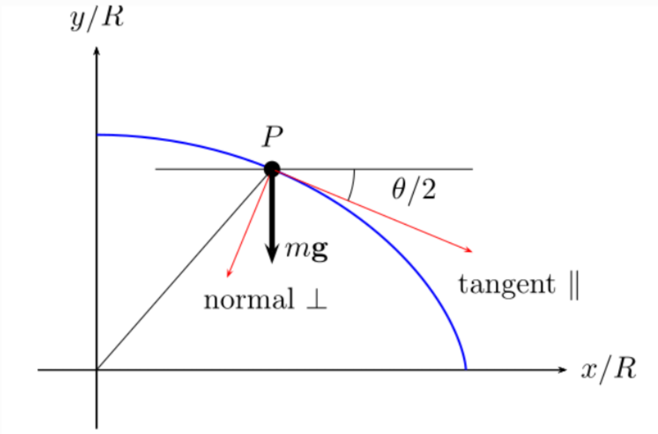
1.5. A particle of mass m is projected with initial velocity $v_0\mathbf{e}_1$ from the top of the cycloid

$$x = R(\theta + \sin\theta), \quad y = R(1 + \cos\theta), \quad 0 \le \theta \le \pi$$

drawn in the figure. Before moving on to study the motion of the particle, it is convenient to become acquainted with some basic properties of cycloids. Prove that the slope of the tangent dy/dx to the curve at the point P specified by an angle θ is -tan($\theta/2$). The distance ds between two points (x, y) and $(x + dx, y + dy)$ is given by $ds^2 = dx^2 + dy^2$. Show that, if the points are on the cycloid, this gives $s = 4R\sin(\theta/2)$. Use the definition $\rho = ds/d\theta$ of curvature radius to prove that $\rho = 2R\cos(\theta/2)$. Note that the angle formed by the y-axis and the OP segment is not linearly related to θ.

Assume now that the only force acting on the particle is the Earth's gravitational force -$m g\mathbf{e}_2$. As long as the component of the gravitational force in the normal direction to the curve is larger than the centrifugal force, the particle stays on the curve. At the point where the centrifugal force become larger, the particle leaves the cycloid and jumps into space describing a parabolic shot. Write the equations of motion while the particle is on the cycloid. Find the angle θ at which the particle leaves it. Find the initial velocity for which there is no sliding over the cycloid and motion is parabolic from the beginning.



$mg\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

a) $\left.\begin{array}{l} x = R(\theta + \sin\theta) \\ y = R(1+\cos\theta) \end{array}\right\} \Rightarrow \left.\begin{array}{l} \frac{dx}{d\theta} = R(1+\cos\theta) \\ \frac{dy}{d\theta} = -R\sin\theta \end{array}\right| \Rightarrow$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = -\frac{\cancel{R}\sin\theta}{\cancel{R}(1+\cos\theta)} = -\frac{2\sin\theta/2\,\cos\theta/2}{2\cos^2\theta/2} = -\tan\frac{\theta}{2}$$

b) $ds^2 = dx^2 + dy^2 = R^2(1+\cos\theta)^2 d\theta^2 + R^2\sin^2\theta\, d\theta^2$

$$= R^2(1+\cos^2\theta + 2\cos\theta + \sin^2\theta)\,d\theta^2 = 2R^2(1+\cos\theta)\,d\theta^2 = 4R^2\cos^2\frac{\theta}{2}\,d\theta^2$$

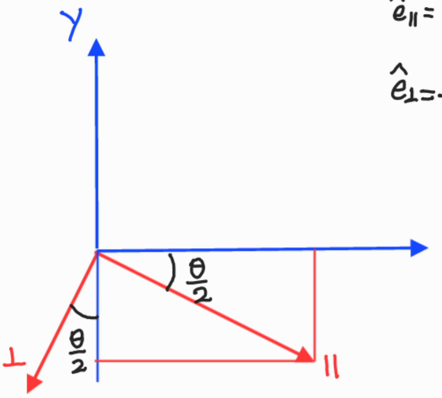
$$\Rightarrow \rho = \frac{ds}{d\theta} = 2R\cos\frac{\theta}{2} \quad \Rightarrow \quad s = 4R\sin\frac{\theta}{2} + \underbrace{\text{const}}_{=0}.$$

