UNIVERSIDAD COMPLUTENSE MADRID

FAMILY NAME ______________________

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Classical Mechanics - Final exam - 16th January 2025

(Time: 3 hours)

1 [2 points]. A particle of mass m moves in a central force field whose potential energy is $V(r) = \frac{1}{2}m\omega^2 r^2$. Write the particle's total energy. Derive from the latter the equation of motion for the radial coordinate. Assume that the particle is initially at r_0 with velocity $\left(\dot{r}(0) = 0,\, \dot{\theta}(0) = \omega_0\right)$. For what values of the energy are the orbits closed? Calculate the turning points in terms of the initial conditions. ω and ω_0 are both constant but different.

The total energy is constant and given by

$$E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + \frac{1}{2}m\omega^2r^2 =: \frac{1}{2}m\dot{r}^2 + U(r) \qquad 0.5 \qquad (1)$$

where

$$\ell = \mu r^2\dot{\theta}^2$$

is the angular momentum, which is also conserved, and

$$U(r) := \frac{\ell^2}{2mr^2} + \frac{1}{2}m\omega^2r^2$$

is the effective potential.

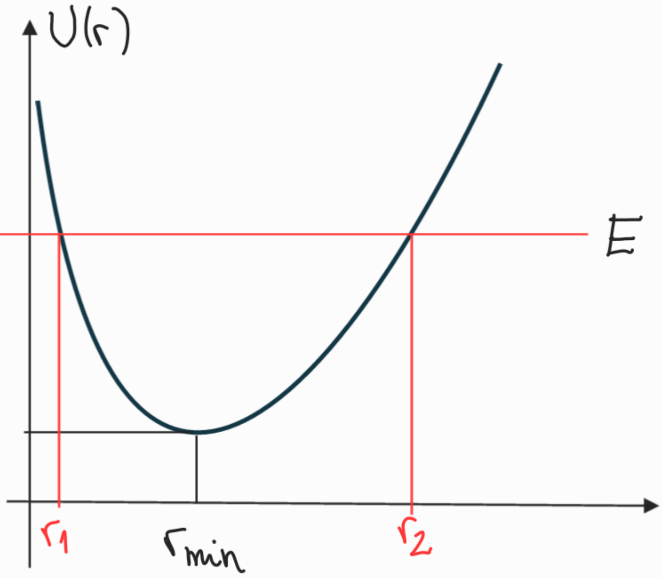
The equation of motion for $r(t)$ is given from eq. (1) by

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left[E - \frac{\ell^2}{2mr^2} - \frac{1}{2}m\omega^2r^2\right]}$$

Alternatively, differentiation of eq. (1) with respect to time gives

$$\dot{r}\left(m\ddot{r} - \frac{\ell^2}{mr^3} + m\omega^2 r\right) = 0 \qquad 0.5$$

The plot of the effective potential $U(r)$ is



Closed orbits occur for r such that $U(r) < E$. It follows that all orbits are closed. At their periapsis (r_1) and apsis (r_2) the velocity $\dot{r}$ vanishes. r_1 and r_2 the solutions to the equation

0.5

$$E = \frac{\ell^2}{2mr^2} + \frac{1}{2} m\omega^2 r^2 \qquad (2)$$

Using that

$$\ell = \ell(t=0) = m r_0^2 \omega_0^2$$

$$E = E(t=0) = \frac{(m r_0^2 \omega_0)^2}{2 m r_0^2} + \frac{1}{2} m\omega^2 r_0^2 = \frac{1}{2} m(\omega_0^2+\omega^2) r_0^2$$

eq. (2) becomes

$$\frac{1}{2} m(\omega_0^2+\omega^2) r_0^2 = \frac{1}{2} m\omega_0^2 \frac{r_0^4}{r^2} + \frac{1}{2} m\omega^2 r^2$$

