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Classical Mechanics - Mid-term exam - November 19th, 2025
(Time: 80 minutes)

1 [3 points]. A particle of mass m moves in a central force field. The particle follows a trajectory $r = ae^{b\theta}$ of known angular momentum L , with a a constant with dimensions of length and b a real parameter. Here r and θ are polar coordinates in the plane on which the motion takes place. Use Binet's equation to find in terms of L and b the potential energy $V(r)$.

(Part of problem 3 of January 17th 2024 final exam. Solution was in the course webpage). Binet's equation is

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2} f\left(\frac{1}{u}\right), \quad u = \frac{1}{r}, \quad f(r) = -\frac{dV(r)}{dr},$$

where $V(r)$ is the potential energy. Using

$$u = \frac{1}{a} e^{-b\theta} \Rightarrow \frac{d^2u}{d\theta^2} = \frac{b^2}{a} e^{-b\theta} = b^2u,$$

and substituting in Binet's equation, it follows that

$$(b^2 + 1)u = -\frac{m}{L^2u^2} f\left(\frac{1}{u}\right).$$

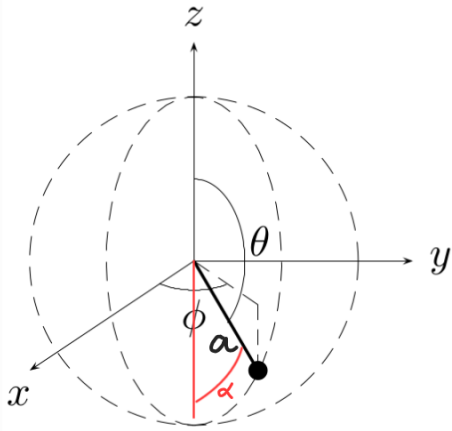
Hence

$$f\left(\frac{1}{u}\right) = -\frac{L^2(b^2 + 1)}{m} u^3 \Leftrightarrow f(r) = -\frac{L^2(b^2 + 1)}{mr^3}.$$

This gives (V_0 is an integration constant)

$$V(r) = -\int dr f(r) = -\frac{L^2(b^2 + 1)}{2mr^2} + V_0.$$

2 [7 points]. A spherical pendulum consists of a mass m attached to a fixed point by an inextensible string of length a subject to a uniform gravitational field, so that it moves on a sphere. Write the Lagrangian. Derive the Euler-Lagrange equations. Find the equilibrium points and study if oscillations about them are stable. Write the Hamiltonian.



The mass coordinates are

$$x = a \sin \theta \cos \phi$$

$$y = a \sin \theta \sin \phi$$

$$z = a \cos \theta$$

The Lagrangian is then

$$L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz = \frac{1}{2} m a^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - m g a \cos \theta \quad (1)$$

The Euler-Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad q^i = \theta, \phi$$

take in our case the form

$$\bullet \quad p_\theta := \frac{\partial L}{\partial \dot{\theta}} = m a^2 \dot{\theta}, \quad \frac{\partial L}{\partial \theta} = m a^2 \sin \theta \cos \theta \dot{\phi}^2 - m g a \sin \theta \Rightarrow$$

$$\Rightarrow m a^2 \ddot{\theta} - m a^2 \sin \theta \cos \theta \dot{\phi}^2 - m g a \sin \theta = 0$$

$$\ddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 - \frac{g}{a} \sin \theta = 0$$

(1)

$$\bullet \quad p_\phi := \frac{\partial L}{\partial \dot{\phi}} = m a^2 \sin^2 \theta \dot{\phi}, \quad \frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{d}{dt} (m a^2 \sin^2 \theta \dot{\phi}) = 0 \Rightarrow$$

$$\Rightarrow p_\phi \text{ is conserved, } J = \sin^2 \theta \dot{\phi} = \text{const.}$$

(2)

In terms of J , eq. (1) can be cast as

$$\ddot{\theta} - \frac{\cos \theta}{\sin^3 \theta} J^2 - \frac{g}{a} \sin \theta = 0$$

(3)

(1.5)