$=0$ since $s=0$ at $\theta=0$

c) The particle's velocity and acceleration are

$$\dot{\vec{x}} = \frac{ds}{dt}\hat{e}_{\parallel}, \qquad |\dot{\vec{x}}| = \frac{ds}{dt} = \frac{ds}{d\theta}\frac{d\theta}{dt} = \rho\dot{\theta} = 2R\cos\frac{\theta}{2}\,\dot{\theta}$$

$$\ddot{\vec{x}} = \frac{d^2s}{dt^2}\hat{e}_\parallel + \frac{ds}{dt}\dot{\hat{e}}_\parallel = \frac{d^2s}{dt^2}\hat{e}_\parallel + \frac{ds}{dt}\frac{\dot\theta}{2}\hat{e}_\perp = \frac{d^2s}{dt^2}\hat{e}_\parallel + \frac{1}{2}\rho\dot\theta^2\hat{e}_\perp$$



$$\left.\begin{aligned}\hat{e}_\parallel &= \hat{e}_1\cos\frac{\theta}{2} - \hat{e}_2\sin\frac{\theta}{2}\\ \hat{e}_\perp &= -\hat{e}_1\sin\frac{\theta}{2} - \hat{e}_2\cos\frac{\theta}{2}\end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned}\hat{e}_1 &= \hat{e}_\parallel\cos\frac{\theta}{2} - \hat{e}_\perp\sin\frac{\theta}{2}\\ \hat{e}_2 &= -\hat{e}_\parallel\sin\frac{\theta}{2} - \hat{e}_\perp\cos\frac{\theta}{2}\end{aligned}\right.$$

$$\Rightarrow \left.\begin{aligned}\dot{\hat{e}}_\parallel &= -\frac{\dot\theta}{2}\left(\hat{e}_1\sin\frac{\theta}{2} + \hat{e}_2\cos\frac{\theta}{2}\right) = \frac{\dot\theta}{2}\hat{e}_\perp\\ \dot{\hat{e}}_\perp &= -\frac{\dot\theta}{2}\left(\hat{e}_1\cos\frac{\theta}{2} - \hat{e}_2\sin\frac{\theta}{2}\right) = -\frac{\dot\theta}{2}\hat{e}_\parallel\end{aligned}\right\}$$

While the particle is on the curve we have

tangent direction $\parallel$: $\quad m\frac{d^2s}{dt^2} = mg\sin\frac{\theta}{2}$

$\rho = 2R\cos\frac{\theta}{2}$

normal direction $\perp$: $\quad m\frac{\rho\dot\theta^2}{2} \geq mg\cos\frac{\theta}{2} \Leftrightarrow R\dot\theta^2 \geq g$

Some authors write $\quad m\frac{\rho\dot\theta^2}{2} = mg\cos\frac{\theta}{2} + \mathcal{F}$

force of particle on the curve $= -$ reaction $= -\mathcal{R}$

Energy is conserved. Hence

$$E(t=0) = E(t) \Leftrightarrow \frac{1}{2}mv_0^2 + 2mgR = \frac{1}{2}mv^2 + mgR(1+\cos\theta) \Leftrightarrow$$

At $t=0$, $V = mgy(\theta=0) = mg2R$

$$\Leftrightarrow v_0^2 + 2gR(1-\cos\theta) = v^2 \qquad (E)$$

At the point $\theta=\theta_*$ at which the particle abandons the curve

$$v^2 = \dot{s}^2 = (\rho\dot\theta)^2 = \rho^2\frac{g}{R} = \left(2R\cos\frac{\theta_*}{2}\right)^2\frac{g}{R} = 4gR\cos^2\frac{\theta_*}{2}$$

$$(E) \Leftrightarrow v_0^2 + 2gR\,2\sin^2\frac{\theta_*}{2} = 4gR\cos^2\frac{\theta_*}{2}$$

$$v_0^2 = 4gR\cos\theta_* \Rightarrow \theta_* = \arccos\left(\frac{v_0^2}{4gR}\right)$$

For $v_0 = 0$, $\theta_* = \frac{\pi}{2}$

To have $\theta_* = 0$, the initial velocity must be $v_0 = 4gR$