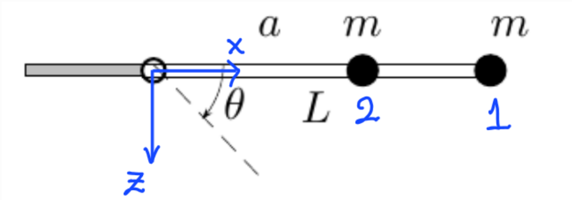
5.10. A stick of length L and negligible mass has a mass m fixed at one end and is attached to a support by a hinge at the other end, so that it may rotate about the hinge axis on a vertical plane. A second mass is fixed at a distance a from the hinge. Find a so that the the stick falls as fast as possible when dropped.



$$x_1 = L\cos\theta \qquad x_2 = a\cos\theta$$
$$z_1 = L\sin\theta \qquad z_2 = a\sin\theta$$

The system is conservative, so energy is conserved.

$$E = \frac{1}{2} m L^2 \dot\theta^2 + \frac{1}{2} m a^2 \dot\theta^2 - (mgL\sin\theta + mga\sin\theta)$$

Using that $\theta(t=0)=0$ and $\dot\theta(t=0)=0$ we have that $E(t)=0$. Hence

$$\frac{1}{2} m (L^2+a^2)\dot\theta^2 - mg(L+a)\sin\theta = 0$$

$$\frac{L^2+a^2}{2g(L+a)\sin\theta}\left(\frac{d\theta}{dt}\right)^2 = 1$$

Taking the square root and noting that t increases as θ increases, so that the plus sign must be taken in the square root, it follows that

$$dt = \frac{d\theta}{\sqrt{2g\sin\theta}}\sqrt{\frac{L^2+a^2}{L+a}}$$

Integrating from $\theta(t=0)=0$ to $\theta(t)=\theta$,

$$t = \left(\int_0^\theta \frac{d\theta'}{\sqrt{2g\sin\theta'}}\right)\sqrt{\frac{L^2+a^2}{L+a}} = t(a)$$

The parenthesis is a positive function of θ. Call it $F(\theta)$. We want to compute at what value of a t has its minimum. Let us do it:

$$\frac{dt}{da} = F(\theta)\,\frac{1}{2}\sqrt{\frac{L+a}{L^2+a^2}}\left[\frac{2a}{L+a} - \frac{L^2+a^2}{(L+a)^2}\right]$$

$$\left.\frac{dt}{da}\right|_{a_0} = 0 \iff 2a_0 - \frac{L^2 + a_0^2}{L + a_0} = 0 \tag{5.10.1}$$

$$a_0^2 + 2a_0 L - L^2 = 0 \implies a_0 = L(-1 \pm \sqrt{2})$$

We discard the negative sign since it is unphysical and keep the positive sign: $a_0 = L(\sqrt{2}-1)$

The second derivative of t at $a = a_0$ is

$$\frac{d^2t}{da^2} = \frac{1}{2}F(\theta)\left\{\frac{1}{2}\sqrt{\frac{L^2+a^2}{L+a}}\underbrace{\left[\frac{1}{L^2+a^2} - \frac{2a(L+a)}{(L^2+a^2)^2}\right]}_{(5.10.1) \implies [\dots]=0 \text{ at } a=a_0}\left[\frac{2a}{L+a} - \frac{L^2+a^2}{(L+a)^2}\right]\right.$$

$$\left. + \sqrt{\frac{L+a}{L^2+a^2}}\left[\frac{2}{L+a} - \frac{4a}{(L+a)^2} + \frac{2(L^2+a^2)}{(L+a)^3}\right]\right\}$$

$$[\dots]|_{a_0} = \frac{2}{L+a_0}\left[1 - \frac{2a}{L+a} + \frac{L^2+a^2}{(L+a)^2}\right] = \text{use } (5.10.1) = \frac{2}{L+a_0}\left[1 - \cancel{\frac{2a}{L+a}} + \cancel{\frac{2a}{L+a}}\right]$$

$$\left.\frac{d^2t}{da^2}\right|_{a_0} = \frac{1}{2}F(\theta)\frac{1}{\sqrt{2a_0}}\frac{2}{L+a_0} = F(\theta)\frac{1}{(L+a_0)\sqrt{2a_0}} > 0 \implies a_0 \text{ is a minimum}$$

We conclude that the shortest time for the stick to reach an angle θ is achieved if the second mass is placed at $a_0 = (\sqrt{2}-1)L$

Note that $t(a)$ can be written as

$$t(a) = L F(\theta)\sqrt{\frac{1+x^2}{1+x}}, \quad x = a/L$$

The plot of the function $f(x) = \sqrt{\frac{1+x^2}{1+x}}$ for $0 \le x \le 1$ ($\iff 0 \le a \le L$) is