UNIVERSIDAD COMPLUTENSE MADRID

FAMILY NAME ____________________

GIVEN NAME ____________________

SIGNATURE ____________________ ID No. ____________________

Classical Mechanics - Final exam - 18th June 2025

(Time: 3 hours)

1 [2.5 points]. A particle of mass m moves along the x-axis subject to a conservative force whose potential energy is given by

$$V(x) = k\left(\frac{a}{x^2} - \frac{2}{x}\right),$$

where k and a are positive constants with suitable units.
(a) Write the equation of motion.
(b) Find the equilibrium points of the system.
(c) Are there stable oscillations about the equilibrium points? If so, determine their angular fequencies.
(d) If the particle is initially at rest at $x(0) = x_R > a$, describe its motion.

Newton's equation is (0.5)

$$\left.\begin{aligned} &m\ddot{x} = F(x) \\ &F(x) = -\frac{dV}{dx}(x) = \frac{2k}{x^2}\left(\frac{a}{x} - 1\right) \end{aligned}\right\} \quad \Rightarrow \quad m\,\ddot{x} = \frac{2k}{x^2}\left(\frac{a}{x} - 1\right).$$

The equilibrium points are its constant solutions $x(t) = x_0$, given by (0.5)

$$\frac{2k}{x_0^2}\left(\frac{a}{x_0} - 1\right) = 0 \quad \Rightarrow \quad x_0 \to \infty,\; x_0 = a\,. \tag{1}$$

Discard $x_0 \to \infty$ since at $x \to \infty$ the particle dose not feel the potential. Setting $x(t) = a + \delta x(t)$, substituting in the equation of motion, expanding in powers of δx and retaining terms of order one, one has

$$m\,\ddot{x} = \frac{2k}{a^2\left(1 + \frac{\delta x}{a}\right)^2}\left(\frac{1}{1 + \frac{\delta x}{a}} - 1\right) = -\frac{2k}{a^3}\,\delta x + O\big(\frac{\delta x^2}{a^2}\big)$$

The equation for small deviations from the equilibrium point is thus

$$\ddot{x} = -\omega^2\,\delta x\,, \qquad \omega^2 = \frac{2k}{ma^3}\,.$$

This is th equation of a harmonic oscillator, whose solution is

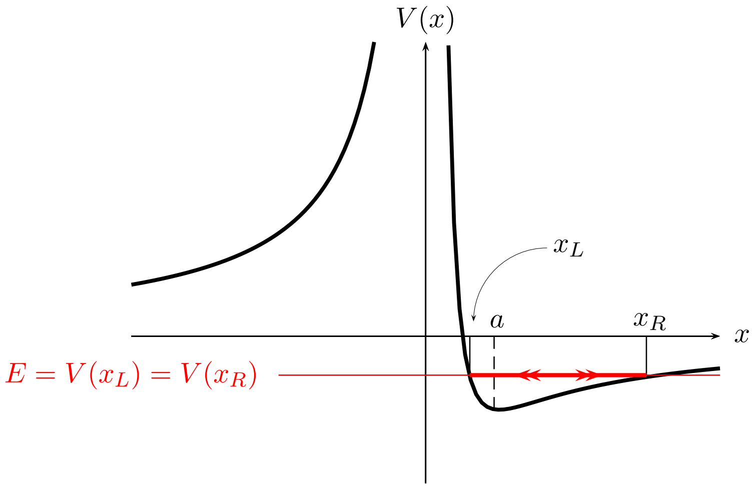
$$\delta x = A\,\cos(\omega t + \varphi)\,,$$

with A and φ constants of integration. Hence there are stable (0.5) oscillations(0.5) about the equilibrium point with angular frequency ω .

Alternatively, the equilibrium points are the extrema of $V(x)$, the local minima corresponding to stable equilibrium points. The square of the angular frecuency of oscillations about a stable equilibrium point is one over the mass times the second derivative of $V(x)$ at the minimum. In the case at hand, the equilibrium points are given by eq. (1) above. At $x_0 = a$ the second derivative of $V(x)$ is positive, so it is a minimum, hence a stable equilibrium point:

$$V''(x) = \frac{2k}{x^4}(3a - 2x) \quad \Rightarrow \quad \omega^2 = \frac{V''(a)}{m} = \frac{2k}{ma^3} > 0\,.$$

The plot of the potential is as in the figure.

