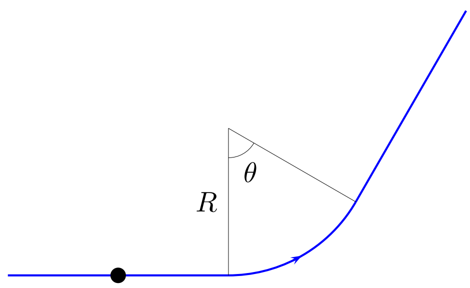
Classical Mechanics II
Problem sheets

Newtonian mechanics

1.1. A particle moves in two dimensions with trajectory given by $x(t) = a\cos(\omega t)$, $y(t) = b\sin(\omega t)$, with ω constant. Find the particle's angular momentum with respect to the origin, the torque of the force acting on the particle and the orthogonal-to-the-trajectory component of the acceleration.

1.2. The trajectory of a particle is formed by two straight half-lines whose endpoints are joined by a circle arc of angle θ and radius R, as shown in the figure. The particle's velocity is $v = kt$, with k constant. Calculate for $\theta = \pi/3$ the highest value of the modulus of the acceleration if the particle enters the curve at time $t = 2$.



1.3. A particle moves on a sphere of radius R subject to a force $\mathbf{F}$. Show that if the radial component of the force vanishes, $F_r = 0$, the particle does not move.

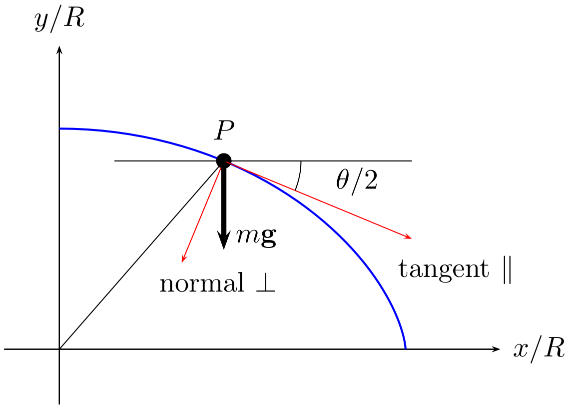
1.4. A particle of mass m is dropped (zero initial velocity) in a medium with friction force mk^2v^2 proportional to the square of the particle's velocity v, k being a constant. show that the particle's velocity reaches a maximum value an calculate it. Consider now the case in which the particle is projected upwards with initial velocity v_0 in the same medium Calculate the height h that it reaches.

1.5. A particle of mass m is projected with initial velocity $v_0\mathbf{e}_1$ from the top of the cycloid

$$x = R\,(\theta + \sin\theta)\,, \quad y = R\,(1 + \cos\theta)\,, \quad 0 \le \theta \le \pi$$

drawn in the figure. Before moving on to study the motion of the particle, it is convenient to become acquainted with some basic properties of cycloids. Prove that the slope of the tangent dy/dx to the curve at the point P specified by an angle θ is -tan$(\theta/2)$. The distance ds between two points (x, y) and $(x + dx, y + dy)$ is given by $ds^2 = dx^2 + dy^2$. Show that, if the points are on the cycloid, this gives $s = 4R\sin(\theta/2)$. Use the definition $\rho = ds/d\theta$ of curvature radius to prove that $\rho = 2R\cos(\theta/2)$. Note that the angle formed by the y-axis and the OP segment is not linearly related to θ.

Assume now that the only force acting on the particle is the Earth's gravitational force -$mg\mathbf{e}_2$. As long as the component of the gravitational force in the normal direction to the curve is larger than the centrifugal force the particle stays on the curve. At the point where the centrifugal force become larger the particle leaves the cycloid and jumps into space describing a parabolic shot. Write the equations of motion while the particle is on the cycloid. Find the angle θ at which the particle leaves it. Find the initial velocity for which there is no sliding over the cycloid and motion is parabolic from the beginning.



