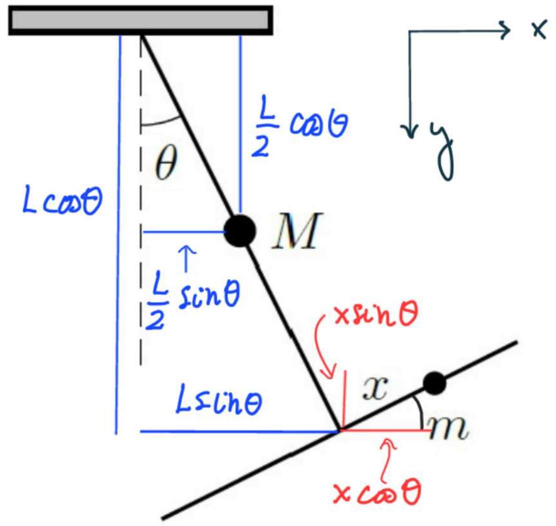
3.6. Consider a long thin rod of length L and neglegible mass with an end hinged at a support. A second rod, also of negligible mass, is welded at the other end. The first rod has a mass M fixed at its mid point. The second rod has a mass m that can freely slide along it. The system can rotate about the hinge in the vertical plane containing the two rods. See the figure. Write its Lagrangian and find the corresponding Euler-Lagrange equations. Discuss the oscillation modes for small oscillations of both masses.

Hint. Use as generalized coordinates the angle θ and the distance x of the second particle to the crossing of the rods.



$$x_M = \frac{L}{2}\sin\theta$$
$$y_M = \frac{L}{2}\cos\theta$$

$$x_m = L\sin\theta + x\cos\theta$$
$$y_m = L\cos\theta - x\sin\theta$$

$$V = -mgy$$

The Lagrangian is

$$L = T - V = \frac{1}{2}M(\dot{x}_M^2 + \dot{y}_M^2) + \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2) - (-Mgy_M - mgy_m)$$

$$\dot{x}_M^2 + \dot{y}_M^2 = \frac{L^2}{4}\dot{\theta}^2$$

$$\dot{x}_m^2 + \dot{y}_m^2 = (L\dot{\theta}\cos\theta + \dot{x}\cos\theta - x\dot{\theta}\sin\theta)^2 + (-L\dot{\theta}\sin\theta - \dot{x}\sin\theta - x\dot{\theta}\cos\theta)^2$$
$$= L^2\dot{\theta}^2 + \dot{x}^2 + x^2\dot{\theta}^2 + 2L\dot{\theta}\dot{x}.$$

$$L = \frac{1}{8}ML^2\dot{\theta}^2 + \frac{1}{2}m(L^2\dot{\theta}^2 + \dot{x}^2 + x^2\dot{\theta}^2 + 2L\dot{\theta}\dot{x})$$
$$+ \frac{1}{2}MgL\cos\theta + mg(L\cos\theta - x\sin\theta)$$

It follows that

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{4}ML^2\dot{\theta} + mL^2\dot{\theta} + mx^2\dot{\theta} + mL\dot{x}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + mL\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -\frac{1}{2} MgL\sin\theta - mgL\sin\theta - mgx\cos\theta$$

$$\frac{\partial L}{\partial x} = mx\dot{\theta}^2 - mg\sin\theta$$

The EL equations for θ and x read

$$\frac{1}{4}ML^2\ddot{\theta} + \overbrace{mL^2\ddot{\theta}}^{1} + 2mx\dot{x}\dot{\theta} + mx^2\ddot{\theta} + \overbrace{mL\ddot{x}}^{2} + \frac{1}{2}MgL\sin\theta + \underbrace{mgL\sin\theta}_{3} + mgx\cos\theta = 0 \qquad (3.6\text{-}1)$$

$$m\ddot{x} + mL\ddot{\theta} - mx\dot{\theta}^2 + mg\sin\theta = 0 \qquad (3.62)$$

