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Classical Mechanics - Final exam - June 17th, 2026

1 [2 points]. A particle of mass m moves in a central force field. The particle follows a trajectory $r = ae^{b\theta}$ of known angular momentum L , with a a constant with dimensions of length and b a real parameter. Assuming that the potential is zero at infinity, find in terms of L , a and b the potential energy. Write the trajectory as a function $r = r(t)$ of time if initially the particle is at r_0 . Compute the trajectory's total energy.

The equation of motion in Binet's form is

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2} f\left(\frac{1}{u}\right), \quad u = \frac{1}{r}, \quad f(r) = -\frac{dV(r)}{dr},$$

where $V(r)$ is the potential energy. Using

$$u = \frac{1}{a} e^{-b\theta} \Rightarrow \frac{d^2u}{d\theta^2} = \frac{b^2}{a} e^{-b\theta} = b^2u,$$

it follows that

$$(b^2 + 1)u = -\frac{m}{L^2u^2} f\left(\frac{1}{u}\right).$$

Hence

$$f\left(\frac{1}{u}\right) = -\frac{L^2(b^2 + 1)}{m} u^3 \Leftrightarrow f(r) = -\frac{L^2(b^2 + 1)}{mr^3}.$$

This gives (1)

$$V(r) = -\int dr f(r) = -\frac{L^2(b^2 + 1)}{2mr^2} + k,$$

with k an integration constant. Using that $V(r \rightarrow \infty) = 0$, we have $k = 0$ and

$$V(r) = -\frac{L^2(b^2 + 1)}{2mr^2}.$$

Upon using conservation of angular momentum $L = mr^2\dot{\theta}$, the time derivative of r reads

$$\dot{r} = \frac{dr}{d\theta} \frac{d\theta}{dt} = bae^{b\theta} \frac{L}{mr^2} = \frac{bL}{mr}.$$

Integration yields

$$rdr = \frac{bL}{m} dt \Rightarrow \frac{1}{2} r^2 = \frac{bL}{m} t + c$$

with c an integration constant. Using that $r = r_0$ at $t = 0$, we have $c = r_0^2/2$, and so

$$r = \sqrt{\frac{2bL}{m} t + r_0^2}.$$

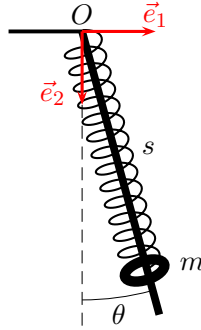
The orbit's energy is

$$E = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} - \frac{L^2(b^2 + 1)}{2mr^2} = \frac{1}{2} m \dot{r}^2 - \frac{L^2 b^2}{2mr^2} = 0,$$

where in the last equality we have used $\dot{r} = bL/mr$.

2 [2 points]. A massless spring (of spring constant k and natural length a) is threaded around a rigid rod. A ring of mass m is attached to the lower end of the spring. The ring slides up and down the rod with no friction. The upper end of the rod is attached to the ceiling through a hinge, so that the whole device may swing like a pendulum. Derive the Lagrangian in the following two cases:

- (a) The rod is masless.
- (b) The rod has mass M and length ℓ .



- (a) The ring's position is specified by coordinates

$$\left. \begin{aligned} x &= s \sin \theta \\ y &= s \cos \theta \end{aligned} \right\} \Rightarrow \left. \begin{aligned} \dot{x} &= \dot{s} \sin \theta + s \dot{\theta} \cos \theta \\ \dot{y} &= \dot{s} \cos \theta - s \dot{\theta} \sin \theta \end{aligned} \right\} \Rightarrow \dot{x}^2 + \dot{y}^2 = \dot{s}^2 + s^2 \dot{\theta}^2.$$

The ring's kinetic energy is

$$T = \frac{1}{2} m^2 (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m^2 (\dot{s}^2 + s^2 \dot{\theta}^2).$$

The potential energy is due the gravitational field and the spring,

$$V = -mgy + \frac{1}{2} k (s - a)^2 = -mgs \cos \theta + \frac{1}{2} k (s - a)^2.$$

