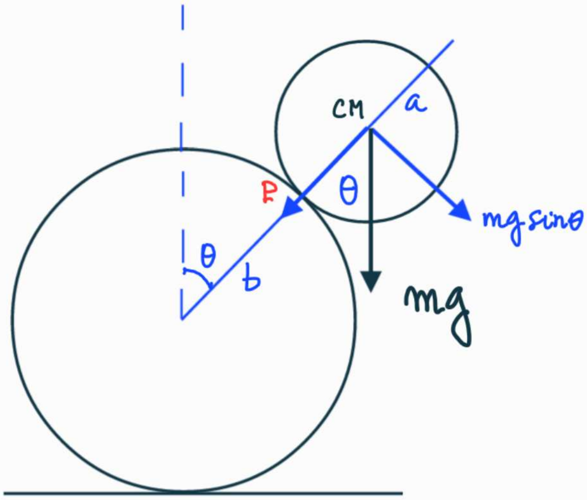
6.2. A ball of radius a at the top of a fixed sphere of radius b is given an infinitesimal kick so it begins to roll without slipping over the sphere. At what point will the ball abandon the sphere? Compare with problem 1.7.



I = moment of inertia of a sphere w.r.t a central axis $= \frac{2}{5} MR^2$

$x_{CM} = (b+a)\sin\theta$

$y_{CM} = b + (b+a)\cos\theta$

The CM of the small ball follows a circular trajectory of radius $b+a$. The gravitational force $m\vec{g}$ has a tangent to the large sphere component and a normal component. The ball leaves the sphere when the centrifugal force becomes equal to the normal component $mg\cos\theta$

$$mg\cos\theta = m(b+a)\dot\theta^2 \Rightarrow \dot\theta^2 = \frac{v_{CM}^2}{(b+a)^2} = \frac{g\cos\theta}{a+b} \qquad (6.2.1)$$

$$\ddot{\vec{x}} = (\ddot r - r\dot\theta^2)\hat e_r + (r\ddot\theta + 2\dot r\dot\theta)\hat e_\theta \Rightarrow (r = b+a) \Rightarrow \text{centrifugal force} = m(b+a)\dot\theta^2\,\hat e_r$$

$$\text{Also centrifugal force} = \frac{m v_{CM}^2}{b+a} = \frac{m(\dot x_{CM}^2 + \dot y_{CM}^2)}{b+a} = m(b+a)\dot\theta^2$$

Conservation of energy states that

$$E_{initial} = mg(2b+a) = \frac{1}{2} m \dot{\vec{x}}_{CM}^2 + \frac{1}{2} I\omega^2 + mg[b + (b+a)\cos\theta] = E_{on\text{-}sphere} \qquad (6.2.2)$$

$I = \frac{2}{5} ma^2 =: \alpha ma^2$; $\alpha = \frac{2}{5}$, ω = angular velocity of ball around its central axis

The linear velocity of the point P of contact of the ball with the sphere is on the one hand $v_p = b\dot\theta$, and on the other $v_p = a\omega$. Hence

$$a\omega = b\theta \Rightarrow \frac{1}{2} I\omega^2 = \frac{1}{2}\alpha m a^2\omega^2 = \frac{1}{2}\alpha m b^2\dot\theta^2.$$

Conservation of energy [eq. (6.2.2)] then reads

$$mg(\cancel{2}b+a) = \frac{1}{2}m(b+a)^2\dot{\theta}^2 + \frac{1}{2}\alpha m b^2\dot{\theta}^2 + mg[\cancel{b} + (b+a)\cos\theta]$$

$$g(b+a)(1-\cos\theta) = \frac{1}{2}[(b+a)^2 + \alpha b^2]\dot{\theta}^2$$

eq. (6.2.1): $\dot{\theta}^2 = \frac{g\cos\theta}{b+a}$

$$(b+a)^2(1-\cos\theta) = \frac{1}{2}[(b+a)^2 + \alpha b^2]\cos\theta$$

$$(b+a)^2 = \left[\frac{3}{2}(b+a)^2 + \alpha b^2\right]\cos\theta$$

$$2\left(1+\frac{a}{b}\right)^2 = \left[3\left(1+\frac{a}{b}\right)^2 + \alpha\right]\cos\theta$$

$$\cos\theta = \frac{2\left(1+\frac{a}{b}\right)^2}{3\left(1+\frac{a}{b}\right)^2+\alpha}$$

Note that for $a \ll b$, $\cos\theta = \frac{2}{3+\alpha}$.