Noting that

$1 + 2 + 3 = L(mL\ddot{\theta} + m\ddot{x} + mg\sin\theta)$ = use eq [x] = $L\,mx\dot{\theta}^2$

eqs. (3.6.1) and (3.6.2) can be written as

$$\frac{1}{2}ML^2\ddot{\theta} + mLx\dot{\theta}^2 + 2mx\dot{x}\dot{\theta} + mx^2\ddot{\theta} + \frac{1}{2}MgL\sin\theta + mgx\cos\theta = 0$$

$$\ddot{x} + L\ddot{\theta} - x\dot{\theta}^2 + g\sin\theta = 0$$

Write these equations as

$$\frac{1}{4}M\ddot{\theta} + m\frac{x}{L}\dot{\theta}^2 + 2m\frac{x}{L}\frac{\dot{x}}{L}\dot{\theta} + m\left(\frac{x}{L}\right)^2\ddot{\theta} + \frac{1}{2}M\frac{g}{L}\sin\theta + m\frac{g}{L}\frac{x}{L}\cos\theta = 0$$

$$\frac{\ddot{x}}{L} + \ddot{\theta} - \frac{x}{L}\dot{\theta}^2 + \frac{g}{L}\sin\theta$$

$\theta(t)=0$, $x(t)=0$ is an equilibrium point. For small oscillations, $\theta \ll 1$ and $x \ll L$, about the equilibrium point, using $\sin\theta = \theta + O(\theta^3)$ and $\cos\theta = 1 + O(\theta^2)$, we have

$$\frac{M}{4}\ddot{\theta} + \frac{Mg}{2L}\theta + \frac{mg}{L}\frac{x}{L} + O\left(\frac{x}{L}\theta^2, \frac{x^2}{L^2}\theta, \theta^3\right) = 0 \qquad (3\text{-}6\text{-}3)$$

$$\frac{\ddot{x}}{L} + \ddot{\theta} + \frac{g}{L}\theta + O\left(\theta^3, \frac{x}{L}\theta^2\right) = 0 \qquad (3\text{-}6\text{-}4)$$

To solve these equations, we make the ansatz

$$\frac{x}{L} = A e^{i\omega_x t} , \qquad \theta = B e^{i\omega_\theta t} .$$

Upon substitution in eqs. (3.6.3) and (3.6.4), we have

$$\left.\begin{aligned} &\left(-\frac{M\omega_\theta^2}{4} + \frac{Mg}{2L}\right)\theta + \frac{mg}{L}\frac{x}{L} = 0 \\ &\left(-\omega_\theta^2 + \frac{g}{L}\right)\theta - \omega_x^2 \frac{x}{L} = 0 \end{aligned}\right\}$$

For these equations to have non-trivial solutions for $\frac{x}{L}$ and θ, the determinant of the matrix of coefficients must be zero

$$\det \begin{pmatrix} -\frac{M\omega_\theta^2}{4} + \frac{Mg}{2L} & \frac{mg}{L} \\ -\omega_\theta^2 + \frac{g}{L} & -\omega_x^2 \end{pmatrix} = 0$$

$$\omega_x^2 \frac{M}{4}\left(\omega_\theta^2 - \frac{2g}{L}\right) = \frac{mg}{L}\left(\frac{g}{L} - \omega_\theta^2\right)$$

$$\omega_x^2 = \frac{4mg}{ML} \frac{\frac{g}{L} - \omega_\theta^2}{\omega_\theta^2 - \frac{2g}{L}} \qquad (3.6.5)$$

For ω_x^2 to be ≥ 0, so that ω_x is real and $\frac{x}{L}$ is indeed an oscillation, we need either

i) $\frac{g}{L} < \omega_\theta^2 < \frac{2g}{L} \iff 1 < \frac{L\omega_\theta^2}{g} < 2$, or

ii) $\frac{g}{L} > \omega_\theta^2$ and $\omega_\theta^2 > \frac{2g}{L}$, which is not possible

We conclude that the equilibrium point $x(t) = \theta(t) = 0$ is stable with oscillations about it of the form

$$\theta(t) = A\cos(\omega_\theta t + \phi) \qquad \frac{x}{L} = B\cos(\omega_x t + \beta)$$

with any ω_θ such that $1 < \frac{L\omega_\theta^2}{g} < 2$ and ω_x^2 given by eq. (3.6.5)