This gives for the Lagrangian

$$L = T - V = \frac{1}{2} m^2 (\dot{s}^2 + s^2 \dot{\theta}^2) + mgs \cos \theta - \frac{1}{2} k (s - a)^2.$$

- (b) The kinetic energy is now as before **plus** the rotational energy of the rod due to its rotataion about the axis orthogonal to the $(\vec{e}_1, \vec{e}_2)$ -plane going through O ,

$$T = T_{(a)} + T_{\text{rot}}, \quad T_{\text{rot}} = \frac{1}{2} I_O \omega^2 = \frac{1}{6} M \ell^2 \dot{\theta}^2.$$

Here we have used that the moment of inertia of a bar of mass M and length ℓ about an axis perpendicular to it at one of its ends is $I_O = \frac{1}{3} M \ell^2$, and that the angular velocity with which the rod rotates is $\dot{\theta}$.

The potential energy is as before **plus** the gravitational pontential energy of the bar,

$$V = V_{(a)} + V_M, \quad V_M = -Mg \frac{\ell}{2} \cos \theta.$$

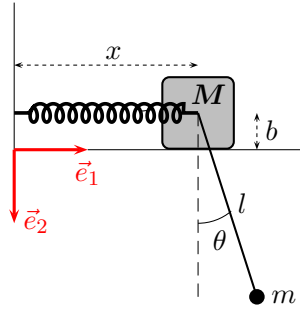
All in all, the Lagrangian is

$$L = L_{(a)} + T_{\text{rot}} - V_M = L_{(a)} + \frac{1}{6} M \ell^2 \dot{\theta}^2 + \frac{1}{2} Mg \ell \cos \theta.$$

3 [3 points]. A block of mass M slides without friction on a table. The block is attached to a wall through a spring of natural length a and spring constant k . A simple pendulum of length l and mass m hangs from its center (see the figure). The Lagrangian of the system is

$$L = \frac{1}{2}(M + m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{x}\dot{\theta}\cos\theta - \frac{1}{2}k(x - a)^2 + mgl\cos\theta - mgb$$

where b is a constant (equal to the distance of the center's block to the table). Derive the system's Euler-Lagrange equations. Find the equilibrium points (x_0, θ_0) of the system. Determine if there are stable oscillations $(\delta x, \delta\theta)$ about them and, if so, find their angular frequencies.



We must find the specific form that the Euler-Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$$

take for L given above with $q = x, \theta$. All we have to do is partially differentiate,

$$\begin{aligned} \frac{\partial L}{\partial x} &= -k(x - a) & \frac{\partial L}{\partial \dot{x}} &= (M + m)\dot{x} + ml\dot{\theta}\cos\theta \\ \frac{\partial L}{\partial \theta} &= -ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta & \frac{\partial L}{\partial \dot{\theta}} &= ml^2\dot{\theta} + ml\dot{x}\cos\theta, \end{aligned}$$

and substitute,

$$\left. \begin{aligned} (M + m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta + k(x - a) &= 0 \\ ml^2\ddot{\theta} + ml\ddot{x}\cos\theta - \cancel{ml\dot{x}\dot{\theta}\sin\theta} - (-\cancel{ml\dot{x}\dot{\theta}\sin\theta} - mgl\sin\theta) &= 0 \end{aligned} \right\}$$

This gives

$$\left. \begin{aligned} (M + m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta + k(x - a) &= 0 \\ ml\ddot{\theta} + m\ddot{x}\cos\theta + mg\sin\theta &= 0 \end{aligned} \right\} \quad (1)$$

The equilibrium points are constant solutions $x = x_0, \theta = \theta_0$ (hence $\dot{x} = \ddot{x} = \dot{\theta} = \ddot{\theta} = 0$) to these equations,

$$\left. \begin{aligned} x_0 - a &= 0 \\ \sin\theta_0 &= 0 \end{aligned} \right\} \Rightarrow x_0 = a, \quad \theta_0 = 0, \quad \pi \leftarrow \text{unstable}.$$

The value $\theta_0 = \pi$ has been discarded since it corresponds to the mass m in the upright position, which is clearly unstable.

