{3}$$

To find the form of the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0\,, \qquad q^i = x, y, z\,, \tag{4}$$

compute the partial derivatives in them,

$$\begin{aligned}
p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} + \frac{qB}{2}\,y\,, & \frac{\partial L}{\partial x} &= -\frac{qB}{2}\,\dot{y}\,,\\
p_y &= \frac{\partial L}{\partial \dot{x}} = m\dot{y} - \frac{qB}{2}\,x\,, & \frac{\partial L}{\partial y} &= \frac{qB}{2}\,\dot{x}\,,\\
p_z &= \frac{\partial L}{\partial \dot{x}} = m\dot{z}\,, & \frac{\partial L}{\partial z} &= 0 \;\Rightarrow p_z = \text{conserved}\,.
\end{aligned}$$

Upon substitution in eqs. (4), this gives (0.5)

$$\begin{aligned}
&m\ddot{x} + qB\,\dot{y} = 0\,,\\
&m\ddot{y} - qB\,\dot{x} = 0\,,\\
&\ddot{z} = 0 \quad\Rightarrow\quad z(t) = z_1 t + z_0, \quad z_0, z_1 = \text{integration constants}
\end{aligned}$$

The initial conditions for the position and the velocity in the z-direction $z(0) = \dot{z}(0) = 0$ imply $z_0, z_1 = 0$, so that $z(t) = 0$. To solve the equations for x and y, consider $u(t) = x(t) + \mathrm{i}y(t)$. It satisfies the equation

$$m\ddot{u} = m\,(\ddot{x} + \mathrm{i}\ddot{y}) = qB\,(-\dot{y} + \mathrm{i}\dot{x}) = \mathrm{i}qB\dot{u} \quad\Leftrightarrow\quad \ddot{u} = \mathrm{i}\omega\dot{u}\,, \quad \omega := \frac{qB}{m}\,.$$

Its solution is $\dot{u} = C_0 e^{\mathrm{i}\omega t}$, with C_0 a complex integration constant. To determine C_0, use the initial conditions for the velocity in the x and y-directions

$$C_0 = \dot{u}(0) = \dot{x}(0) + \mathrm{i}\dot{y}(0) = v_0 \,.$$

It follows that

$$\dot{u}(t) = \dot{x}(t) + \mathrm{i}\dot{y}(t) = v_0 e^{\mathrm{i}\omega t} \quad \begin{cases} \dot{x}(t) = v_0 \cos\omega t \quad \Rightarrow \quad x(t) = \dfrac{v_0}{\omega}\sin\omega t + k_1 \,, \\[2ex] \dot{y}(t) = v_0 \sin\omega t \quad \Rightarrow \quad y(t) = -\dfrac{v_0}{\omega}\cos\omega t + k_2 \,, \end{cases}$$

with k_1 and k_2 integration constants. To determine them, impose the initial conditions for the position in the x and y-directions,

$$x(0) = y(0) = 0 \quad \Rightarrow \quad k_1 = 0, \quad k_2 = \frac{v_0}{\omega} \,.$$

All in all, the solution is (0.5)

$$x(t) = \frac{v_0}{\omega}\sin\omega t \,, \qquad y(t) = \frac{v_0}{\omega}\big(1 - \cos\omega t\big) \,, \qquad z(t) = 0 \,.$$

The variation of the Lagrangian under the transformation (2) is

$$\delta L = m\Big(\dot{x}\,\frac{d\delta x}{dt} + \dot{y}\,\frac{d\delta y}{dt}\Big) + \frac{qB}{2}\,\Big(\frac{d\delta x}{dt}\,y + \dot{x}\delta y - \frac{d\delta y}{dt}\,x - \dot{y}\delta x\Big) \,, \tag{5}$$

where we have sued that $\delta z = 0$. Using that

$$\delta x = y\delta\theta, \quad \delta y = -x\delta\theta \quad \Rightarrow \quad \frac{d\delta x}{dt} = \dot{y}\delta\theta \,, \qquad \frac{d\delta y}{dt} = -\dot{x}\delta\theta$$

each one of the two parenthesis in eq. (5) vanishes. Hence $\delta L = 0$ and the transformation (2) is a symmetry of the system (0.75). Its associated conserved quantity is

$$Q = p_i \delta q^i = p_x \delta x + p_y \delta y = (p_x y - p_y x)\,\delta\theta$$

Since $\delta\theta$ is arbitrary, the conserved quantity is $L_z = p_x y - p_y x$, which is the z-component of the angular momentum. From $z = 0$ and $p_z = 0$ it follows that the x and y-components of the particle's angular momentum vanish, $L_x = y\,p_z - z\,p_y = 0$ and $L_y = z\,p_x - x\,p_z = 0$. Hence, the particle's angular momentum $\mathbf{L} = \mathbf{x} \wedge \mathbf{p}$ is conserved (0.75).

