$ds^2 = -dt^2 + h(r)\,dr^2 + r^2 d\Omega_2^2$. $T^t{}_t = \rho$ $T^r{}_r = T^\theta{}_\theta = T^\phi{}_\phi = P$

$ds^2 = -dt^2 + h(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

$$g_{\mu\nu} = \begin{array}{c} \\ t \\ r \\ \theta \\ \phi \end{array}\begin{array}{c} t \\ \begin{pmatrix} -1 & & & \\ & h(r) & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix} \end{array} \qquad g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1/h(r) & & \\ & & 1/r^2 & \\ & & & 1/r^2\sin^2\theta \end{pmatrix}$$

It's a particular case of static spherically symmetric metric with $g_{tt} = -1$. In the notation used in the lectures, $f = 1$.

(Hence, the only $\Gamma^\mu{}_{\nu\lambda}$ different from zero are

$$\Gamma^r{}_{rr} = \frac{h'}{2h} \qquad \Gamma^r{}_{\theta\theta} = -\frac{r}{h} \qquad \Gamma^r{}_{\phi\phi} = -\frac{r\sin^2\theta}{h}$$

$$\Gamma^\theta{}_{r\theta} = \frac{1}{r} \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi{}_{r\phi} = \frac{1}{r} \qquad \Gamma^\phi{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \quad)$$

The Ricci tensor has components

$$R_{tt} = 0$$

$$R_{rr} = \frac{h'}{rh}$$

$$R_{\theta\theta} = 1 - \frac{1}{h} + \frac{rh'}{2h^2}$$

$$R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}$$

$$R_{\mu\nu} = 0 \quad \text{for } \mu \neq \nu.$$

The Ricci scalar is

$$R = \frac{2h'}{rh^2} + \frac{2}{r^2}\left(1 - \frac{1}{h}\right)$$

The Einstein equations become

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu} \qquad (0)$$

$$\begin{cases} \frac{1}{2} R - \Lambda = -8\pi G \rho & (1) \\ \frac{h'}{rh} - \frac{1}{2} hR + \Lambda h = 8\pi G p h & (2) \\ 1 - \frac{1}{h} + \frac{rh'}{2h^2} - \frac{1}{2} r^2 R + \Lambda r^2 = 8\pi G p r^2 & (3) \end{cases}$$

It follows

$$(2) \Rightarrow \quad \frac{rh'}{h^2} = \left(8\pi G p - \Lambda + \frac{1}{2} R\right) r^2$$

$$(3): \quad 1 - \frac{1}{h} + \frac{1}{2}\left(8\pi G p - \Lambda + \frac{1}{2} R\right) r^2 - \frac{1}{2} R r^2 + \Lambda r^2 = 8\pi G p r^2$$

$$1 - \frac{1}{h} = \left(4\pi G p - \frac{1}{2}\Lambda + \frac{1}{4} R\right) r^2 = \text{use } (1)$$

$$= \left[4\pi G p + \frac{1}{2}\left(-8\pi G \rho\right)\right] r^2$$

$$= 4\pi G\,(p - \rho)\, r^2$$

$$\boxed{h = \frac{1}{1 - 4\pi G (p - \rho) r^2}}$$

$$\left.\begin{array}{l} (0) \Rightarrow \quad -R + 4\Lambda = 8\pi G (\rho + 3p) \\ (1) \Rightarrow \quad -\frac{R}{2} + \Lambda = +8\pi G \rho \end{array}\right\} \Rightarrow \quad \begin{array}{l} \Lambda = 4\pi G (3p - \rho) \\ R = 24\pi G (p - \rho) \end{array}$$

$$ds^2 = -dt^2 + a^2(t)\left[h(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta)\,d\phi^2\right]$$

$$T_{tt} = \rho \qquad T_{rr} = p\,h(r) \qquad T_{\theta\theta} = p r^2 \qquad T_{\phi\phi} = p r^2 \sin^2\theta$$

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu{}_{\mu\alpha} T^\alpha{}_\nu - \Gamma^\alpha{}_{\mu\nu} T^\mu{}_\alpha$$

$$T^\mu{}_\nu: \quad T^t{}_t = -\rho \qquad T^r{}_r = T^\theta{}_\theta = T^\phi{}_\phi = p$$

$$\nu = t \quad \nabla_\mu T^\mu{}_t = \partial_t T^t{}_t + \Gamma^\mu{}_{\mu t} T^t{}_t - \Gamma^\alpha{}_{\mu t} T^\mu{}_\alpha$$

$$= \partial_t \rho - \frac{3\dot a}{a}\rho + \Gamma^t{}_{tt}\rho - \left(\Gamma^r{}_{rt} + \Gamma^\theta{}_{\theta t} + \Gamma^\phi{}_{\phi t}\right)p$$

