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Classical Mechanics - Final exam - January 15th, 2026

1 [2.5 points]. The trajectory of a particle in a central force field is

$$r(\theta) = \frac{r_0}{b\theta + 1},$$

where r_0 and b are constants, and θ is the polar angle. Find the potential energy in terms of r and of the angular momentum L of the trajectory. Write the trajectory as a function $r = r(t)$ if initially the particle is at r_0 . Compute the trajectory's total energy. Assume that the potential energy at $r \rightarrow \infty$ vanishes.

Binet's equation is

$$\frac{d^2 u}{d\theta^2} + u = -\frac{m}{L^2 u^2} f\left(\frac{1}{u}\right) \quad u(\theta) = \frac{1}{r(\theta)} \quad (1.1)$$

Using the expression for the trajectory,

$$u(\theta) = \frac{b\theta + 1}{r_0} \Rightarrow \frac{du}{d\theta} = \frac{b}{r_0} \Rightarrow \frac{d^2 u}{d\theta^2} = 0,$$

and substituting in eq. (1.1), one has

$$\frac{1}{r} = -\frac{m}{L^2} r^2 f(r) \Rightarrow f(r) = -\frac{L^2}{mr^3}. \quad \Rightarrow$$

The potential energy is then

$$V(r) = -\int f(r) dr = \frac{L^2}{m} \int \frac{dr}{r^3} = -\frac{L^2}{2mr^2} + V_0. \quad 0.75$$

To determine the integration constant V_0 , use the data that $V(r \rightarrow \infty) = 0$. This gives $V_0 = 0$, so that 0.25

$$V(r) = -\frac{L^2}{2mr^2}$$

To find $r(t)$ we first find $\frac{dr}{dt}$ and then integrate:

$$\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} = -\frac{b}{r_0} r^2 \frac{L}{mr^2} = -\frac{bL}{mr_0} \Rightarrow dr = -\frac{bL}{mr_0} dt \quad (2.1)$$

$$\frac{dr}{d\theta} = \frac{d}{d\theta} \left(\frac{r_0}{b\theta+1} \right) = -\frac{r_0 b}{(b\theta+1)^2} = -\frac{b}{r_0} r^2$$

$$L = mr^2 \dot{\theta} \Rightarrow \dot{\theta} = \frac{L}{mr^2}$$

Integration of (2.1) gives

$$r = -\frac{bL}{mr_0} t + C \quad 0.75$$

The integration constant C is determined from the initial condition

$$r(t=0) = r_0 \Rightarrow C = r_0 \Rightarrow r = -\frac{bL}{mr_0} t + r_0 \quad 0.25$$

Finally the total energy is

$$E = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} + V(r) \Rightarrow E = \frac{b^2 L^2}{2mr_0^2} \quad 0.5$$

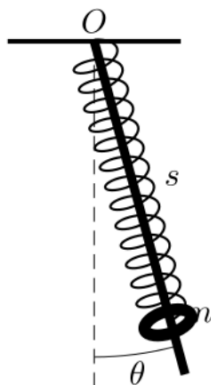
$$V(r) = -\frac{L}{2mr^2} ; \quad \dot{r} = -\frac{bL}{mr_0}$$

2 [4 points]. A massless spring (of spring constant k and natural length a) is threaded around a massless rigid rod. A ring of mass m is attached to the lower end of the spring. The ring slides up and adown the rod with no friction. The upper end of the rod is attached to the ceiling through a hinge, so that the whole device may swing like a pendulum. The Lagrangian of the system is

$$L = \frac{1}{2} m (\dot{s}^2 + s^2 \dot{\theta}^2) - \frac{1}{2} k (s - a)^2 + mgs \cos \theta,$$

where s is the distance along the rod from the hinge to the ring.

Write the Euler-Lagrange equations. Find the equilibrium points. Find the frequencies of small oscillation about them. Compute the Hamiltonian. Obtain the Hamilton equations.



