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Classical Mechanics - Midterm exam - 13th November 2024

1 [3 points]. A particle of mass μ moves in an atractive central potential whose force is given by

$$\vec{\mathbf{F}} = f(r)\,\frac{\vec{\mathbf{x}}}{r}\,, \qquad f(r) < 0$$

with $f(r)$ a known function of r. Find the condition for the existence of an $r = a$ circular orbit. Study if there are small fluctuacions about it and find their frequency.

→ Problem 2.6 (in problem sheets) solved in full detail in a lecture

The total energy is

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) = \frac{1}{2}\mu\dot{r}^2 + U(r)\,, \qquad U(r) = \frac{L^2}{2\mu r^2} + V(r)$$

$$\vec{F} = f(r)\frac{\vec{x}}{r} = -\vec{\nabla}V = -\frac{dV}{dr}\frac{\vec{x}}{r} \Rightarrow \frac{dV}{dr} = -f(r)$$

Equilibrium stable points occur at minima $r=a$ of $U(r)$:

$$\left.\frac{dU}{dr}\right|_a = 0 \quad \text{with} \quad \left.\frac{d^2U}{dr^2}\right|_a > 0$$

In our case

$$\frac{dU}{dr} = -\frac{L^2}{\mu r^3} + \frac{dV}{dr} = -\frac{L^2}{\mu r^3} - f(r) \Rightarrow U'(a) = 0 \quad \text{becomes} \quad \boxed{-\frac{L^2}{a^3\mu} = f(a)}$$

$$\frac{d^2U(r)}{dr^2} = \frac{3L^2}{\mu r^4} - \frac{df(r)}{dr} \Rightarrow U''(a) = \frac{3}{a}\frac{L^2}{a^3\mu} - f'(a) = -\frac{3f(a)}{a} - f'(a) \quad (1)$$

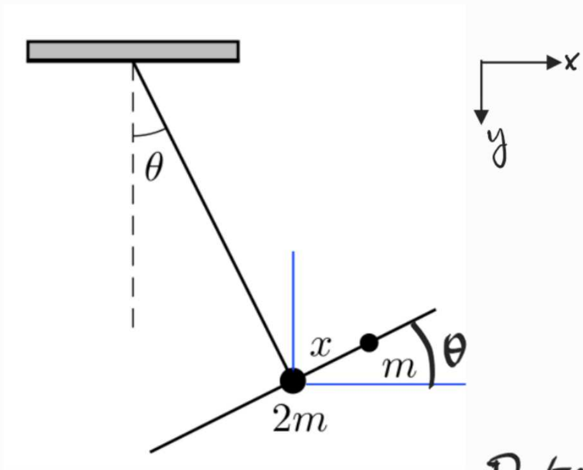
$$\Rightarrow U''(a) > 0 \text{ becomes } \boxed{-\frac{3f(a)}{af'(a)} > 1}$$

The equation for small oscillations $r = a + \delta r$ in a 1-dimensional problem is

$$\mu\ddot{r} = -U''(a)\,\delta r \iff \mu\ddot{r} = \left[\frac{3f(a)}{a} + f'(a)\right]\delta r$$

The oscillations angular frequency is $\Omega = \sqrt{\underbrace{-\frac{1}{\mu}\left(\frac{3f(a)}{a} + f'(a)\right)}_{>0}}$

2 [4 points]. Consider a rod of length L and negligible mass with an end hinged at a support and a mass $2m$ at its other end. Welded to the free end there is a very long second rod, also of negligible mass. A mass m can freely slide along the second rod. The system can rotate about the hinge in the vertical plane containing the two rods. See the figure. Assume that the sliding of m along the second bar is not obstructed by the mass $2m$. Write the system's Lagrangian and find the corresponding Euler-Lagrange equations. Discuss if there are stable small oscillations about ($x_0 = 0$, $\theta_0 = 0$). Find the Hamiltonian of the system.