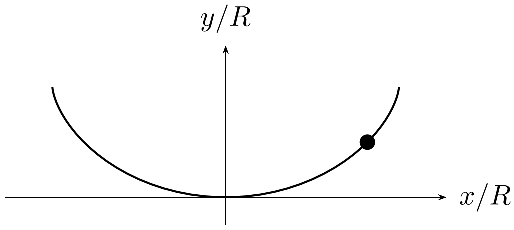
1.6. A particle of mass m freely slides on a wire with the shape of a cycloid

$$x = R\,(\theta + \sin\theta)\,, \quad y = R\,(1 - \cos\theta)\,, \quad -\pi \le \theta \le \pi\,.$$

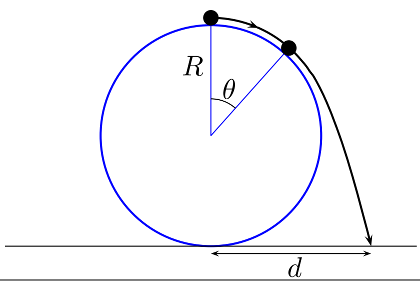
The wire is fixed with its y-axis pointing vertically upwards. Show that the the total energy is

$$E = mR^2\,(1 + \cos\theta)\,\dot{\theta}^2 + mgR\,(1 - \cos\theta)\,.$$

Write it in terms of the distance s over the cycloid, show that s oscillates harmonically and find its oscillation period.



1.7. A particle of mass m freely slides over a sphere of radius R that remains fixed to the floor, as depicted in the figure. Assuming that the only force that acts on the particle is the uniform gravitational force on the Earth's surface, find the angle θ at which the particle leaves the sphere and the distance d at which it hits the floor.



1.8. A particle of mass m travels with constant velocity $\mathbf{v}_-$ in the lower half-space $z < 0$, in which the potential energy takes the constant value V_-. It goes across the surface $z = 0$ and enters the upper half-space $z > 0$, where the potential energy takes the constant value V_+. Use conservation of energy and momentum to find the change in the direction of motion and velocity.

Central Forces

2.1. Write the perihelion and the aphelion distances to the Sun of a planet's orbit in terms of the orbit's eccentricity. What is the velocity at them?

2.2. The average distance of a planet to the Sun is defined as the mean value of the distance over a period. Compute it in terms of the eccentricity.

Hint. The result can be expressed in terms of the integral

$$I = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(1 + e\cos\theta)^3} = \frac{e^2+2}{2\,(1-e^2)^{5/2}}\ .$$

2.3. [Taylor, example 8.6]. Elliptic orbits around the Earth are specified by their energies E and their angular momenta L. To change from one orbit to another it is therefore enough to change E and L.

Assume a satellite in en elliptic orbit around the Earth. At the perigee and apogee (the equivalent to the perihelion and aphelion in a planetary orbit) the radial component of the velocity vanishes, $\dot{r} = 0$. At these points, a change in the velocity becomes a change in its angular component $r\dot{\theta}$, which in turns transforms into a change in the angular momentum $L = \mu r^2\dot{\theta}$, the new angular momentum being constant over a different orbit.

This suggests that to change the orbit, it is best to fire the satellite's rockets at the perigee or apogee, so that the angular momentum changes and the satellite places itself in a new orbit. Say that the firing changes the velocity from v_1 to $v_2 = \lambda v_1$. Assume that r does not change during the firing and the the firing does not change the satellite's mass. Under these assumptions, the angular momentum changes from L_1 to $L_2 = \lambda L_1$. The constant α in the orbit's equation

$$\frac{1}{r} = \frac{1}{\alpha}\,(1 + e\cos\theta)\,, \qquad \frac{1}{\alpha} = \frac{GMm}{L^2}$$

changes from α_1 to $\alpha_2 = \lambda^2\,\alpha_1$. Show that, if the firing is done at the perigee, the change in the eccentricity is

$$e_2 = \lambda^2 e_1 + \lambda^2 - 1\,.$$

To jump to an orbit of eccentricity ϵ_2, it is then enough to choose a convenient λ. The factor λ is called thrust factor. If $\lambda > 1$, the thrust is forward and the satellite speeds up. If $\lambda < 1$, the thrust is backwards and the satellite slows down.