$$\Gamma^\mu{}_{\mu\alpha} = \frac{1}{\sqrt{-g}}\partial_\alpha \sqrt{-g}$$

$$\Gamma^\mu{}_{\mu t} = \frac{3\dot a}{a} \qquad \Gamma^\mu{}_{\mu r} = \frac{1}{r^2\sqrt{h}}\partial_r\left(r^2\sqrt{h}\right) = \frac{h'}{2h} + \frac{2}{r}$$

$$\Gamma^\mu{}_{\mu\theta} = \frac{\cos\theta}{\sin\theta} \qquad \Gamma^\mu{}_{\mu\phi} = 0$$

$$= -\partial_t \rho - \frac{3\dot a}{a}\rho + \Gamma^t{}_{tt}\rho + \Gamma^t{}_{tt} p - \Gamma^\mu{}_{\mu t} p$$

$$= -\partial_t \rho - \frac{3\dot a}{a}(\rho + p) + \Gamma^t{}_{tt}(\rho + p)$$

$$= -\partial_t \rho - \frac{3\dot a}{a}(\rho + p)$$

$$\Gamma^t{}_{tt} = \frac{1}{2} g^{t\rho}\left(\partial_t g_{\rho t} + \partial_t g_{t\rho} - \partial_\rho g_{tt}\right) = 0$$

$$\nu = r \quad \nabla_\mu T^\mu{}_r = \partial_r T^r{}_r + \Gamma^\mu{}_{\mu r} T^r{}_r - \Gamma^\alpha{}_{\mu r} T^\mu{}_\alpha$$

$$= \partial_r p + \Gamma^\mu{}_{\mu r}\, p + \Gamma^t{}_{tr}\, \rho - \left(\Gamma^r{}_{rr} + \Gamma^\theta{}_{\theta r} + \Gamma^\phi{}_{\phi r}\right) p$$

$$= \partial_r p + \cancel{\Gamma^\mu{}_{\mu r}\, p} + \Gamma^t{}_{tr} (\rho + p) - \cancel{\Gamma^\mu{}_{\mu r}\, p}$$

$$= \partial_r p$$

$$\Gamma^t{}_{tr} = \frac{1}{2} g^{tt} \left(\partial_t g_{tr} + \partial_r g_{tt} - \partial_t g_{tr}\right) = 0$$

$$\nu = \theta \quad \nabla_\mu T^\mu{}_\theta = \partial_\theta T^\theta{}_\theta + \Gamma^\mu{}_{\mu\theta} T^\theta{}_\theta - \Gamma^\alpha{}_{\mu\theta} T^\mu{}_\alpha$$

$$= \partial_\theta p + \Gamma^\mu{}_{\mu\theta}\, p + \Gamma^t{}_{t\theta}\, \rho - \left(\Gamma^r{}_{r\theta} + \Gamma^\theta{}_{\theta\theta} + \Gamma^\phi{}_{\phi\theta}\right) p$$

$$= \partial_\theta p + \cancel{\Gamma^\mu{}_{\mu\theta}\, p} + \Gamma^t{}_{t\theta} (\rho + p) - \cancel{\Gamma^\mu{}_{\mu\theta}\, p}$$

$$= \partial_\theta p$$

$$\Gamma^t{}_{t\theta} = \frac{1}{2} g^{tt} \left(\partial_t g_{t\theta} + \partial_\theta g_{tt} - \partial_t g_{t\theta}\right) = 0$$

$$\nu = \phi \quad \nabla_\mu T^\mu{}_\phi = \partial_\phi T^\phi{}_\phi + \Gamma^\mu{}_{\mu\phi} T^\phi{}_\phi - \Gamma^\alpha{}_{\mu\phi} T^\mu{}_\alpha$$

$$= \partial_\phi p + \cancelto{0}{\Gamma^\mu{}_{\mu\phi}\, p} + \Gamma^t{}_{t\phi}\, \rho - \left(\Gamma^r{}_{r\phi} + \Gamma^\theta{}_{\theta\phi} + \Gamma^\phi{}_{\phi\phi}\right) p$$

$$= \partial_\phi p + \Gamma^t{}_{t\phi} (\rho + p) - \cancelto{0}{\Gamma^\mu{}_{\mu\phi}\, p}$$

$$= \partial_\phi p$$

$$\Gamma^t{}_{t\phi} = 0$$

$\nabla_\mu T^\mu{}_\nu = 0$ requires i) $\partial_r p = \partial_\theta p = \partial_\phi p = 0 \Rightarrow p = p(t)$

ii) $-\partial_t \rho - \frac{3\dot{a}}{a}(\rho + p) = 0$

For $\rho = \text{const}$, $\frac{3\dot{a}}{a}(\rho + p) = 0 \Rightarrow$ a) $\dot{a} = 0 \Rightarrow a = \text{const}.$

b) $\rho + p = 0.$