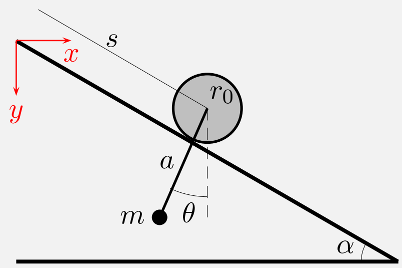


If initially the particle is at rest at $x(0) = x_R$ (to the right of the minimum) the particle's energy is at that time purely potential and given by

$$E = V(x_R) = k\Big(\frac{a}{x_R^2} - \frac{2}{x_R}\Big)\,.$$

The particle will start moving to the left, loosing potential energy and gaining kinetic energy until it reaches a point x_L at which the the kinetic energy vanishes and the energy is again purely potential and equal to $V(x_R)$, so that $V(x_L) = V(x_R)$. At this point, it will start moving rightward, until it reaches X_R ans so on. It will thus move in between X_L and X_R (0.5).

2 [2 points]. A disk of mass M and radius r_0 rolls without slipping over an incline with slope $\tan\alpha$ (see the figure). A simple pendulum of mass m and length a hangs from its center. Write the Lagrangian of the system and its Euler-Lagrange equations. Call s to the distance from the top of the incline to the disk center and take the incline's top as origin of potential energies.



The disk's center of mass and the mass m coordinates are

$$\left.\begin{array}{r} x_{CM} = s\cos\alpha \\ y_{CM} = s\sin\alpha \end{array}\right\} \qquad \left.\begin{array}{r} x_m = x_{CM} - a\sin\theta = s\cos\alpha - a\sin\theta \\ y_m = y_{CM} + a\cos\theta = s\sin\alpha + a\cos\theta \end{array}\right\}.$$

The kinetic energy of the disk is the sum of the translational kinetic energy of its center of mass and the kinetic energy of rotaion about the axis going through its center of mass,

$$T^{\text{disk}} = T^{\text{disk}}_{CM} + T^{\text{disk}}_{rot} = \frac{1}{2}\,M\,(\dot{x}^2_{CM} + \dot{y}^2_{CM}) + \frac{1}{2}\,I\,\omega^2\,.$$

Here I is the moment of inertia with respect to the axis orthogonal to the disk that goes through its center and ω is the angular velocity of the rotation. Recall that

$$I = \frac{1}{2}\,Mr_0^2\,.$$

To find ω, note that the angle $d\beta$ rolled by the disk is $d\beta = \omega dt$, and that the distance advanced by the point of contact of the disk with the incline is $ds = r_0 d\beta$. Hence

$$\dot{s} = r_o\dot{b} = r_o\omega \quad \Rightarrow \quad \omega = \frac{\dot{s}}{r_0}\,.$$

It follows that (0.5)

$$T^{\text{disk}} = T^{\text{disk}}_{CM} + T^{\text{disk}}_{rot} = \frac{1}{2}\,M\dot{s}^2 + \frac{1}{2}\left(\frac{1}{2}\,M^2r_0^2\right)\frac{\dot{s}^2}{r_0^2} = \frac{3}{4}\,M\dot{s}^2\,.$$

In turn, the kinetic energy of the pendulum is (0.5)

$$T_m = \frac{1}{2}\,m\,(\dot{x}^2_m + \dot{y}^2_m) = \frac{1}{2}\,m\left[\dot{s}^2 + a^2\dot{\theta}^2 - 2\,a\dot{s}\dot{\theta}\cos(\theta-\alpha)\right]$$

The total potential energy is (0.5)

$$V = V_M + V_m = -Mgy_{cm} - mgy_m = -Mgs\sin\alpha - mg(s\sin\alpha + a\cos\theta)$$

The Lagrangian is then

$$L = T - V = \frac{3}{4}\,M\dot{s}^2 + \frac{1}{2}\,m\left[\dot{s}^2 + a^2\dot{\theta}^2 - 2\,a\dot{s}\dot{\theta}\cos(\theta-\alpha)\right] + (M+m)\,gs\sin\alpha + mga\cos\theta\,.$$

To write the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad q = s, \theta\,,$$

we need the partial derivatives

$$\left.\begin{aligned} \frac{\partial L}{\partial \dot{s}} &= \frac{3}{2}\,M\dot{s} + m\dot{s} - ma\dot{\theta}\cos(\theta-\alpha) \\ \frac{\partial L}{\partial \dot{\theta}} &= ma^2\dot{\theta} - ma\dot{s}\cos(\theta-a) \end{aligned}\right\} \qquad \left.\begin{aligned} \frac{\partial L}{\partial s} &= (M+m)\,g\sin\alpha \\ \frac{\partial L}{\partial \theta} &= ma\dot{s}\dot{\theta}\sin(\theta-\alpha) - mga\sin\theta \end{aligned}\right\}.$$

