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$$ds^2 = 2\,du\,dv + f(v)\,dw^2 + 2\,dw\,dz$$

$$\Gamma^{\mu}{}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}\left(\partial_\nu g_{\sigma\lambda} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}\right)$$

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}{}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}{}_{\beta\mu} + \Gamma^{\alpha}{}_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\mu\beta}$$

$$g_{\mu\nu} = \begin{array}{c} \\ u \\ v \\ w \\ z \end{array} \begin{array}{c} \begin{array}{cccc} u & v & w & z \end{array} \\ \left(\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & & \\ \hline & & f(v) & 1 \\ & & 1 & 0 \end{array}\right) \end{array} \qquad g^{\mu\nu} = \begin{array}{c} \begin{array}{cccc} u & v & w & z \end{array} \\ \left(\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & & \\ \hline & & 0 & 1 \\ & & 1 & -f(v) \end{array}\right) \end{array} \begin{array}{c} \\ u \\ v \\ w \\ z \end{array}$$

$$\Gamma^{\mu}{}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}\big( \delta_{\nu v}\, \partial_v g_{ww}\, \delta_{\sigma w}\delta_{\lambda w}$$
$$+ \delta_{\lambda v}\, \partial_v g_{ww}\, \delta_{\nu w}\delta_{\sigma w}$$
$$- \delta_{\sigma v}\, \partial_v g_{ww}\, \delta_{\nu w}\delta_{\lambda w}\big)$$
$$= \frac{1}{2} f'(v)\left( g^{\mu w}\delta_{\nu v}\delta_{\lambda w} + g^{\mu w}\delta_{\lambda v}\delta_{\nu w} - g^{\mu v}\delta_{\nu w}\delta_{\lambda w}\right)$$

$$\Gamma^{u}{}_{\nu\lambda} = -\frac{1}{2} f' g^{uv}\delta_{\nu w}\delta_{\lambda w} \quad \Rightarrow \quad \Gamma^{u}{}_{ww} = -\frac{1}{2} f'$$

$$\Gamma^{v}{}_{\nu\lambda} = 0$$

$$\Gamma^{w}{}_{\nu\lambda} = 0$$

$$\Gamma^{z}{}_{\nu\lambda} = \frac{1}{2} f'\left(\delta_{\nu v}\delta_{\lambda w} + \delta_{\lambda v}\delta_{\nu w}\right) \Rightarrow \Gamma^{z}{}_{vw} = \frac{1}{2} f'$$

$$R_{\beta\nu} = \partial_\alpha \Gamma^{\alpha}{}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}{}_{\beta\alpha} + \Gamma^{\alpha}{}_{\alpha\sigma}\Gamma^{\alpha}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\alpha\beta}$$

$$\Gamma^\alpha{}_{\beta\alpha} = 0 \quad \Rightarrow \quad R_{\beta\nu} = \partial_\alpha \Gamma^\alpha{}_{\beta\nu} - \Gamma^\alpha{}_{\nu\sigma}\Gamma^\sigma{}_{\alpha\beta}$$

$$= \partial_v \underbrace{\Gamma^v{}_{\beta\nu}}_{\substack{\parallel \\ 0}} - \Gamma^u{}_{\nu\sigma}\Gamma^\sigma{}_{u\beta} - \Gamma^z{}_{\nu\sigma}\Gamma^\sigma{}_{z\beta}$$

$$R_{\beta\nu} = -\Gamma^u{}_{ww}\underbrace{\Gamma^w{}_{u\beta}}_{\substack{\parallel \\ 0}}\delta_{\nu w} - \Gamma^z{}_{\nu w}\underbrace{\Gamma^w{}_{z\beta}}_{\substack{\parallel \\ 0}}\delta_{\nu v} - \Gamma^z{}_{wv}\underbrace{\Gamma^v{}_{z\beta}}_{\substack{\parallel \\ 0}}\delta_{\nu w}$$

$R_{\mu\nu} = 0$ for all $f(v)$ $\Rightarrow$ $R = 0$ for all $f(v)$

Einstein's equations hold for all $f(v)$.

For the metric to be flat, the Riemann tensor must vanish. That is, $R_{\alpha\beta\mu\nu} = 0$. Now

$R^v{}_{\dots} = 0$ trivial

$R^w{}_{\dots} = 0$ trivial

$$R^u{}_{\beta\mu\nu} = \partial_\mu \Gamma^u{}_{\beta\nu} - \underbrace{\partial_\nu \Gamma^u{}_{\beta\mu}}_{\delta_{\nu v}\delta_{\beta w}\delta_{\mu w}\partial_v\Gamma^u{}_{ww}} + \underbrace{\Gamma^u{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\beta} - \Gamma^u{}_{\nu\sigma}\Gamma^\sigma{}_{\mu\beta}}_{\text{require } \sigma = w,\ \text{but } \Gamma^w{}_{\dots} = 0}$$

$\partial_\mu \Gamma^u{}_{\beta\nu}$: $\delta_{\mu v}\delta_{\beta w}\delta_{\nu w}\partial_v \Gamma^u{}_{ww}$

$$R^u{}_{\beta\mu\nu} = -\tfrac{1}{2}f''\left(\delta_{\mu v}\delta_{\beta w}\delta_{\nu w} - \delta_{\nu v}\delta_{\beta w}\delta_{\mu w}\right)$$

$$R^z{}_{\beta\mu\nu} = \partial_\mu \Gamma^z{}_{\beta\nu} - \partial_\nu \Gamma^z{}_{\beta\mu} + \underbrace{\Gamma^z{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\beta} - \Gamma^z{}_{\nu\sigma}\Gamma^\sigma{}_{\mu\beta}}_{\substack{\text{require } \sigma = v, w, \\ \text{but } \Gamma^v{}_{\dots} = \Gamma^w{}_{\dots} = 0}}$$

$$R^{z}{}_{\beta\mu\nu} = \delta_{\mu v}(\delta_{\beta v}\delta_{\nu w} + \delta_{\beta w}\delta_{\nu v})\,\tfrac{1}{2}f''$$
$$\quad - \delta_{\nu v}(\delta_{\beta v}\delta_{\mu w} + \delta_{\beta w}\delta_{\mu v})\,\tfrac{1}{2}f''$$

Hence

$$R^{\alpha}{}_{\beta\mu\nu} = 0 \iff f'' = 0 \implies f(v) = c_1 v + c_0$$

The metric is flat for $f(v) = c_1 v + c_0$

non flat for any other $f(v)$