$$\omega^2 r^4 - (\omega_0^2+\omega^2) r_0^2 r^2 + \omega_0^2 r_0^4 = 0$$

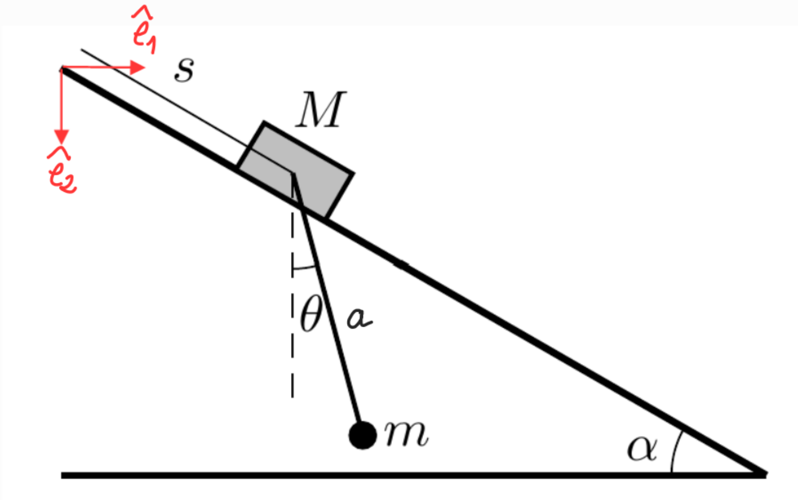
$$r^2 = \frac{(\omega_0^2+\omega^2) r_0^2 \pm \sqrt{(\omega_0^2+\omega^2)^2 r_0^4 - 4\omega^2\omega_0^2 r_0^4}}{2\omega^2} = \frac{r_0^2}{2\omega^2}\left[(\omega_0^2+\omega^2) \pm (\omega_0^2-\omega^2)\right]$$

$$r^2 = r_0^2 \frac{\omega_0^2}{\omega^2},\ r_0^2 \underset{r>0}{\Rightarrow} r_{1,2} = r_0,\ \frac{\omega_0 r_0}{\omega}$$

0.5

Motion starts with zero radial velocity at $r_1 = r_0$, the particle increases its kinetic energy (and decreases its potential energy), goes through the minimum r_{min} (whatever that is), decreases its kinetic energy (increases its potential energy) and reaches $r_2 = \frac{\omega_0}{\omega} r_0$, and starts its way back.

2 [1 point]. A body of mass M freely slides down a frictionless slope that forms an angle α with the floor. A simple pendulum of mass m hangs from the box. Write the system's Lagrangian. Call s to the distance of the mass M to the top of the incline.



a = pendulum's length

$V = -$ mass $g\, y$

The masses coordinates are

$$\left.\begin{aligned} x_M &= s\cos\alpha \\ y_M &= s\sin\alpha \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{x}_M &= \dot{s}\cos\alpha \\ \dot{y}_M &= s\sin\alpha \end{aligned}$$

$$\left.\begin{aligned} x_m &= x_M + a\sin\theta \\ y_m &= y_M + a\cos\theta \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{x}_m &= \dot{s}\cos\alpha + a\dot{\theta}\cos\theta \\ \dot{y}_m &= \dot{s}\sin\alpha - a\dot{\theta}\sin\theta \end{aligned}$$

Hence

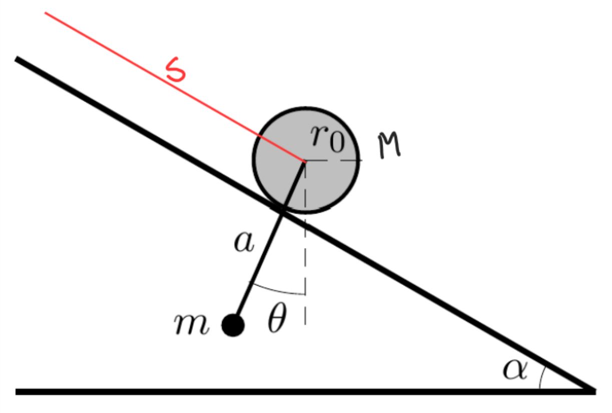
$$L = T - V = \frac{1}{2}M(\dot{x}_M^2 + \dot{y}_M^2) + \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2) - (-Mg\,y_M - mg\,y_m)$$

$$L = \frac{1}{2}M\dot{s}^2 + \frac{1}{2}m\left[\dot{s}^2 + a^2\dot{\theta}^2 + 2a\dot{s}\dot{\theta}\,(\cos\alpha\cos\theta - \sin\alpha\sin\theta)\right] \quad 0.5$$
$$+ Mgs\sin\alpha + mg(s\sin\alpha + a\cos\theta) \quad 0.5$$

$$L = \frac{1}{2}M\dot{s}^2 + \frac{1}{2}m\left[\dot{s}^2 + a^2\dot{\theta}^2 + 2a\dot{s}\dot{\theta}\cos(\theta+\alpha)\right] + (M+m)gs\sin\alpha + mga\cos\theta$$

[Solved on the lectures]

3 [1 point]. A disk of mass M and radius r_0 rolls without slipping over an incline with slope $\tan\alpha$ (see the figure). A simple pendulum of mass m and length a hangs from its center. Write the system's Lagrangian.



