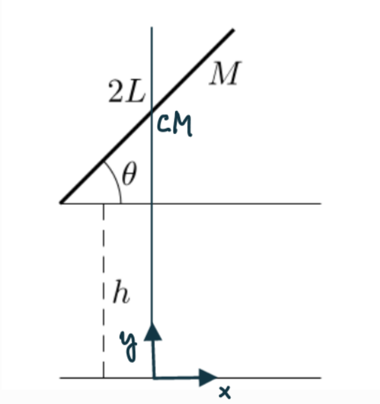
**6.7.** A thin rigid uniform rod of length $2L$ and mass $M$ is initally held at a height $h$ above the floor, forming an angle $\theta_0$ with it. The rod is released. Write the Lagrangian and the corresponding Euler-Lagrange equations. Calculate the time that the CM takes to reach the floor for $h=0$.




$x$

$y = h + L\sin\theta =: y$

$I_{CM} = \frac{1}{12} ML^2$

Take as translational degrees of freedom the coordinates $x$ and $y$ of the CM, and the angle $\theta$ as rotational degree of freedom. The Lagrangian is

$$L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_{CM}\,\omega^2 - Mgy$$

$$= \frac{1}{2}M(\dot{x}^2 + \dot{h}^2 + L^2\dot{\theta}^2\cos^2\theta + 2\dot{h}L\dot{\theta}\cos\theta) + \frac{1}{24}ML^2\dot{\theta}^2 - Mg(h + L\sin\theta)$$

$$= \frac{1}{2}M\left[\dot{x}^2 + \dot{h}^2 + L^2\dot{\theta}^2\left(\frac{1}{12} + \cos^2\theta\right) + 2L\dot{h}\dot{\theta}\cos\theta\right] - Mg(h + L\sin\theta)$$

The EL equations that follow from $L$ are

$$\left.\begin{aligned} &M\ddot{x} = 0 \Rightarrow x = v_0 t + x_0. \\ &ML\frac{d}{dt}\left[L\dot{\theta}\left(\frac{1}{12} + \cos^2\theta\right) + \dot{h}\cos\theta\right] + MgL\cos\theta = 0 \\ &M\frac{d}{dt}(\dot{h} + L\dot{\theta}\cos\theta) + Mg = 0 \end{aligned}\right\}$$

Since the rod is initially at rest, $v_0 = 0$. Taking the initial $x$-coordinate of the CM as the origin of the $x$-axis we set $x_0 = 0$. This gives

$$x(t) = 0$$

The other two equations read

$$\left.\begin{aligned} &L\left[\ddot{\theta}\left(\frac{1}{12}+\cos^2\theta\right)-2\cos\theta\sin\theta\,\dot{\theta}^2\right]+\ddot{h}\cos\theta-\sin\theta\,\dot{h}\,\dot{\theta}+g\sin\theta=0\\ &\ddot{h}+L\ddot{\theta}\cos\theta-L\sin\theta\,\dot{\theta}^2+g\cos\theta=0\end{aligned}\right\}$$

For $h=0$, the lower end of the rod is already on the floor, and $\dot{h}=\ddot{h}=0$. The total energy of the system is conserved. After setting $h=0$ and using $x(t)=0$, the energy is

$$E=\frac{1}{2}ML^2\dot{\theta}^2\left(\frac{1}{12}+\cos^2\theta\right)+MgL\sin\theta$$

Using that at $t=0$, $\theta(0)=\theta_0$ and $\dot{\theta}(0)=0$, we have

$$\frac{1}{2}\cancel{M}L^2\dot{\theta}^2\left(\frac{1}{12}+\cos^2\theta\right)+\cancel{M}gL\sin\theta=E(t=0)=\cancel{M}gL\sin\theta_0$$

$$\left(\frac{d\theta}{dt}\right)^2=\frac{2g}{L}\,\frac{\sin\theta_0-\sin\theta}{\frac{1}{12}+\cos^2\theta}$$

$$dt=-\sqrt{\frac{L}{24g}\,\frac{1+12\cos^2\theta}{\sin\theta_0-\sin\theta}}\,d\theta \leftarrow$$

Since $\theta$ decreases a $t$ increases the negative sign must be chosen when taking the square root.

Integrate over $t$ to obtain the time $T$ that it takes the rod to reach the floor

$$T=-\int_{\theta_0}^{0}d\theta\,\sqrt{\frac{L}{24g}\,\frac{1+12\cos^2\theta}{\sin\theta_0-\sin\theta}}$$