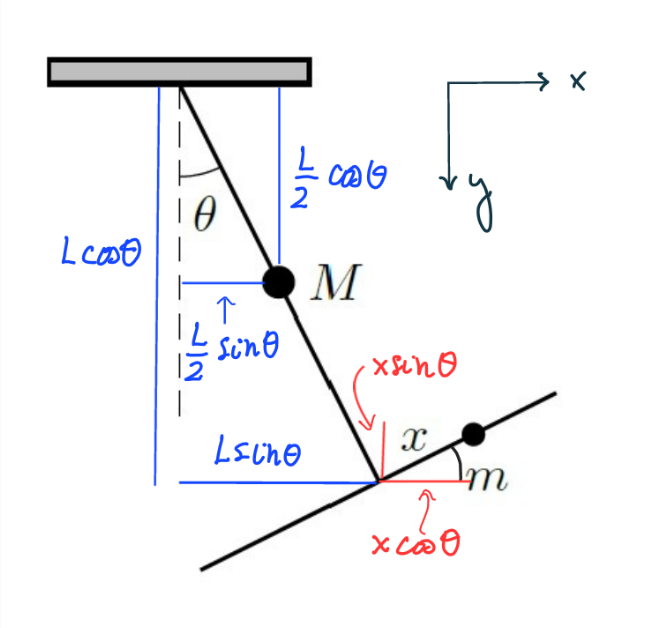
3.6. Consider a long thin rod of length L and neglegible mass with an end hinged at a support. A second rod, also of negligible mass, is welded at the other end. The first rod has a mass M fixed at its mid point. The second rod has a mass m that can freely slide along it. The system can rotate about the hinge in the vertical plane containing the two rods. See the figure. Write its Lagrangian and find the corresponding Euler-Lagrange equations. Discuss the oscillation modes for small oscillations of both masses with $M=2m$.

Hint. Use as generalized coordinates the angle θ and the distance x of the second particle to the crossing of the rods.



$$x_M = \frac{L}{2}\sin\theta$$
$$y_M = \frac{L}{2}\cos\theta$$

$$x_m = L\sin\theta + x\cos\theta$$
$$y_m = L\cos\theta - x\sin\theta$$

$$V = -mgy$$

The Lagrangian is

$$L = T - V = \frac{1}{2}M(\dot{x}_M^2 + \dot{y}_M^2) + \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2) - (-Mgy_M - mgy_m)$$

$$\dot{x}_M^2 + \dot{y}_M^2 = \frac{L^2}{4}\dot{\theta}^2$$

$$\dot{x}_m^2 + \dot{y}_m^2 = (L\dot{\theta}\cos\theta + \dot{x}\cos\theta - x\dot{\theta}\sin\theta)^2 + (-L\dot{\theta}\sin\theta - \dot{x}\sin\theta - x\dot{\theta}\cos\theta)^2$$
$$= L^2\dot{\theta}^2 + \dot{x}^2 + x^2\dot{\theta}^2 + 2L\dot{\theta}\dot{x}.$$

$$L = \frac{1}{8}ML^2\dot{\theta}^2 + \frac{1}{2}m(L^2\dot{\theta}^2 + \dot{x}^2 + x^2\dot{\theta}^2 + 2L\dot{\theta}\dot{x})$$
$$+ \frac{1}{2}MgL\cos\theta + mg(L\cos\theta - x\sin\theta)$$

It follows that

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{4}ML^2\dot{\theta} + mL^2\dot{\theta} + mx^2\dot{\theta} + mL\dot{x}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + mL\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -\frac{1}{2} MgL\sin\theta - mgL\sin\theta - mgx\cos\theta$$

$$\frac{\partial L}{\partial x} = mx\dot\theta^2 - mg\sin\theta$$

The EL equations for θ and x read

$$\frac{1}{4}ML^2\ddot\theta + \overbrace{mL^2\ddot\theta}^{1} + 2mx\dot x\dot\theta + mx^2\ddot\theta + \overbrace{mL\ddot x}^{2} + \\ + \frac{1}{2}MgL\sin\theta + \underbrace{mgL\sin\theta}_{3} + mgx\cos\theta = 0 \qquad (3.6\text{-}1)$$

$$m\ddot x + mL\ddot\theta - mx\dot\theta^2 + mg\sin\theta = 0 \qquad (3.6\text{-}2)$$