The disk's kinetic energy is

$$T_{disk} = \frac{1}{2}M\dot{s}^2 + \frac{1}{2}I\omega^2$$

$$I = \left\{ \begin{array}{l} \text{moment of inertia of disk w.r.t} \\ \text{an axis going through its center} \end{array} \right\} = \frac{1}{2}Mr_0^2$$

If $d\beta = \omega dt$ is the angle rolled by the disk, the distance advanced by the point of contact of the disk with the incline is $ds = r_0 d\beta$. Hence

$$\dot{s} = r_0\dot{\beta} = r_0\omega \implies \omega = \frac{\dot{s}}{r_0}$$ 0.5

and

$$T_{disk} = \frac{1}{2}M\dot{s}^2 + \frac{1}{2}\left(\frac{1}{2}Mr_0^2\right)\frac{\dot{s}^2}{r_0} = \frac{3}{4}M\dot{s}^2$$

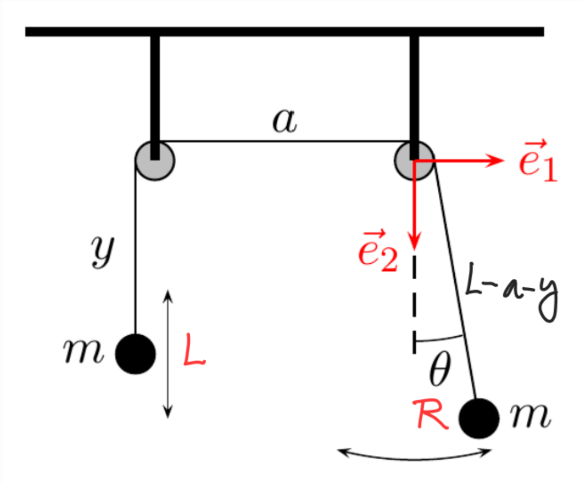
The pendulum's kinetic energy and the potential energies are as in the previous problem. Thus (θ in problem 2 is $-\theta$ now)

$$L = \frac{3}{4}M\dot{s}^2 + \frac{1}{2}m\left[\dot{s}^2 + a^2\dot{\theta}^2 - 2a\dot{s}\dot{\theta}\cos(\theta-\alpha)\right] + (M+m)gs\sin\alpha + mga\cos\theta$$

0.5

[Solved on the lectures]

4 [3 points]. Two particles of mass m are connected by an inextensible string of length L going through two pulleys of negligible size as indiciated in the figure. The pulleys are separated by a fixed distance a. The mass on the left may only move vertically, whereas that on the right may oscillate sideways in the vartical plane containing the string and the pulleys. Write the Lagrangian of the system. Find the Euler-Lagrange equations. Determine the equilibrium points. Are there any stable oscillations about them? If so, determine their angular frequency. Call y to the distance from the mass on the left to the left pulley.



L, R for left, right masses

$x_L = -a$

$y_L = y$

$x_R = (L-a-y)\sin\theta$

$y_R = (L-a-y)\cos\theta$

$V = -$ mass $g\,y$

Using

$\sin\theta$

$\dot{x}_L = 0, \quad \dot{y}_L = \dot{y}$

$\dot{x}_R = (L-a-y)\dot{\theta}\cos\theta - \dot{y}\sin\theta, \qquad \dot{y}_R = -(L-a-y)\dot{\theta}\sin\theta - \dot{y}\cos\theta$

the Lagrangian is written as

$$L = T - V = \frac{1}{2}m(\dot{x}_L^2 + \dot{y}_L^2) + \frac{1}{2}m(\dot{x}_R^2 + \dot{y}_R^2) - (-mg\,y_L - mg\,y_R)$$

$$L = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\left[(L-a-y)^2\dot{\theta}^2 + \dot{y}^2\right] + mgy + mg(L-a-y)\cos\theta$$ 1

Since

$$\frac{\partial L}{\partial \dot{y}} = 2m\dot{y}, \qquad \frac{\partial L}{\partial y} = -m(L-a-y)\dot{\theta}^2 + mg - mg\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m(L-a-y)^2\dot{\theta}, \qquad \frac{\partial L}{\partial \theta} = -mg(L-a-y)\sin\theta$$

the Euler-Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0$ read

$$\left.\begin{aligned} &2\ddot{y} + (L-a-y)\dot{\theta}^2 - g(1-\cos\theta) = 0 \\ &(L-a-y)^2\ddot{\theta} - 2(L-a-y)\dot{y}\dot{\theta} + g(L-a-y)\sin\theta = 0 \end{aligned}\right\}$$

0.5 (1)

0.5 (2)

Eq. (2) has two solutions:

(i) $L-a-y=0 \quad \Rightarrow \quad y = L-a = \text{const}$ (3)

(2i) $(L-a-y)\ddot{\theta} - 2\dot{y}\dot{\theta} + g\sin\theta = 0$ (4)

For solution (3) the mass on the right is at the right pulley so that it does not move. Let us look at solution (4).