2 [2 points]. In reference frame S an event 1 takes place at time $ct_1 = 1$ in position $\mathbf{x}_1 = (2, 0, 0)$. Two other events, 2 and 3, take place at

$$a) \quad (ct_2, \mathbf{x}_2) = (4, 1, 0, 0) \qquad\qquad b) \quad (ct_3, \mathbf{x}_3) = (3, 5, 0, 0)$$

Can they be causally connected with event 1? In case they can, find an inertial frame S' in which events take place at the same position but at different times. In case they can't, find an inertial frame S' in which events occur at the same time but in different positions.

(See problem 6.2 solved in the lectures). Event 1 is at $x_1^\mu = (1, 2, 0, 0)$. For two events at $x_1^\mu = (ct_1, \mathbf{x}_1)$ and $x_2^\mu = (ct_2, \mathbf{x}_2)$ to be causally connected, the vector $x_{21}^\mu := x_1^\mu - x_2^\mu$ must be timelike. Let us check if this is the case here

$$a) \quad x_{21}^2 = \eta_{\mu\nu}\, x_{21}^\mu\, x_{21}^\nu = (4-1)^2 - (1-2)^2 = 8 > 0 \;\Rightarrow\; x_{21}^\mu \text{ is timelike}\,,$$
$$b) \quad x_{31}^2 = \eta_{\mu\nu}\, x_{31}^\mu\, x_{31}^\nu = (3-1)^2 - (5-2)^2 = -5 < 0 \;\Rightarrow\; x_{31}^\mu \text{ is spacelike}\,.$$

Event 2 can then be causally connected with event 1, but event 3 cannot (0.5+0.5).

Let us now look for an inertial frame S' at which events 1 and 2 take place at the same position but at different times. To do this, recall that a Lorentz boost along the x axis with velocity v goes from S to S' and transforms coordinates through

$$x' = \gamma\,(x - vt) \qquad ct' = \gamma\,(ct - \frac{v}{c}\,x), \qquad y' = y, \qquad z' = z, \qquad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

We want to find v such that $x'_{21} = x'_2 - x'_1 = 0$, i. e.

$$0 = \gamma\left(x_{21} - \frac{v}{c}\,ct_{21}\right) \quad\Leftrightarrow\quad 0 = (-1 - 3\,\frac{v}{c}) \quad\Rightarrow\quad v = -\frac{c}{3}\,.$$

Hence, in a frame that moves with velocity $-c/3$ in the x-direction of frame S events 1 and 2 occur at the same position but at different times (0.5) .

Let us now look for an inertial frame S' at which events 1 and 3 take place at the same time but at different space points. To do this, recall that a Lorentz boost along the x axis with velocity v goes from S to S' and transforms coordinates through

$$x' = \gamma\,(x - vt) \qquad ct' = \gamma\,(ct - \frac{v}{c}\,x), \qquad y' = y, \qquad z' = z, \qquad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

We want to find v such that $ct'_{31} = c(t'_3 - t'_1) = 0$, i. e.

$$0 = \gamma\,(ct_{31} - \frac{v}{c}\,x_{31}) \quad\Leftrightarrow\quad 0 = (2 - 3\,\frac{v}{c}) \quad\Rightarrow\quad v = \frac{2c}{3}\,.$$

where we have used that $ct_{31} = 2$ and $x_{31} = -3$. In a frame that moves with velocity $2c/3$ in the x-direction of frame S events 1 and 3 occur at the same time but at different positions (0.5).

3 [2 points]. A particle of mass m moves in a central force field. The particle follows a trajectory $r = ae^{b\theta}$ of known angular momentum L, with a a constant with dimensions of length and b a real parameter. Find in terms of L a and b the potential energy $V(r)$. Write the coordinate $r(t)$ of the trajectory as a function of time.

(Compare with problem 2.5 solved in the lectures). The equation of motion in Binet's form is

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}\, f\left(\frac{1}{u}\right), \qquad u = \frac{1}{r}\,, \qquad f(r) = -\frac{dV(r)}{dr}\,,$$

where $V(r)$ is the potential energy. Using

$$u = \frac{1}{a}\, e^{-b\theta} \quad \Rightarrow \quad \frac{d^2u}{d\theta^2} = \frac{b^2}{a}\, e^{-b\theta} = b^2 u\,,$$

it follows that

$$(b^2+1)\, u = -\frac{m}{L^2u^2}\, f\left(\frac{1}{u}\right).$$

Hence

$$f\left(\frac{1}{u}\right) = -\frac{L^2(b^2+1)}{m}\, u^3 \quad \Leftrightarrow \quad f(r) = -\frac{L^2(b^2+1)}{mr^3}\,.$$