From the lagrangian's expression we have

$$p_s = \frac{\partial L}{\partial \dot{s}} = m\dot{s}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ms^2\dot{\theta} \quad (3.1)$$

$$\frac{\partial L}{\partial s} = m\dot{\theta}^2 - k(s-a) + mg\cos\theta$$

$$\frac{\partial L}{\partial \theta} = -mgs\sin\theta$$

The Euler-Lagrange equations then read

$$EL_s: m\ddot{s} - m\dot{\theta}^2 + k(s-a) - mg\cos\theta = 0$$

$$EL_\theta: ms^2\ddot{\theta} + 2ms\dot{s}\dot{\theta} + mgs\sin\theta = 0$$

0-75

To find the equilibrium points set $s=s_0=\text{const.}$ and $\theta=\theta_0=\text{const.}$ in equations EL_s and EL_θ :

$$\left. \begin{array}{l} k(s_0-a) - mg\cos\theta_0 = 0 \\ s_0\sin\theta_0 = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} s_0=0 \Rightarrow \text{spring fully compressed. Unstable.} \\ \theta_0=0, \pi \rightarrow \theta_0=0 \Rightarrow k(s_0-a) - mg = 0 \end{array} \right.$$

Discard $\theta_0=\pi$. It is unstable since the mass is straight up.

The only equilibrium point which may have stable oscillation is

$$s_0 = \frac{mg}{k} + a, \quad \theta_0 = 0$$

0.75

To study oscillations about it, set

$$s = s_0 + \delta s, \quad \theta = \theta_0 + \delta\theta = \delta\theta,$$

substitute in eqs EL_s and EL_θ , and linearize (= keep terms of order up to 1 in δs and $\delta\theta$):

$$EL_s: \quad m\ddot{\delta s} - m(s_0 + \delta s)\dot{\delta\theta}^2 + \underbrace{k(s_0 - a + \delta s)}_{mg + k\delta s} - \underbrace{mg \cos \delta\theta}_{mg[1 - \frac{1}{2}(\delta\theta)^2 + \dots]} = 0$$

$\uparrow$ order 1 in δ $\uparrow$ order 2

$$m\ddot{\delta s} + k\delta s + mg - mg + O(\delta^2) = 0$$

$$EL_\theta: \quad \underbrace{(s_0 + \delta s)^2 \ddot{\delta\theta}}_{s_0^2 \ddot{\delta\theta} + O(\delta^2)} + 2(s_0 + \delta s) \underbrace{\dot{\delta s} \dot{\delta\theta}}_{\text{order 2}} + \underbrace{g(s_0 + \delta s) \sin \delta\theta}_{g s_0 [\delta\theta + O(\delta^2)]} = 0$$

$$s_0 \ddot{\delta\theta} + g \delta\theta = 0$$

Hence

$$\left. \begin{aligned} \delta\ddot{s} &= -\frac{k}{m} \delta s \\ \delta\ddot{\theta} &= -\frac{g}{s_0} \delta\theta \end{aligned} \right\} \Rightarrow$$

stable oscillations about
 $s_0 = a + \frac{mg}{k}$ with frequency $\omega_s = \sqrt{\frac{k}{m}}$ and
 $\theta_0 = 0$ with frequency $\omega_\theta = \sqrt{\frac{g}{s_0}} = \sqrt{\frac{gk}{mg + ka}}$

From eqs. (3.1) it follows that

$$\dot{s} = \frac{p_s}{m}, \quad \dot{\theta} = \frac{p_\theta}{ms^2}$$

This gives for the Hamiltonian

$$H = p_s \dot{s} + p_\theta \dot{\theta} - L = p_s \frac{p_s}{m} + p_\theta \frac{p_\theta}{ms^2} - \left[\frac{1}{2} m \left(\frac{p_s^2}{m^2} + s^2 \frac{p_\theta^2}{m^2 s^4} \right) - \frac{1}{2} k(s-a)^2 + mgs \cos \theta \right]$$

Lagrangian must be written on terms of p_s, p_θ, s, θ

$$H = \frac{p_s^2}{2m} + \frac{p_\theta^2}{2ms^2} + \frac{1}{2} k(s-a)^2 - mgs \cos \theta$$

0.5

Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad - \dot{p}_i = \frac{\partial H}{\partial q_i}$$

are

$$\dot{s} = \frac{\partial H}{\partial p_s}$$

$$\dot{s} = \frac{p_s}{m}$$

(1)

$$-\dot{p}_s = \frac{\partial H}{\partial s}$$

$$-\dot{p}_s = -\frac{p_\theta^2}{ms^3} + k(s-a) - mg \cos \theta$$

(2)

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta}$$

$$\dot{\theta} = \frac{p_\theta}{ms^2}$$

(3)

$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta}$$

$$-\dot{p}_\theta = mg \sin \theta$$

2.5

(4)