Compare with 36 in problem sheets, solved in a lecture and uploaded at the course's webpage

Potential energy $V = -mgy$

From the figure we read

$$x_{2m} = L\sin\theta \qquad \dot{x}_{2M} = L\dot{\theta}\cos\theta$$

$$y_{2m} = L\cos\theta \qquad \dot{y}_{2m} = -L\dot{\theta}\sin\theta$$

$$x_m = L\sin\theta + x\cos\theta \qquad \dot{x}_m = L\dot{\theta}\cos\theta + \dot{x}\cos\theta - x\dot{\theta}\sin\theta$$

$$y_m = L\cos\theta - x\sin\theta \qquad \dot{y}_m = -L\dot{\theta}\sin\theta - \dot{x}\sin\theta - x\dot{\theta}\cos\theta$$

The Lagrangian is

$$L = \frac{1}{2}2m(\dot{x}_{2m}^2 + \dot{y}_{2m}^2) + \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2) - (-2mgy_{2m} - mgy_m)$$

$$L = \frac{1}{2}m(3L^2\dot{\theta}^2 + \dot{x}^2 + x^2\dot{\theta}^2 + 2L\dot{x}\dot{\theta}) + mg(3L\cos\theta - x\sin\theta)$$

The Euler-Lagrange equations are

$$\frac{\partial L}{\partial\dot{\theta}} = 3mL^2\dot{\theta} + mx^2\dot{\theta} + mL\dot{x} \qquad \frac{\partial L}{\partial\theta} = mg(-3L\sin\theta - x\cos\theta)$$

$$\frac{\partial L}{\partial\dot{x}} = m(\dot{x} + L\dot{\theta}) \qquad \frac{\partial L}{\partial x} = mx\dot{\theta}^2 - mg\sin\theta$$

$$3L^2\ddot{\theta} + x^2\ddot{\theta} + 2x\dot{x}\dot{\theta} + L\ddot{x} + 3gL\sin\theta + gx\cos\theta = 0 \qquad (1)$$

$$\ddot{x} + L\ddot{\theta} - x\dot{\theta}^2 + g\sin\theta = 0 \qquad (2)$$

Using eq. (2), eq. (1) reduces to

$$2L^2\ddot{\theta} + x^2\ddot{\theta} + 2x\dot{x}\dot{\theta} + \underbrace{L(L\ddot{\theta} + \ddot{x} + g\sin\theta)}_{x\dot{\theta}^2} + 2gL\sin\theta + gx\cos\theta = 0$$

Hence

$$2L^2\ddot{\theta} + x^2\ddot{\theta} + 2x\dot{x}\dot{\theta} + x\dot{\theta}^2 + 2gL\sin\theta + gx\cos\theta = 0 \qquad (3)$$

$$\ddot{x} + L\ddot{\theta} + g\sin\theta = 0 \qquad (4)$$

To discuss oscillations about $(\theta_0 = 0, x_0 = 0)$ set

$$\theta = \theta_0 + \delta\theta = \delta\theta \quad , \quad x = x_0 + \delta x = \delta x$$

in eqs. (3) and (4), expand in powers of $\delta\theta$, δx and keep terms of order one in $\delta\theta$, δx. This gives

$$2L^2\delta\ddot{\theta} + 2gL\,\delta\theta + g\,\delta x = 0 \qquad (5)$$

$$\delta\ddot{x} + L\delta\ddot{\theta} + g\delta\theta = 0 \implies L\delta\ddot{\theta} + g\delta\theta = -\delta\ddot{x} \qquad (6)$$

Upon substitution of eq. (6) in eq. (5) one has

$$-2L\,\delta\ddot{x} + g\,\delta x = 0 \implies \begin{cases} \delta\ddot{x} = \frac{g}{2L}\delta x \implies \text{unstable since } \frac{g}{2L} > 0 \\ \delta x = 0 \overset{\implies}{\scriptstyle (6)} \delta\ddot{\theta} = -\frac{g}{L}\delta\theta \end{cases}$$

Small oscillations of a simple pendulum of mass 3m at the welding point.