Consider now a satellite in a circular orbit of radius R. Its crew wishes to transfer to a circular orbit of radius $2R$. To do this, it uses two consecutive boosts. The first one is to transfer to an elliptic orbit whose perigee is R and whose apogee is $2R$. When the satellite reaches the apogee of the elliptic orbit, the crew fires the rockets again so the satellite jumps to a circular orbit of radius $2R$. Find the thrust factors of both firings.

2.4. A particle moves in an attractive central field force with potential energy

$$V(r) = -\frac{k}{r} \qquad k = const > 0\,,$$

The particle is initially at a distance a and has zero velocity. What are the energy and the angular momentum of the particle? Solve the equation of motion and argue that the particle falls into the origin. Calculate the time it will take the particle to reach the origin.

2.5. [Goldstein, 3.13]. A particle describes a circular orbit under the influence of an attractive central force directed towards a point on the circle.

(a) Show that the force dependence on r is $1/r^5$.

(b) Show that in this orbit the particle's total energy is zero.

(c) Find the period of the motion.

(d) Find $\dot{x}$, $\dot{y}$ and v as functions of the angle around the circle and show that they become infinite when the particle goes through the center of force.

2.6. A particle of mass μ moves under the influence of a central force $\mathbf{F} = f(r)\hat{\mathbf{x}}$. Show that the condition for the existence of a circular orbit of radius r_0 is that the particle's angular momentum L satisfy the condition

$$f(r_0) = -\frac{L^2}{\mu r_0^3}\,.$$

Prove that the orbit is stable if

$$\left.\frac{df}{dr}\right|_{r=r_0} < -\frac{3}{r_0}\,f(r_0)\,.$$

Find the values of n for which the field force with $f(r) = -k/r^n$ has stable circular orbits.

2.7. Consider a particle of mass μ under the influence of a central field force $\mathbf{F} = -\mu k^2 \mathbf{x}/r^4$. Is this force attractive or repulsive? Find the potential energy of the particle. Write the equation of motion for $u = 1/r(\theta)$. Assume that the particle is initially at a distance r_0 with velocity $\mathbf{v}_0$ and that the velocity forms an angle α with the axis joining the origin to the initial position. Write the initial data for $r\,,\theta$ and their time derivatives. Solve the equation of motion to obtain the the particle's trajectory. Take $\alpha = \pi/2$ and with the help of a computer draw the orbits for

$$\frac{k^2}{r_0^2\,v_0^2} = \frac{1}{2}\,,\frac{144}{169}\,,\frac{35}{36}\,.$$

2.8. Consider the same central force as in the previous problem, but assume now that the particle approaches from far away ($|\mathbf{x}_0| \to \infty$) with impact parameter b and velocity $\mathbf{v}_0$. Write the initial data for $r\,,\theta$ and their time derivatives. The equation of motion is the same as in the previous problem. Show that it can now be written as

$$\frac{du^2}{d\theta^2} + \Big(1 - \frac{k^2}{b^2 v_0^2}\Big)u = 0\,,$$

Solve the equation and discuss the trajectories for

$$\frac{k^2}{b^2 v_0^2} = \frac{1}{2}, \frac{144}{169}, \frac{35}{36}.$$

Help yourself with a plotting program.

2.9. A comet of mass μ moves in the Sun's gravitational field (potential $V(r) = -k/r$ with $k = \mu M_\odot G$). The comet approaches the Sun with impact parameter b and velocity v_0. Write thee equation of motion for $u = 1/r(\theta)$ and solve it. Find the scattering angle with respect to the incoming direction. Find the distance of maximum approach.

2.10. Derive Rutherford's expression for the cross section of a repulsive Coulomb potential starting from the equations

$$\psi_\infty = \pi - 2\chi\,, \qquad \chi = \int_0^{s_0} \frac{ds}{\sqrt{1 - s^2 - \frac{V(b/s)}{E}}}\,,$$

where ψ_∞ is the scattering angle, b is the impact parameter and χ is the angle between the incoming direction and the line joining the origin with the periapsis.