Equilibrium points are constant solutions $y=y_0$ $\theta=\theta_0$ to eqs. (1) and (4). Upon substitution in the latter, we have

$$\left.\begin{array}{l} 1-\cos\theta_0 = 0 \\ \sin\theta_0 = 0 \end{array}\right\} \Rightarrow \quad \theta_0 = 0$$

and y_0 arbitrary. Setting

$$\theta = \theta_0 + \delta\theta = \delta\theta \quad [\Rightarrow \sin\theta = \delta\theta + O(\delta\theta^3), \quad \cos\theta = 1 + O(\delta\theta^2)], \qquad y = y_0 + \delta y \quad ,$$

expanding eqs. (1) and (4) in powers of $\delta\theta$ and δy and keeping terms of order one, we have

$$\left.\begin{array}{l} (2) \to \quad 2\delta\ddot{y} = 0 \\ (4) \to \quad (L-a-y_0)\delta\ddot{\theta} + g\,\delta\theta = 0 \end{array}\right\} \Rightarrow \quad \delta\ddot{\theta} = -\omega^2\delta\theta \ , \quad \omega^2 = \frac{g}{L-a-y_0}$$

oscillations in θ with angular frequency ω 1

$$\Rightarrow \begin{cases} \delta y = At + B \\ \delta\theta = C\cos(\omega t + \varphi) \end{cases} , \quad A, B, C, \varphi = \text{integration constants}$$

Note that ω^2 depends on y_0 but $y(t) = y_0 + \delta y$ changes in time, so that ω^2 changes in time. For $A>0$, $y(t)$ increases there is a smaller $L-a-y_0$ and ω^2 increases

5 [1 point]. Given a dynamical variable $F\big(t,\, q^1(t), \cdots, q^n(t),\, p_1(t), \cdots, p_n(t)\big)$ defined in phase space, show that its time derivative is given by

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F,\, H\}\,,$$

where $\{F, H\}$ is the Poisson bracket of F with the Hamiltonian H.

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i}\frac{dq^i}{dt} + \frac{\partial F}{\partial p_i}\frac{dp_i}{dt} \right) = \left\{ \begin{array}{l} \text{use Hamilton's equations} \\ \dot{q}^i(t) = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i(t) = -\frac{\partial H}{\partial q^i} \end{array} \right\}$$

$$= \frac{\partial F}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i} \right) = \{\text{definition of Poisson bracket}\}$$

$$= \frac{\partial F}{\partial t} + \{F, H\}$$

[From the lectures]. 1

6 [2 points]. A free particle of mass m moves in an inertial frame S along the x-axis with energy

$$E = \frac{3}{2\sqrt{2}} mc^2 .$$

Compute its velocity $\vec{v} = d\vec{x}/dt$. A second intertial frame S' moves realtive to S in the x-direction with velocity $c/2$. Calculate the particle's velocity as observed from S'.

Using

$$E^2 = (mc^2)^2 + \vec{p}^{\,2} c^2 ,$$

$$\vec{p} = m \gamma_v \vec{v} , \qquad \gamma_v = \frac{1}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} , \qquad \vec{v} = \frac{d\vec{x}}{dt}$$

we have

$$\frac{9}{8}(mc^2)^2 = (mc^2)^2 + m^2 \gamma_v^2 \left(\frac{d\vec{x}}{dt}\right)^2 c^2 \Rightarrow \frac{c^2}{8} = \frac{\vec{v}^2}{1 - \frac{\vec{v}^2}{c^2}} \Rightarrow$$

$$\Rightarrow c^2 - \vec{v}^2 = 8\vec{v}^2 \Rightarrow |\vec{v}| = \frac{c}{3} \Rightarrow \vec{v} = \frac{d\vec{x}}{dt} = \frac{c}{3}\hat{e}_1$$ 1

In frame S'

$$\left.\begin{aligned} x' &= \gamma_u (x - ut) \\ ct' &= \gamma_u \left(ct - \frac{u}{c} x\right) \end{aligned}\right\} \quad \gamma_u = \frac{1}{\sqrt{1 - \frac{\vec{u}^2}{c^2}}} \qquad \vec{u} = \frac{c}{2}\hat{e}_1 = \text{velocity of } S' \text{ relative to } S$$

It follows that

$$\vec{u}' = u' \vec{e}_1 = -\frac{c}{5} \vec{e}_1$$

$$u' = \frac{dx'}{dt'} = \frac{\gamma_u (dx - u\,dt)}{\gamma_u \left(dt - \frac{u}{c^2} dx\right)} = \frac{\frac{dx}{dt} - u}{1 - \frac{u}{c^2}\frac{dx}{dt}} = \frac{\frac{c}{3} - \frac{c}{2}}{1 - \frac{c}{2c^2}\frac{c}{3}} = -\frac{c}{5}$$ 1