Radial free fall in Reissner-Nordstrom

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r} + \frac{q}{r^2}} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

[[Solution to $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$

$$T_{\mu\nu} = 2\left(F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right)$$

$$\nabla_\mu F^{\mu\nu} = 0$$

with $F = dA$, $A = \frac{q}{r} dt$ $\left(A_t = \frac{q}{r}, \; F_{rt} = -\frac{q}{r^2} \right)$]]

ds^2 is of the form $-f(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$,

for which

$$\Gamma^t{}_{tr} = \frac{f'}{2f} \qquad \text{''} \qquad f = 1 - \frac{2m}{r} + \frac{q^2}{r^2}$$

$$\Gamma^r{}_{tt} = \frac{1}{2} f f' \qquad \Gamma^r{}_{rr} = -\frac{f'}{2f} \qquad \Gamma^r{}_{\theta\theta} = -rf \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta \, f$$

$$\Gamma^\theta{}_{r\theta} = \frac{1}{r} \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi{}_{r\phi} = \frac{1}{r} \qquad \Gamma^\phi{}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$$

Geodesic equations $\ddot{x}^\mu + \Gamma^\mu{}_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda$ are (recall $\theta = \frac{\pi}{2}$)

$$\ddot{t} + \frac{f'}{f} \dot{t} \dot{r} = 0 \quad \Rightarrow \quad f \frac{dt}{d\tau} = E \text{ const.}$$

$$\ddot{r} + \frac{1}{2} f f' \dot{t}^2 - \frac{f'}{2f} \dot{r}^2 - r f \dot{\phi}^2 = 0$$

equation for θ is trivially satisfied

$r^2 \dot{\phi} = L = \text{const}.$

a) Mass shell condition for a massive particle $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -1$ becomes

$$-f\dot{t}^2 + \frac{1}{f}\dot{r}^2 + 0 + r^2\dot{\phi}^2 = -1 \qquad (\text{recall } \theta = \frac{\pi}{2})$$

$$-\frac{E^2}{f} + \frac{\dot{r}^2}{f} + \frac{L^2}{r^2} = -1$$

$$E^2 = \dot{r}^2 + f\left(1 + \frac{L^2}{r^2}\right)$$

$$\hookrightarrow f = 1 - \frac{2m}{r} + \frac{q^2}{r^2}$$

$$\underbrace{\frac{E^2 - 1}{2}}_{E_{\text{eff}}} = \frac{1}{2}\dot{r}^2 \underbrace{- \frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL^2}{r^3} + \frac{q^2}{2r^2}\left(1 + \frac{L^2}{r^2}\right)}_{V_{\text{eff}}}$$

effective total energy

effective potential energy

effective angular momentum $r^2\dot{\phi} = \frac{L^2}{r^2}$

[recall that in Kepler's problem: angular momentum $l = r^2\dot{\phi}$]

Radially infalling particle: $\dot{\phi} = 0 \Rightarrow L = 0$

$$V_{\text{eff}} = -\frac{m}{r} + \frac{q^2}{2r^2}$$

Newtonian attractive potential

Repulsive potential

As $r \to 0$ $\frac{q^2}{2r^2}$ (repulsive) dominates over $-\frac{m}{r}$ (attractive)

so the particle never reaches $r=0$.

b) At the turning point $\dot{r}=0$ so

$$\frac{E^2-1}{2} = -\frac{m}{r} + \frac{q^2}{2r^2}$$

If the particle is at rest ($\dot{r}=0$) at infinity ($r \to \infty$), $E^2=1$ and r_{min} is the solution to

$$-\frac{m}{r_{min}} + \frac{q^2}{2r_{min}^2} = 0 \quad \Rightarrow \quad r_{min} = \frac{q^2}{2m}$$

Now

$$1 - \frac{2m}{r} + \frac{q^2}{r^2} = 0 \quad \Rightarrow \quad r = r_\pm = m \pm \sqrt{m^2-q^2}$$

If $m^2 > q^2$ $r_- := m - \sqrt{m^2-q^2} < m + \sqrt{m^2-q^2} =: r_+$. It is clear that

$$r_- > r_{min} \iff m - \sqrt{m^2-q^2} > \frac{q^2}{2m}$$

$$\iff m - \frac{q^2}{2m} > \sqrt{m^2-q^2} > 0$$

$$\iff \left(m - \frac{q^2}{2m}\right)^2 > m^2-q^2$$

$$\iff m^2 - q^2 + \frac{q^4}{4m^2} > m^2 - q^2 \quad \text{which holds true}$$