2.11. Calculate the total cross section for a central potential $V(r) = k/r^2$, with $k > 0$.

2.12. Write the total energy of a particle of mass μ moving under a central force with potential energy $V(r) = -k/r^2$ where $k > 0$. Show that if

$$k > \frac{L^2}{2\mu}\,,$$

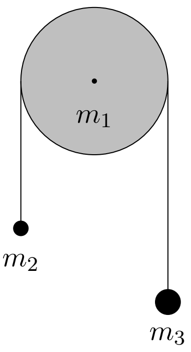
the particle falls down to $r \to 0$. Assume that the particle approaches the origin from infinity with impact parameter b and velocity v_0. Find the minimum value of b with which the particle must approach for it to escape the potential. Argue that one can assign an effective total cross section to the potential for the infall of particles given by $\sigma = \pi b_{\text{min}}^2$.

2.13. A particle of mass μ approaches a potential $V = -k/r^3$, with $k > 0$, from infinity with energy E. What is the total cross section for the particle to fall into the origin?

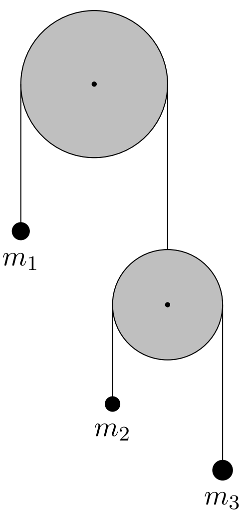
Hint. A plot of the effective potential will help you to understand what's going on.

Lagrangian formalism

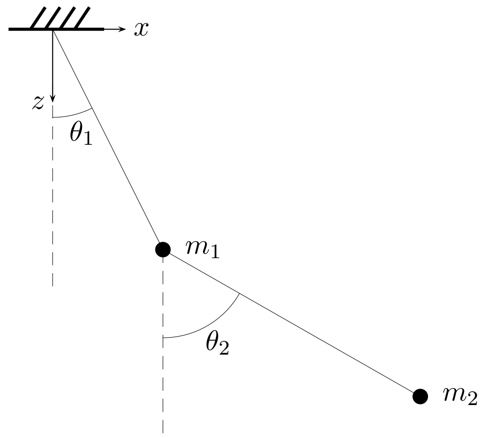
3.1. A pulley of mass m_1 can rotate freely about its axis which is fixed in a horizontal position. Two masses m_2 and m_3 hang at the ends of a light inextensible string going around the pulley. Assuming that the system moves in the vertical plane containing the pulley, find the Lagrangian of the system, write its equation of motion and determine the accelerations of the masses.



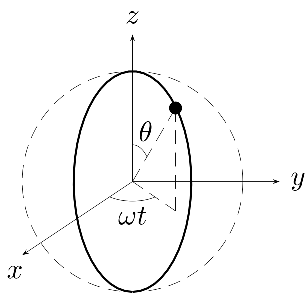
3.2. A light pulley (negligible mass) can rotate freely about its axis which is fixed in a horizontal position. A mass m_1 hangs at one end of a light inextensible string that goes over the pulley. A second light pulley hangs at the other end. And a second light inextensible string goes over the second pulley, with masses m_2 and m_3 hanging at its ends. Write the Euler-Lagrange equations and find the accelerations of the masses.



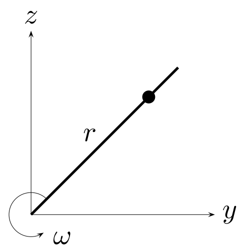
3.3. Consider the double pendulum with masses m_1 and m_2 in the figure. Find the constraints. Make a sensible choice of generalized coordinates. Write the system's Lagrangian and the equations of motion.



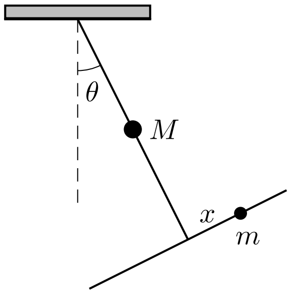
3.4. A hoop of radius a contained in a vertical plane rotates with constant angular velocity about its North-South diameter, see the attached figure. A bead of mass m is threaded to the hoop and moves without friction along it. Write the Lagrangian of the system. Find the equations of motion. Determine the equilibrium points and discuss their stability.