This gives (1)

$$V(r) = -\int dr\, f(r) = -\frac{L^2(b^2+1)}{2mr^2}$$

Upon using conservation of angular momentum $L = mr^2\dot{\theta}$, the time derivative of r reads

$$\dot{r} = \frac{dr}{d\theta}\,\frac{d\theta}{dt} = bae^{b\theta}\,\frac{L}{mr^2} = \frac{bL}{mr}\,.$$

Hence, upon integration from $t = 0$ to $t = t$,

$$rdr = \frac{bL}{m}\, dt \;\; \Rightarrow \;\; r = \sqrt{\frac{2bL}{m}\,(t + r_0^2)}\,,$$

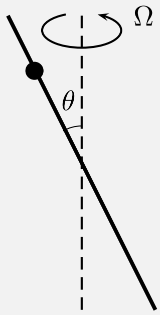
with r_0 an integration constant, the initial radial coordinate.

Note that the orbit's energy is zero since

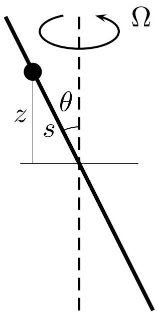
$$E = \frac{1}{2}\, m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = \frac{1}{2}\, m\dot{r}^2 - \frac{L^2b^2}{2mr^2} = 0\,,$$

where in the last equality we have used $\dot{r} = bL/mr$.

4 [3 points]. A rod of mass M and length L rotates with constant angular velocity Ω about a vertical axis that goes through its center. Rotation takes place so that the angle θ formed by the rotation axis and the rod remains constant. A bead of mass m can freely slide along the rod. The system is subject to the Earth's constant gravitational field. Write the Lagrangian, derive the Euler-Lagrange equations and solve them. Explain the solution. What is the rôle that the rod plays in the system?



Take the rod's center as origin of coordinates and the rotation axis as the z-axis. As x-axis take the line orthogonal to the rotation axis that goes through the bead's initial position. Call s to the distance on the rod from the bead to the origin.



The bead coordinates are (0.5)

$$x = s\sin\theta\cos\Omega t, \qquad y = s\sin\theta\sin\Omega t, \qquad z = s\cos\theta$$

It follows that

$$\dot{x} = \sin\theta\,(\dot{s}\cos\Omega t - s\Omega\sin\Omega t), \qquad \dot{y} = \sin\theta\,(\dot{s}\sin\Omega t + s\Omega\cos\Omega t), \qquad \dot{z} = \dot{s}\cos\theta.$$

The Lagrangian is then (0.5)

$$L = T - V = \frac{1}{2}m\left(\dot{s}^2 + s^2\,\Omega^2\sin^2\theta\right) + \frac{1}{2}I\Omega^2 - mgs\cos\theta\,.$$

The moment of inertia I with respect to the rotation axis is a constant, hence the term $I\Omega^2/2$ in the Lagrangian is a constant and does not enter the Euler-Lagrange equation. For the sake of completeness, let us compute I. By definition,

$$I = \int_{-L/2}^{L/2} ds\,\rho\,d_s^2 = \frac{M\sin^2\theta}{L}\int_{-L/2}^{L/2} ds\,s^2 = \frac{1}{12}\sin^2\theta M L^2\,.$$

where $\rho = M/L$ is the rod's mass density and $d_s = s\sin\theta$ is the distance to the rotation axis from the rod's segment $[s, s+ds]$. Note that $I = a(\theta)ML^2$, with $a = \sin^2\theta/12$=const, as one expects on dimensional grounds.

The system only has one degree of freedom, namely the distance s on the rod, so there is only one Euler-Lagrange equation,

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{s}}\Big) - \frac{\partial L}{\partial s} = 0\,.$$

Using

$$\frac{\partial L}{\partial \dot{s}} = m\dot{s}\,, \qquad \frac{\partial L}{\partial s} = ms\Omega^2\sin^2\theta - mg\cos\theta\,,$$

one has (0.5)

$$m\ddot{s} - ms\Omega^2\sin^2\theta + mg\cos\theta = 0\,.$$

Its solution is (0.5)

$$s(t) = A\,e^{-t\Omega\sin\theta} + B\,e^{t\Omega\sin\theta} + \frac{g\cos\theta}{\Omega^2\sin^2\theta}\,, \tag{6}$$

where A and B are integration constants. The sum of the first two terms in the solution is the general solution to the homogeneous equation. The third term is a particular solution to the complete equation. The first term approaches zero exponentially as time increases, whereas the second one blows up. Only for very specific initial conditions, the integration constant B will be equal to zero and the trajectory will approach for very large times to a circle of radius $g\tan\theta/\Omega^2$. If the initial conditions are such that B is different from zero, the bead will leave the rod after a finite time (0.5).

The solution does no depend on the rod's parameters, M and L. The rotating rod constraints the particle to move on a "double-cone" with half-angle at its tip equal to θ, with the particle's angular velocity around the cone axis being constant. (0.5).