It is clear that

$$(2): -\dot{p}_s = [\text{eq. (1)}] = -\frac{d}{dt}(m\dot{s}) = -m\ddot{s}$$

$$\parallel$$

$$-\frac{p_\theta^2}{ms^3} + k(s-a) - mg \cos \theta = [\text{eq. (3)}] = -\frac{(ms^2\dot{\theta})^2}{ms^3} + k(s-a) - mg \cos \theta$$

$$-m\ddot{s} = -ms\dot{\theta}^2 + k(s-a) - mg \cos \theta \Leftrightarrow EL_s$$

0.5

$$(4): -\dot{p}_\theta = \frac{d}{dt}(ms^2\dot{\theta}) = ms^2\ddot{\theta} + 2ms\dot{s}\dot{\theta}$$

$$\parallel$$

$$-mg \sin \theta$$

$$\left. \begin{array}{l} ms^2\ddot{\theta} + 2ms\dot{s}\dot{\theta} + mg \sin \theta = 0 \\ \Downarrow \\ EL_\theta \end{array} \right\}$$

3 [1.5 points]. Consider a rigid body on which no external force or torque acts. Use the Euler equations to prove that its kinetic energy is conserved.

The Euler equations for a rigid solid upon which no external force or torque acts are (in the principal frame)

$$I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\omega}_2 + \omega_3 \omega_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\omega}_3 + \omega_1 \omega_2 (I_2 - I_1) = 0$$

The solid's kinetic energy (in the principal frame) is

$$T = \frac{1}{2} \omega_i I_{ij} \omega_j = (\text{principal frame}) = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$$

Differentiation with respect to time gives

$$\frac{dT}{dt} = I_1 \omega_1 \dot{\omega}_1 + I_2 \omega_2 \dot{\omega}_2 + I_3 \omega_3 \dot{\omega}_3 = \{ \text{use Euler's equations} \}$$

$$= -\omega_1 \omega_2 \omega_3 (I_3 - I_2) - \omega_2 \omega_3 \omega_1 (I_1 - I_3) - \omega_3 \omega_1 \omega_2 (I_2 - I_1) = 0 \Rightarrow$$

$$\Rightarrow T \text{ is constant on time (conserved)}$$

1.5

The question asked is: Prove $T = \text{const.}$ using Euler equations.

4 [2 points]. In reference frame S an event 1 takes place at time $ct_1 = 1$ in position $\mathbf{x}_1 = (0, 0, 0)$. A second event takes place at $(ct_2, \mathbf{x}_2) = (4, 1, 0, 0)$. Can they be causally connected? In case they can, find an inertial frame S' in which events take place at the same position but at different times. If not, find an inertial frame S' in which events occur at the same time but in different positions.

Answer the same question if the events on S take place at

$$(ct_1, \mathbf{x}_1) = (0, 1, 0, 0)$$

$$(ct_2, \mathbf{x}_2) = (3, 5, 0, 0)$$

a) The 4-vector x^μ from P_1 to P_2 is

$$x^\mu = (c(t_2 - t_1), \vec{x}_2 - \vec{x}_1) = (4 - 1, 1, 0, 0) = (3, 1, 0, 0)$$

It is time-like, since

$$x^2 = 3^2 - 1^2 - 0 - 0 = 8 > 0 \Rightarrow \text{The two events can be causally connected}$$

Relative to a frame S' that moves with velocity v along the x -axis, one has

$$ct' = \gamma(ct - \frac{v}{c}x)$$

$$x' = \gamma(x - \frac{v}{c}ct)$$

$$\text{with } ct=3, x=1.$$

We want $x'=0$:

$$1 - \frac{v}{c}3 = 0 \Rightarrow v = \frac{c}{3} \text{ In } S' \text{ both events are seen at the same point but different times (one after the other)}$$

b) In the second case, we have

$$x^\mu = (c(t_2 - t_1), \vec{x}_2 - \vec{x}_1) = (3 - 0, 5 - 1, 0, 0) = (3, 4, 0, 0).$$

This is space-like since

$$s^2 = 3^2 - 4^2 - 0 - 0 = -7 < 0 \Rightarrow \text{The two events cannot be causally connected}$$

In a frame S'

$$ct' = \gamma(ct - \frac{v}{c}x)$$

$$x' = \gamma(x - \frac{v}{c}ct)$$

$$\text{with } ct=3, x=4$$

We want $ct'=0$:

$$3 - \frac{v}{c}4 = 0 \Rightarrow v = \frac{3}{4}c. \text{ In } S' \text{ both events are seen at the same coordinate time but different points.}$$