3.5. A particle of mass m slides over a rigid straight wire. The wire has one end fixed at the origin of the yz plane and rotates counterclockwise about the x axis. The particle is initially at a distance a from the origin and at rest with respect to the wire. Write the Lagrangian, find the equations of motion. Determine the equilibrium points and discuss their stability. Take as y-axis the axis initially containing the wire. Assume that the Earth's gravitational force is equal to -$m g \mathbf{e}_3$.

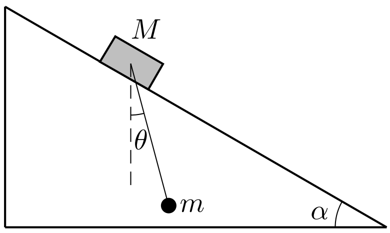


3.6. Consider a long thin rod of length L and negligible mass with an end hinged at a support. A second rod, also of negligible mass, is welded at the other end. The first rod has a mass M fixed at its mid point. The second rod has a mass m that can freely slide along it. The system can rotate about the hinge in the vertical plane containing the two rods. See the figure. Write its Lagrangian and find the corresponding Euler-Lagrange equations. Discuss the oscillation modes for small oscillations of both masses.



Hint. Use as generalized coordinates the angle θ and the distance x of the second particle to the crossing of the rods.

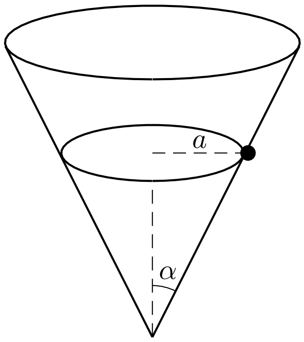
3.7. A body of mass M freely slides down a frictionless inclined plane forming an angle α with the floor. A simple pendulum of mass m hangs from the box. Write the Lagrangian and the equations of motion. Find the oscillating modes and their frequencies for small oscillations.



Hint. Use as generalized coordinates the distance along the plane and the angle θ.

3.8. A cone is fixed to the ground upside down, with its axis normal to the floor. A particle slides freely on its inner side under the action of gravity. Write the Lagrangian and the equations of motion. Prove that the particle may move in a circle of radius a around the cone's axis. Find the angular frequency of this motion. Assume now that the particle is slightly perturbed from this motion. What is the frequency of the oscillations about the constant radius orbit. Find the condition under which both frequencies agree.

Hint. Use as generalized coordinates the distance of the particle to the cone's axis and the angle around the axis.



3.9. Consider the previous problem. Show that the angle about the cone's axis is a cyclic coordinate and find its associated conserved quantity.

3.10. The Lagrangian of a particle with mass m and charge q that moves in an EM field $(\mathbf{E}, \mathbf{B})$ is

$$L = \frac{1}{2}\, m\, \dot{\mathbf{x}}^2 - q\phi + q\dot{\mathbf{x}}\cdot\mathbf{A},$$

where ϕ and $\mathbf{A}$ are the electric and magnetic potentials, from which $\mathbf{E}$ and $\mathbf{B}$ are obtained as

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi, \qquad \mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A}\,.$$

The Euler-Lagrange equations are

$$m\ddot{\mathbf{x}} = q\left(\mathbf{E} + \dot{\mathbf{x}} \wedge \mathbf{B}\right).$$

The potentials ϕ, $\mathbf{A}$ are not unique, since any other potentials ϕ', $\mathbf{E}'$ obtained as

$$\phi' = \phi - \frac{\partial f}{\partial t}, \qquad \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla} f\,, \qquad f = f(\mathbf{x}, t) = \text{arbitrary function,} \tag{3.1}$$

give the same $\mathbf{E}$ and $\mathbf{B}$. The equations of motion thus remain invariant under the transformations (3.1). This makes us suspect that such transformation are a symmetry of the system. Prove that, under transformations (3.1), the Lagrangian changes by a total time derivative. Transformations (3.1) are called gauge transformations and constitute a symmetry of the EM field called gauge invariance.