Upon subsitution the following Euler-Lagrange equations are obtained (1.0)

$$\left.\begin{aligned} &\frac{3}{2}\,M\ddot{s} + m\ddot{s} - ma\ddot{\theta}\cos(\theta-\alpha) + ma\dot{\theta}^2\sin(\theta-a) - (M+m)\,g\sin\alpha = 0 \\ &ma^2\ddot{\theta} - ma\ddot{s}\cos(\theta-a) + mga\sin\theta = 0 \end{aligned}\right\}.$$

(This problem was solved in detail the lectures. It was part of the June exam and its solution was available in the course's webpage).

3 [2.5 points]. A particle of mass m and charge q moves in the electromagnetic field produced by the potential

$$\phi = 0\,, \qquad \vec{A} = \frac{\mu}{|\vec{x}|^3}\,(\vec{e}_3 \wedge \vec{x})$$

where μ is a constant. Write the Lagrangian in cartesian coordinates. Write it in cylindrical coordinates $\{r, \phi, z\}$. Restrict yourself fron now on to $z = 0$. Write the Euler-Lagrange equations for r and ϕ. What is the particle's total energy? Give two conserved quantities. Show that the energy can be written as the energy of a particle moving in an effective central potential and find the effective potential.

The Lagrangian is

$$L = \frac{1}{2}\,m\dot{\vec{x}}^2 + q\,\dot{\vec{x}}\vec{A}\,.$$

In cartesian coordinates,

$$A_x = -\frac{\mu y}{|\vec{x}|^3}\,, \quad A_y = \frac{\mu x}{|\vec{x}|^3}\,, \quad A_z = 0$$

and one has (0.5)

$$L = \frac{1}{2}\,m\big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\big) + \frac{q\mu}{|\vec{x}|^3}\,\big(-\dot{x}y + \dot{y}x\big)\,.$$

In cylindrical coordinates

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \qquad \left.\begin{aligned} \dot{x} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\ \dot{z} &= \dot{z} \end{aligned}\right\}$$

the Lagrangian takes the form (0.5)

$$L = \frac{1}{2}\,m\big(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\big) + \frac{q\mu r^2\dot{\theta}}{(r^2+z^2)^{3/2}}\,.$$

For $z = 0$, it reduces to

$$L = \frac{1}{2}\,m\big(\dot{r}^2 + r^2\dot{\theta}^2\big) + \frac{q\mu\dot{\theta}}{r}\,.$$

To find the Euler-Lagrange equations, compute the partial derivatives

$$\left.\begin{aligned} \frac{\partial L}{\partial \dot{r}} &= m\dot{r} \\ \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta} + \frac{q\mu}{r} \end{aligned}\right\} \qquad \left.\begin{aligned} \frac{\partial L}{\partial r} &= mr\dot{\theta}^2 - \frac{q\mu\dot{\theta}}{r^2} \\ \frac{\partial L}{\partial \theta} &= 0 \end{aligned}\right\}$$

and substitute in

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0\,, \qquad q = r, \theta\,.$$

This gives (0.5)

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{q\mu\dot{\theta}}{r^2} = 0 \tag{2}$$

$$\frac{d}{dt}\left(mr^2\dot{\theta} + \frac{q\mu}{r}\right) = 0 \;\;\Rightarrow\;\; J := mr^2\dot{\theta} + \frac{q\mu}{r} = \text{const} \;\;\Rightarrow\;\; \dot{\theta} = \frac{1}{mr^2}\left(J - \frac{q\mu}{r}\right). \tag{3}$$

The quantity J is clearly conserved (0.25). As discussed in the lectures, for a particle moving in an arbitrary $(\theta, \vec{A})$, the energy is given by

$$E = \frac{1}{2} m \dot{\vec{x}}^2 + q\phi \,,$$

is equal to the Hamiltonian and is conserved. In the case at hand,

$$E = \frac{1}{2} m \dot{\vec{x}}^2 = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) .$$