Noting that

$$1 + 2 + 3 = L(mL\ddot\theta + m\ddot x + mg\sin\theta) = \text{use eq } [x] = Lmx\dot\theta^2$$

eqs. (3.6.1) and (3.6.2) can be written as

$$\frac{1}{2}ML^2\ddot\theta + mLx\dot\theta^2 + 2mx\dot x\dot\theta + mx^2\ddot\theta + \frac{1}{2}MgL\sin\theta + mgx\cos\theta = 0$$

$$\ddot x + L\ddot\theta - x\dot\theta^2 + g\sin\theta = 0$$

Write these equations as

$$\frac{1}{4}M\ddot\theta + m\frac{x}{L}\dot\theta^2 + 2m\frac{x}{L}\frac{\dot x}{L}\dot\theta + m\left(\frac{x}{L}\right)^2\ddot\theta + \frac{1}{2}M\frac{g}{L}\sin\theta + m\frac{g}{L}\frac{x}{L}\cos\theta = 0$$

$$\frac{\ddot x}{L} + \ddot\theta - \frac{x}{L}\dot\theta^2 + \frac{g}{L}\sin\theta$$

$\theta(t)=0$, $x(t)=0$ is an equilibrium point. For small oscillations, $\theta \ll 1$ and $x \ll L$, about the equilibrium point, using $\sin\theta = \theta + O(\theta^3)$ and $\cos\theta = 1 + O(\theta^2)$, we have

$$\frac{M}{4}\ddot\theta + \frac{Mg}{2L}\theta + \frac{mg}{L}\frac{x}{L} + O\left(\frac{x}{L}\theta^2, \frac{x^2}{L^2}\theta, \theta^3\right) = 0 \qquad (3.6\text{-}3)$$

$$\frac{\ddot x}{L} + \ddot\theta + \frac{g}{L}\theta + O\left(\theta^3, \frac{x}{L}\theta^2\right) = 0 \qquad (3.6\text{-}4)$$

To solve these equations, we write them as

$$\left.\begin{aligned} L\ddot{\theta} &= -\frac{2g}{L}L\theta - \frac{4mg}{LM}x \\ L\ddot{\theta}+\ddot{x} &= -\frac{g}{L}L\theta \end{aligned}\right\} \Leftrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} L\ddot{\theta} \\ \ddot{x} \end{pmatrix} = \frac{g}{L}\begin{pmatrix} -2 & -\frac{4m}{M} \\ -1 & 0 \end{pmatrix}\begin{pmatrix} L\theta \\ x \end{pmatrix}$$

$$\begin{pmatrix} L\ddot{\theta} \\ \ddot{x} \end{pmatrix} = \frac{g}{L}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -2 & -\frac{4m}{M} \\ -1 & 0 \end{pmatrix}\begin{pmatrix} L\theta \\ x \end{pmatrix}$$

$$\begin{pmatrix} L\ddot{\theta} \\ \ddot{x} \end{pmatrix} = \frac{g}{L}\underbrace{\begin{pmatrix} -2 & -\frac{4m}{M} \\ 1 & \frac{4m}{M} \end{pmatrix}}_{F}\begin{pmatrix} L\theta \\ x \end{pmatrix}$$

The eigenvalues λ^2 of F are the solutions of

$$\det(F-\lambda^2\mathbb{1})=0 \Leftrightarrow \left(-\frac{2g}{L}-\lambda^2\right)\left(\frac{4mg}{LM}-\lambda^2\right)+\frac{4mg^2}{L^2M}=0$$

$$\lambda^4 - \lambda^2\frac{2g}{L}\left(\frac{2m}{M}-1\right)-\frac{4mg^2}{L^2M}=0$$

$$\lambda^2 = \frac{g}{L}\left(\frac{2m}{M}-1\right) \pm \sqrt{\frac{g^2}{L^2}\left(\frac{2m}{M}-1\right)^2+\frac{4mg^2}{L^2M}}$$

For e.g. $M=2m$,

$M=2m \Rightarrow \lambda^2 = \pm\frac{g}{L}\sqrt{2}$ $\begin{cases} \text{Positive sign corresponds to no oscillations} \\ \text{Negative sign: } \lambda^2=-\frac{g}{L}\sqrt{2} \Rightarrow \lambda=\pm i\omega,\ \omega=\left(\frac{\sqrt{2}g}{L}\right)^{1/2} \end{cases}$

The eigenvector associated to $\lambda^2=-\frac{g}{L}\sqrt{2}$ is

$$\frac{g}{L}\begin{pmatrix} -2 & -2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\frac{g}{L}\sqrt{2}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \begin{aligned} -2v_1-2v_2 &= \sqrt{2}v_1 \\ v_1+2v_2 &= \sqrt{2}v_2 \end{aligned} \Rightarrow$$

$$\Rightarrow v_2 = -\left(1+\frac{1}{\sqrt{2}}\right)v_1$$

There is only one normal oscillation mode, given by

$$\begin{pmatrix} L\theta \\ x \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ \sqrt{2}+1 \end{pmatrix} A\cos(\omega t+\varphi) \qquad \omega = \left(\frac{\sqrt{2}g}{L}\right)^{1/2}$$

with A,φ integration constants.