Equilibrium points have $\theta = \theta_0$. For them eq. (3) reads

$$\frac{\cos \theta_0}{\sin^3 \theta_0} J^2 + \frac{g}{a} \sin \theta_0 = 0 \quad \Leftrightarrow \quad \sin \theta_0 \left(\cos \theta_0 \dot{\phi}^2 + \frac{g}{a} \right) = 0 \quad \text{eq. (2)}$$

There are two solutions

1) $\sin \theta_0 = 0 \Rightarrow \theta_0 = 0, \pi$

$\theta_0 = 0$ is unstable (mass at the sphere's north pole)

$\theta_0 = \pi$. In this case, $J = \sin^2 \theta_0 \dot{\phi} = 0$ and eq. (3) becomes

$$\ddot{\theta} - \frac{g}{a} \sin \theta = 0 \quad (4)$$

which is the equation for the simple pendulum. Note that the usual formulation of the simple pendulum is made in terms of the angle α given by $\theta = \pi - \alpha$, so that

$$\ddot{\alpha} + \frac{g}{a} \sin \alpha = 0 \quad (1)$$

The simple pendulum has small oscillations of frequency $\Omega = \sqrt{\frac{g}{a}}$

2. $\cos \theta_0 \dot{\phi}^2 + \frac{g}{a} = 0$. This implies $\frac{\pi}{2} < \theta_0 \leq \pi$ and $\dot{\phi} = \sqrt{-\frac{g}{a \cos \theta_0}} =: \omega$

In this case the mass rotates about the z -axis with constant angular velocity $\dot{\phi}$. To study the stability of this orbit, set

$\theta = \theta_0 + \delta\theta$ and linearize eq. (3) (1)

$$\delta\ddot{\theta} - \frac{\cos(\theta_0 + \delta\theta)}{\sin^3(\theta_0 + \delta\theta)} J^2 - \frac{g}{a} \sin(\theta_0 + \delta\theta) = 0 \quad \left. \begin{array}{l} \text{expand in powers of } \delta\theta \\ \downarrow \end{array} \right\}$$

$$\delta\ddot{\theta} - \left[\frac{\cos \theta_0}{\sin^3 \theta_0} - \left(\frac{1}{\sin^2 \theta_0} + \frac{3 \cos^2 \theta_0}{\sin^4 \theta_0} \right) \delta\theta \right] J^2 - \frac{g}{a} (\sin \theta_0 + \cos \theta_0 \delta\theta) + O(\delta\theta^2) = 0$$

$$J^2 = \sin^4 \theta_0 \dot{\phi}^2 = - \frac{\sin^4 \theta_0 g}{\cos \theta_0 a}$$

$$\delta\ddot{\theta} - \frac{g}{a} \left[(\sin^2\theta_0 + 3\cos^2\theta_0) \frac{1}{\cos\theta_0} + \cos\theta_0 \right] \delta\theta = 0$$

$$\delta\ddot{\theta} + \frac{g}{a\cos\theta_0} (1+3\cos^2\theta_0) \delta\theta = 0 \Leftrightarrow \delta\ddot{\theta} = -\Omega^2 \delta\theta$$

$$\Omega^2 := - \frac{g(1+3\cos^2\theta_0)}{a\cos\theta_0} = \omega^2 (1+3\cos\theta_0)$$

There are therefore stable oscillations about θ_0 with $\omega\theta_0 = - \frac{g\sin^4\theta_0}{a\omega^2}$.

(1.5)

The Hamiltonian is given by

$$H = \dot{q}^i p_i - L = \dot{\theta} p_\theta + \dot{\phi} p_\phi - L$$

$$= \frac{p_\theta}{ma^2} p_\theta + \frac{p_\phi}{ma^2 \sin^2\theta} p_\phi - \left\{ \frac{1}{2} ma^2 \left[\left(\frac{p_\theta}{ma^2} \right)^2 + \sin^2\theta \left(\frac{p_\phi}{ma^2 \sin^2\theta} \right)^2 \right] - mga \cos\theta \right\}$$

$$H = \frac{1}{2ma^2} \left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} \right) + mga \cos\theta = \text{function of } p_i \text{ and } q^i, \text{ not of } \dot{q}^i \text{ and } \dot{q}_i.$$

(1)