UNIVERSIDAD COMPLUTENSE DE MADRID
DEPARTAMENTO DE FISICA TEORICA I

Física computacional

Example of computation of local error for multistep explicit numerical schemes.

Consider the initial value problem

$$\left.\begin{array}{l} y'(t) = f\big(t, y(t)\big) \\ y(0) = y_0 \end{array}\right\} .$$

Find the local error for the numerical scheme

$$y_{n+1} = -4y_n + 5y_{n-1} + \delta t\Big[4\,f\big(t_n, y_n\big) + 2\,f\big(t_{n-1}, y_{n-1}\big)\Big] .$$

This is a two-step scheme since it provides the numerical solution y_{n+1} at t_{n+1} in terms of the solutions y_n and y_{n-1} at the two previous steps t_n and t_{n-1}. It is explicit since no equation for y_{n+1} must be solved.

The local error e_{n+1} is defined as the difference between the numerical solution y_{n+1} and the true solution $y(t_{n+1})$, provided there is no error at previous steps. That is,

$$e_{n+1} := y_{n+1} - y(t_{n+1}) \quad \text{for} \quad e_k = y_k - y(t_k) = 0 \quad k \leq n\,. \tag{1}$$

For the scheme at hand, we have

$$e_{n+1} = -4\,y_n + 5y_{n-1} + \delta t\Big[\,4\,f(y_n, t_n) + 2\,f(y_{n-1}, t_{n-1})\,\Big] - y(t_{n+1})\,.$$

Since by assumption y_n and y_{n-1} are equal to $y(t_n)$ and $y(t_{n-1})$, this equation becomes

$$e_{n+1} = -4\,y(t_n) + 5\,y(t_n - \delta t) + \delta t\left[\,4\,f\big(y(t_n), t_n\big) + 2\,f\big(y(t_n - \delta t), t_n - \delta t\big)\,\right] - y(t_n + \delta t)\,. \tag{2}$$

Furthemore, since

$$f\big(t_k, y(t_k)\big) = y'(t_k)\,,$$

we may write

$$e_{n+1} = -y(t_n + \delta t) - 4\,y(t_n) + 5\,y(t_n - \delta t) + \delta t\left[\,4\,y'(t_n) + 2\,y'(t_n - \delta t)\,\right]. \tag{3}$$

We expand the right-hand side in powers of δt up to order 4. To do so, we need the expansions of $y(t_n \pm \delta t)$ to order 4, and that of $y'(t_n + \delta t)$ to order 3. Using that, by assumption, $y_k = y(t_k)$, $y'_k = y'(t_k)$, $y''_k = y''(t_k), \ldots$ for $k \leq n$, we have

$$y(t_n \pm \delta t) = y_n \pm \delta t\, y'_n + \frac{\delta t^2}{2}\, y''_n \pm \frac{\delta t^3}{6}\, y'''_n + \frac{\delta t^4}{24}\, y''''_n + O(\delta t^5) \tag{4}$$

and

$$y'(t_n \pm \delta t) = y'_n \pm \delta t\, y''_n + \frac{\delta t^2}{2}\, y'''_n \pm \frac{\delta t^3}{6}\, y'''_n + O(\delta t^4)\,. \tag{5}$$

Substituting now eqs. (4) and (5) in eq. (3), we finally obtain

$$|e_{n+1}| = \frac{h^4}{6}\, y''''_n\,. \tag{6}$$