For small deviations near the equilibrium points, $x = a + \delta x$ and $\theta = \delta\theta$. Substitution in eqs. (1) and expansion in powers of δx and $\delta\theta$ yields

$$\left. \begin{aligned} (M+m)\delta\ddot{x} + ml\delta\ddot{\theta} [1 + O(\delta\theta^2)] - ml\delta\dot{\theta}^2 [\delta\theta + O(\delta\theta^3)] + k\delta x &= 0 \\ ml\delta\ddot{\theta} + m\delta\ddot{x} [1 + O(\delta\theta^2)] + mg[\delta\theta + O(\delta\theta^3)] &= 0 \end{aligned} \right\}.$$

Keeping terms of order one in δx and $\delta\theta$, one has

$$\left. \begin{aligned} (M+m)\delta\ddot{x} + ml\delta\ddot{\theta} + k\delta x &= 0 \\ ml\delta\ddot{\theta} + m\delta\ddot{x} + mg\delta\theta &= 0 \end{aligned} \right\}. \quad (2)$$

Eqs. (2) are a normal mode problem. They can be written in matrix form as

$$\begin{pmatrix} M+m & m \\ m & m \end{pmatrix} \begin{pmatrix} \delta\ddot{x} \\ l\delta\ddot{\theta} \end{pmatrix} = \begin{pmatrix} -k & 0 \\ 0 & -m\frac{g}{l} \end{pmatrix} \begin{pmatrix} \delta x \\ l\delta\theta \end{pmatrix}.$$

Multiplying both sides from the left with the inverse of the matrix on the left side,

$$\begin{pmatrix} M+m & m \\ m & m \end{pmatrix}^{-1} = \frac{1}{Mm} \begin{pmatrix} m & -m \\ -m & M+m \end{pmatrix},$$

one obtains

$$\begin{pmatrix} \delta\ddot{x} \\ l\delta\ddot{\theta} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} -k & m\frac{g}{l} \\ k & -(M+m)\frac{g}{l} \end{pmatrix} \begin{pmatrix} \delta x \\ l\delta\theta \end{pmatrix}.$$

The eigenvalues λ of the matrix on the right side are

$$\begin{aligned} 0 = \det \begin{pmatrix} -\frac{k}{M} - \lambda & \frac{mg}{Ml} \\ \frac{k}{M} & -\left(1 + \frac{m}{M}\right)\frac{g}{l} - \lambda \end{pmatrix} &= \lambda^2 + \lambda \left[\frac{k}{M} + \frac{g}{l} \left(1 + \frac{m}{M}\right) \right] + \frac{kg}{Ml} \Rightarrow \\ \Rightarrow \lambda &= \frac{1}{2} \left\{ -\frac{k}{M} - \frac{g}{l} \left(1 + \frac{m}{M}\right) \pm \sqrt{\left[\frac{k}{M} + \frac{g}{l} \left(1 + \frac{m}{M}\right) \right]^2 - \frac{4kg}{Ml}} \right\}. \end{aligned}$$

Stable oscillation occur for λ real and negative with angular frequency ω given by $\omega^2 = -\lambda$. Note that $\lambda < 0$ requires the argument of the square root to be nonnegative,

$$\left[\frac{k}{M} + \frac{g}{l} \left(1 + \frac{m}{M}\right) \right]^2 \geq \frac{4kg}{Ml}.$$

In the grading I have positively considered any reasonable comment concerning stability.

4 [1 point]. Write in terms of $q(t)$ and $\dot{q}(t)$ the variations $\delta q(t)$ and $\delta[\dot{q}^2(t)]$ under a time translation $t \rightarrow t + \epsilon$.