3.11. A particle of mass m and charge q moves in an uniform magnetic field oriented in the positive z axis, $\mathbf{B} = B\mathbf{e}_3$. Recalling that the magnetic vector potential $\mathbf{A}$ producing such a field can be taken in Cartesian coordinates as

$$\mathbf{A} = \frac{B}{2}\left(y\mathbf{e}_1 - x\mathbf{e}_2\right),$$

write the Lagrangian in these coordinates. Find the equations of motion and solve them. Assume that the particle is initially at the origin with velocity $\mathbf{v} = v_0\mathbf{e}_2$. Are there any cyclic coordinates? Consider the same scenario in cylindrical coordinates.

3.12. Consider a particle with mass m and Lagrangian

$$L = mc^2 \sqrt{1 - \left(\frac{1}{c}\frac{dx}{dt}\right)^2}\,,$$

where c is the velocity of light. This Lagrangian is not of the form $m\dot{x}^2/2$ that we have been using so far, but let's go with it nevertheless! Use the variational principle to obtain the system's equation of motion. How does L transform under the change of generalized coordinate and time $(x, t) \to (\tilde{x}, \tilde{t})$ given by

$$x = \tilde{x}\,\cosh\theta + c\tilde{t}\,\sinh\theta\,,$$
$$ct = \tilde{x}\,\sinh\theta + c\tilde{t}\,\cosh\theta\,,$$

where θ is the arbitrary parameter characterizing the transformation?

Hamiltonian formalism

4.1. A particle of mas m in a uniform gavitational field can slide over a helical wire with equations

$$x = a\cos\theta\,, \qquad y = a\sin\theta\,, \qquad z = b\theta\,,$$

a and b being positive constants and $0 \leq \theta \leq 2\pi$. Find the system's Hamiltonian and its Hamilton equations.

4.2. The spherical pendulum consists of a particle of mass m attached to a fixed point by a light inextensible string of length L moving under uniform gravity. As compared to the simple pendulum, the motion is not restricted to take place in a vertical plane. Find the Lagrangian, the Hamiltonian and the Hamilton equations.

4.3. A particle is suspended from a suport by an inextensible string of negligible mass. The string goes through a samll ring which moves in the vertical direction below the support with displacement $a(t)$. Assume that the particle's motion is contained in a vertical plane and that during motion the string remains tight. Find the Lagrangian, the Hamiltonian and Hamiltons equation. Is energy conserved?

Motion of rigid bodies

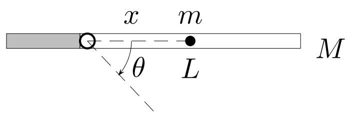
5.1. Consider an equilateral tirangle with side a and calculate its moments of inertia relative to (i) an orthogonal axis that goes through its center and (ii) along an orthogonal axis that goes through any of its vertices. Calculate the same moments of intertia for a a rectangle of sides a and b.

5.2. A ball of radius a at the top of a fixed sphere of radius b is given an infinitesimal kick so it begins to roll without slipping over the sphere. At what point will the ball abandon the sphere? Compare with problem 1.7.

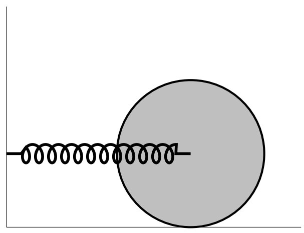
5.3. A ladder of mas m, length L and uniform mass density is initally standing on a frictionless floor, leaning against a frictionless wall. It is released and its upper end starts sliding down the wall while its lower one slides along the floor. What is the horizontal component of the velocity of its center of mass when the bar loses contact with the wall?

Hint. A convenient way to proceed is to consider the translational motion of the bar's center of mass and find the trajectory that it follows while the bar is contact with the wall, and then write the condition for the center of mass to abandon this trajectory.

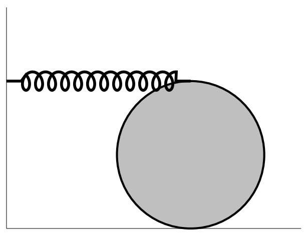
5.4. A hollow tube of mass M and length L has a ball of mass m inside it. The tube can swing around a pivot at one end. It is held initally in a horizontal position and then realesed. If θ denotes the angle formed by the tube axis and the horizontal, and x the distance that the mass travels inside the tube, find the Euler-Lagrange equations for both coordinates.