To find the Hamiltonian, compute first the canonical momenta:

$$\left.\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} + mL\dot{\theta} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = 3mL^2\dot{\theta} + mx^2\dot{\theta} + mL\dot{x} \end{aligned}\right\} \implies \begin{aligned} &\dot{x} = \frac{p_x}{m} - L\dot{\theta} \qquad (7) \\ &\frac{p_\theta}{m} = (3L^2 + x^2)\dot{\theta} + L\left(\frac{p_x}{m} - L\dot{\theta}\right) \implies \end{aligned}$$

$$\implies \frac{1}{m}(p_\theta - Lp_x) = (2L^2 + x^2)\dot{\theta} \implies \dot{\theta} = \frac{1}{m}\,\frac{p_\theta - Lp_x}{2L^2 + x^2} \qquad (8)$$

$$\dot{x} = \frac{p_x}{m} - \frac{1}{m}\,\frac{Lp_\theta - L^2 p_x}{2L^2 + x^2}$$

Substitute next in the expression for H:

$$H = p_x \dot{x} + p_\theta \dot{\theta} - L = \frac{p_x^2}{m} - \frac{L p_x p_\theta - L^2 p_x^2}{m(2L^2+x^2)} + \frac{p_\theta^2 - L p_\theta p_x}{m(2L^2+x^2)}$$

$$- \left[\frac{p_x^2}{2m} + \frac{(p_\theta - L p_x)^2}{2m(2L^2+x^2)} + mg\,(3L\cos\theta - x\sin\theta) \right]$$

$$= \frac{p_x^2}{2m} + \frac{1}{m(2L^2+x^2)} \left(-L p_x p_\theta + L^2 p_x^2 + p_\theta^2 - \cancel{L p_\theta p_x} - \frac{p_\theta^2}{2} - \frac{L^2 p_x^2}{2} + \cancel{L p_\theta p_x} \right)$$

$$- mg\,(3L\cos\theta - x\sin\theta)$$

$$= \frac{p_x^2}{2m} + \frac{(p_\theta - L p_x)^2}{2m(2L^2+x^2)} - mg\,(3L\cos\theta - x\sin\theta) = \text{which is } T+V$$

Here we have used that the Lagrangian $L(q, \dot{q}(q,p))$ can be cast as

$$L = \frac{1}{2} m (L\dot{\theta} + \dot{x})^2 + \frac{1}{2} m (2L^2+x^2)\dot{\theta}^2 + mg\,(3L\cos\theta - x\sin\theta) = (7),(8) =$$

$$= \frac{1}{2} m \left(\frac{p_x}{m}\right)^2 + \frac{1}{2} m (2L^2+x^2) \frac{1}{m^2} \left(\frac{p_\theta - L p_x}{2L^2+x^2} \right)^2 + mg\,(3L\cos\theta - x\sin\theta)$$

$$= \frac{p_x^2}{2m} + \frac{(p_\theta - L p_x)^2}{2m(2L^2+x^2)} + mg\,(3L\cos\theta - x\sin\theta)$$

Alternative approach to oscillations:

$$\left.\begin{aligned} &(8)\quad 2L^2 \delta\ddot{\theta} + 2gL\,\delta\theta + g\,\delta x = 0 \\ &(9)\quad L\,\delta\ddot{\theta} + \delta\ddot{x} + g\,\delta\theta = 0 \end{aligned}\right\} \longrightarrow \left.\begin{aligned} &2L(-L\omega_\theta^2 + g)\,\delta\theta + g\,\delta x = 0 \\ &(-L\omega_\theta^2 + g)\,\delta\theta - \omega_x^2\,\delta x = 0 \end{aligned}\right\} \quad (S)$$

Oscillations are of the form

$$\left.\begin{aligned} \delta\theta &= A_\theta \cos(\omega_\theta t + \varphi_\theta) \\ \delta x &= A_x \cos(\omega_x t + \varphi_x) \end{aligned}\right\} \Rightarrow \begin{aligned} \delta\ddot{\theta} &= -\omega_\theta^2\,\delta\theta \\ \delta\ddot{x} &= -\omega_x^2\,\delta x \end{aligned}$$

For (S) to have non-trivial solutions

$$0 = \det(\text{coeff}) = \det \begin{pmatrix} 2L(-L\omega_\theta^2 + g) & g \\ -L\omega_\theta^2 + g & -\omega_x^2 \end{pmatrix} = (-L\omega_\theta^2 + g)(-2L\omega_x^2 - g) = 0 \Rightarrow$$

$$\Rightarrow \omega_\theta^2 = \frac{g}{L}, \quad \omega_x^2 = -\frac{g}{2L} \Rightarrow \begin{cases} \text{Stable oscillations about } \theta_0 = 0 \\ \text{No} \quad '' \quad '' \quad '' \quad x_0 = 0 \end{cases}$$