One may check its conservation by showing that its time derivative vanishes. Let us do it (0.25):

$$\begin{aligned}\frac{dE}{dt} &= m\dot{r}\ddot{r} + m r \dot{r} \dot{\theta}^2 + m r^2 \dot{\theta} \ddot{\theta} = \text{use for } \ddot{r} \text{ eq. (2)} \\ &= 2 m r \dot{r} \dot{\theta}^2 - \frac{q\mu \dot{\theta} \dot{r}}{r^2} + m r^2 \dot{\theta} \ddot{\theta} = \dot{\theta} \frac{d}{dt} \left(m r^2 \dot{\theta} + \frac{q\mu}{r} \right) = \dot{\theta} \frac{dJ}{dt} = 0 \,.\end{aligned}$$

Noting eq. (3), the energy can be written as

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2mr^2} \left(J - \frac{q\mu}{r} \right)^2 ,$$

This is a central force problem with effective potential (0.5)

$$U(r) = \frac{1}{2mr^2} \left(J - \frac{q\mu}{r} \right)^2 .$$

(This problem was solved in detail the lectures. It was part of June's exam and its solution was available in the course's webpage).

The following is not part of the exam, but since we are already here let us find (for completeness) if there are any stable circular orbits. For a circular orbit $r = \text{const}$, $\dot{r} = 0$ and $U(r)$ must have a minimum. The first and second derivatives of $U(r)$ are

$$\frac{dU}{dr} = -\frac{1}{mr^7} \left(Jr^2 - \mu q \right) \left(Jr^2 - 3\mu q \right) , \qquad \frac{d^2U}{dr^2} = \frac{1}{mr^8} \left(3J^2 r^4 - 20 J r^2 \mu q + 21 \mu^2 q^2 \right) .$$

The first derivative vanishes at r_0 and r_1 given by

$$J r_0^2 = \mu q \,, \qquad J r_1^2 = 3\mu q \,.$$

Since

$$\left. \frac{d^2U}{dr^2} \right|_{r_1} = 4 \frac{\mu^2 q^2}{m r_0^8} > 0 \,, \qquad \left. \frac{d^2U}{dr^2} \right|_{r_1} = -22 \frac{\mu^2 q^2}{m r_1^8} < 0 \,,$$

$r = r_0$ is a minimum and $r = r_1$ is a maximum. There is a stable circular orbit at $r = r_0$.

4 [2 points]. An aircraft moves with constant velocity along the x-axis of an inertial frame S with energy

$$E = \frac{3}{2\sqrt{2}}\, mc^2 \,.$$

Compute its velocity $\vec{v} = d\vec{x}/dt$ as measured by an observer in the inertial frame. Assume that at time $t = 0$ the aircraft was at $x = y = z = 0$. The aircraft carries a clock (which measures proper time). According to this clock, the aircraft emits a light ray at time 2 after going through $x = y = z = 0$. What is the time of emission as meassured by the observer in the intertial frame? What is the equation of the light wave front as seen from the inertial frame?

From

$$E = \frac{mc^2}{\sqrt{1 - \frac{\dot{\vec{x}}^2}{c^2}}}\,, \qquad \vec{v} = \frac{d\vec{x}}{dt}$$

and the data it follows that (1.0)

$$\frac{8}{9} = 1 - \frac{\vec{v}^2}{c^2} \quad \Rightarrow \quad |\vec{v}| = \frac{c}{3} \quad \Rightarrow \quad \vec{v} = \frac{c}{3}\,\vec{e}_1 \,.$$

The relation between the aircraft's proper time interval $\Delta\tau$ and the coordinate time interval Δt is (1.0)

$$\Delta\tau = \sqrt{1 - \frac{\vec{v}^2}{c^2}}\, dt = \sqrt{\frac{8}{9}}\,\Delta t \quad \Rightarrow \quad \Delta t = \frac{3}{2\sqrt{2}}\,\Delta\tau = \frac{3}{\sqrt{2}}\,,$$

where in the last equality we have used that the aircraft's proper time interval is $\Delta\tau = 2$. The inertial frame coordinates of emission are

$$t = 0 + \Delta t = \frac{3}{\sqrt{2}}, \qquad x = 0 + \frac{c}{3}\,\Delta t = \frac{c}{\sqrt{2}}, \qquad y = z = 0\,.$$

The light wave fron equation as seen from the inertial frame is then (0.5)

$$c^2\left(t - \frac{3}{\sqrt{2}}\right)^2 - \left(x - \frac{c}{\sqrt{2}}\right)^2 - y^2 - z^2 = 0\,.$$

(Calculation of $\vec{v}$ taken from June's final exam –solution was available in the course's webpage).