By definition of variation, and upon expanding in powers of ϵ , one has

$$\delta \dot{q}(t) = \frac{d}{dt} q(t + \epsilon) - \frac{d}{dt} q(t) = \frac{d}{dt} [\cancel{q(t)} + \dot{q}(t) \epsilon + O(\epsilon^2) - \cancel{q(t)}] = \epsilon \ddot{q}(t) + O(\epsilon^2).$$

Similarly

$$\begin{aligned} \delta[\dot{q}(t)]^2 &= \left[\frac{d}{dt} q(t + \epsilon) \right]^2 - \left[\frac{d}{dt} q(t) \right]^2 = \left[\frac{d}{dt} \left\{ q(t) + \dot{q}(t) \epsilon + O(\epsilon^2) \right\} \right]^2 - \left[\frac{d}{dt} q(t) \right]^2 \\ &= \left[\dot{q}(t) + \ddot{q}(t) \epsilon + O(\epsilon^2) \right]^2 - \dot{q}(t)^2 = \cancel{\dot{q}(t)^2} + 2\epsilon \dot{q}(t) \ddot{q}(t) + O(\epsilon^2) - \cancel{\dot{q}(t)^2} \\ &= \frac{\epsilon}{2} \frac{d}{dt} \dot{q}(t)^2 + O(\epsilon^2). \end{aligned}$$

5 [1.5 points]. The Hamiltonian of a particle is given by

$$H = \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) + V(r),$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the radial distance, x, y, z are Cartesian generalized coordinates, p_x, p_y, p_z are their conjugate momenta, and $V(r)$ is a central potential. Write in terms of x, y, z, p_x, p_y and p_z the y -th component L_y of the particle's angular momentum. The variation with time of a dynamical variable $f(t; q, p)$ is given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\},$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. Using this, show that L_y is conserved.

From the definition of angular momentum $\vec{L} = \vec{x} \wedge \vec{p}$, it follows that $L_y = z p_x - x p_z$, which does not depend on time explicitly. Hence

$$\frac{\partial L_y}{\partial t} = 0 \Rightarrow \frac{dL_y}{dt} = \{L_y, H\}.$$

Recall that the Poisson bracket of two dynamical variables $f(t; q_i, p_i)$ and $g(t; q_i, p_i)$ is

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right),$$

where q_i are the system's generalized coordinates and p_i their conjugate momenta. In our case $q_i = x, y, z$; $p_i = p_x, p_y, p_z$ and L_y does not depend on y , nor p_y , so that

$$\frac{dL_y}{dt} = \{L_y, H\} = \frac{\partial L_y}{\partial x} \frac{\partial H}{\partial p_x} - \frac{\partial L_y}{\partial p_x} \frac{\partial H}{\partial x} + \frac{\partial L_y}{\partial z} \frac{\partial H}{\partial p_z} - \frac{\partial L_y}{\partial p_z} \frac{\partial H}{\partial z}. \quad (3)$$

The partial derivatives in this expressions are easily evaluated

$$\begin{aligned} \frac{\partial L_y}{\partial x} &= -p_z, & \frac{\partial L_y}{\partial z} &= p_x, & \frac{\partial L_y}{\partial p_x} &= z, & \frac{\partial L_y}{\partial p_z} &= -x, \\ \frac{\partial H}{\partial x^i} &= \frac{dV(r)}{dr} \frac{\partial r}{\partial x^i} = V'(r) \frac{x^i}{r} & \frac{\partial H}{\partial p_i} &= \frac{p_i}{m} & (i = x, y, z). \end{aligned}$$

Upon substituting in eq. (3). it is trivial that

$$\frac{dL_y}{dt} = -p_z \frac{p_x}{m} - z V'(r) \frac{x}{r} + p_x \frac{p_z}{m} + x V'(r) \frac{x}{r} = 0,$$

which implies constancy (conservation) in time of L_y .

6 [0.5 points]. An observer at the origin of an inertial frame emits light at $t = 0$. At what time an observer located in $\vec{x} = (0, 0, \frac{1}{2})$ will detect it?

At time t the light front wave is at $|\vec{x}| = ct$. The light will reach an observer at $\vec{x} = (0, 0, \frac{1}{2})$ at a time $ct = \frac{1}{2}$.