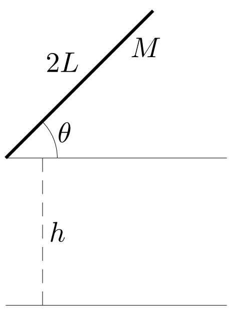
5.5. The axis of a disc with mass m, radius a and uniform mass density is connected to a spring with spring-constant k. If the disc rolls without slipping find the frequency of oscillations.



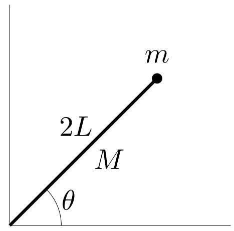
5.6. Consider the disc of the previous problem and assume now that it is the top of the disc that is connected to the spring at its equilibrium point. Find the frequency of the small oscillations.



5.7. A thin rigid uniform rod of length $2L$ and mass M is initally held at a height h above the floor, forming an angle θ_0 with it. The rod is released. Write the Lagrangian and the corresponding Euler-Lagrange equations. Calculate the time that the CM takes to reach the floor.



5.8. A thin rigid uniform rod of length $2L$ and mass M is initially placed with an end against the base of a wall forming an angle θ with the floor. A mass m is attached to its upper end and the rod is released. Write the Lagrangian and the Euler lagrange equations, and calculate the time that it takes the rod to reach the floor.



Special relativity

6.1. From an intertial frame S', the trajectory of a photon is specified in spherical coordinates by angles θ and ϕ. Frame S' moves along the x'-axis with velocity v with respect to frame S. Caculate the components of the photon's velocity in S and verify that its modulus is c.

6.2. In an inertial frame an event occurs at $t_1 = 0$, $\mathbf{x}_1 = 0$. A different event occurs at $t_2 = 2$, $\mathbf{x}_2 = (1, 0, 0)$. Can they be causally connected? If so, find a reference frame at which both events accur at the same space point. Find in this frame the time lapse between these two events. Solve the same problem for $t_2 = 1$, $\mathbf{x}_2 = (2, 0, 0)$.

6.3. The Lagrangian for a particle of mass m is given by

$$L = mc\sqrt{\eta_{\mu\nu}\,\frac{dx^\mu}{d\tau}\,\frac{dx^\nu}{d\tau}}\,,$$

where τ is the particle's proper time. Taking into account that the spacetime coordinates x^α are generalized coordinates, find the Euler Lagrange equations.

Assume now that the particle has charge q and is coupled to an EM field. The Lagrangian is then gven by

$$L = \sqrt{\eta_{\mu\nu}\,\frac{dx^\mu}{d\tau}\,\frac{dx^\nu}{d\tau}} + q\,\eta_{\mu\nu}\,\frac{dx^\mu}{d\tau}\,A^\nu(x),$$

where $A^\nu(x^\alpha)$ is the EM potential whose time component is $1/c$ times the electric potential, $A^0(x^\alpha) = \phi(x^\alpha)/c$, and whose space components form the magnetic potential $\mathbf{A}(x^\alpha)$. Racall that in terms of them the electric and magnetic field are given by

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi \quad\Leftrightarrow\quad \frac{E^i}{c} = \frac{\partial A^i}{\partial x^0} - \frac{\partial A^0}{\partial x^i}$$

$$\mathbf{B} = \boldsymbol{\nabla}\wedge\mathbf{A} \quad\Leftrightarrow\quad B^i = \epsilon^i{}_{jk}\,\frac{\partial A^j}{\partial x^k}\,.$$

Find the Euler-Lagrange equations.

6.4. The action of a relativistic particle is

$$S = mc \int_1^2 d\tau \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\ ,$$

Proper time can be viewd as a (convenient) parameter worldline curves $x^\mu = x^\mu(\tau)$ in configuration space. Show that under an arbitrary smooth change of parameter $\tau \to \lambda = \lambda(\tau)$ the action is invariant. Such a change is called reparametrization. You have just proved reparameterization is a symmetry of your model.

Motion relative to a non-inertial frame