Particle in the magnetic field of a magnetic dipole $\vec{\mu}$ – Orbits in $z=0$ plane

$$\vec{A} = \frac{\vec{\mu}\wedge\vec{x}}{|\vec{x}|^3} \qquad A_1 = \frac{\mu_2 x_3 - \mu_3 x_2}{|\vec{x}|^3} \qquad A_2 = \frac{\mu_3 x_1 - \mu_1 x_3}{|\vec{x}|^3} \qquad A_3 = \frac{\mu_1 x_2 - \mu_2 x_1}{|\vec{x}|^3}$$

Take $\vec{\mu} = \mu\hat{e}_3$ and use cylindrical coordinates

$$\left.\begin{aligned} x_1 &= r\cos\phi \\ x_2 &= r\sin\phi \\ x_3 &= z \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{x}_1 &= \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\ \dot{x}_2 &= \dot{r}\sin\phi + r\dot{\phi}\cos\phi \\ \dot{x}_3 &= 0 \end{aligned}$$

$$A_1 = -\frac{\mu r\sin\phi}{(r^2+z^2)^{3/2}} \qquad A_2 = \frac{\mu r\cos\phi}{(r^2+z^2)^{3/2}} \qquad A_3 = 0$$

The Lagrangian is

$$L = \frac{1}{2}m\dot{\vec{x}}^2 + q\dot{\vec{x}}\vec{A} = \frac{1}{2}m(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2) + q(\dot{x}_1A_1+\dot{x}_2A_2+\dot{x}_3A_3)$$

$$= \frac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2+\dot{z}^2) + \frac{\mu q r^2\dot{\phi}}{(r^2+z^2)^{3/2}}$$

The Euler-Lagrange equations are

$$m\ddot{r} - mr\dot{\phi}^2 - \frac{2\mu q r\dot{\phi}}{(r^2+z^2)^{3/2}} + 3\frac{\mu q r^3\dot{\phi}}{(r^2+z^2)^{5/2}} = 0$$

$$\frac{d}{dt}\left[mr^2\dot{\phi} + \frac{\mu q r^2}{(r^2+z^2)^{3/2}}\right] = 0 \Rightarrow mr^2\dot{\phi} + \frac{\mu q r^2}{(r^2+z^2)^{3/2}} = \text{const.} = \ell$$

$$m\ddot{z} - 3\frac{\mu q r^2 z\dot{\phi}}{(r^2+z^2)^{5/2}} = 0$$

Solutions with $z=0$ have

$$mr^2\dot{\phi} + \frac{\mu q}{r} = \ell \implies \dot{\phi} = \frac{1}{mr^2}\left(\ell - \frac{\mu q}{r}\right)$$

$$m\ddot{r} - mr\dot{\phi}^2 + \frac{\mu q\dot{\phi}}{r^2} = 0 \iff m\ddot{r} - mr\frac{1}{m^2r^4}\left(\ell - \frac{\mu q}{r}\right)^2 + \frac{\mu q}{r^2}\frac{1}{mr^2}\left(\ell - \frac{\mu q}{r}\right) = 0$$

$$m\ddot{r} - \frac{1}{mr^4}\left(r\ell^2 - 3\mu q\ell + \frac{2\mu q^2}{r}\right) = 0$$

Recall from lesson 1 that for a particle in an EM field, the energy was conserved and given by

$$E = \frac{1}{2}m\dot{\vec{x}}^2 + V(\vec{x})$$

Alternatively

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 = \frac{1}{2} m \dot{\vec{x}}^2 \quad \text{is conserved}$$

In our case $V(\vec{x}) = 0$, so that for $z = 0$

$$E = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) = \text{conserved}$$

$$E = \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{1}{2} m r^2 \left[\frac{1}{mr^2} \left(\ell - \frac{\mu q}{r} \right) \right]^2}_{U(r)}$$

Case $\ell > 0$: $\quad \frac{\mu q}{\ell r} = \frac{1}{s} \quad \Rightarrow \quad U(r) = \frac{\ell^2}{2mr^2} \left(1 - \frac{\mu q}{\ell r}\right)^2 = \overbrace{\frac{\ell^4}{2m^2 \mu^2 q^2}}^{k>0} \frac{1}{s^2} \left(1 - \frac{1}{s}\right)^2$

$$U'(s) = -\frac{2k}{s^3}\left(1 - \frac{1}{s}\right)^2 + \frac{2k}{s^2}\left(1 - \frac{1}{s}\right)\frac{1}{s^2} = \frac{2k}{s^5}(-s^2 + 3s - 2)$$

$$U''(s) = -\frac{10k}{s^6}(-s^2 + 3s - 2) + \frac{2k}{s^5}(-2s + 3) = \frac{2k}{s^6}(3s^2 - 12s + 10)$$

$$U'(s_0) = 0 \quad \Rightarrow \quad -s_0^2 + 3s_0 - 2 = 0 \quad \Rightarrow \quad s_0 = 1, 2$$

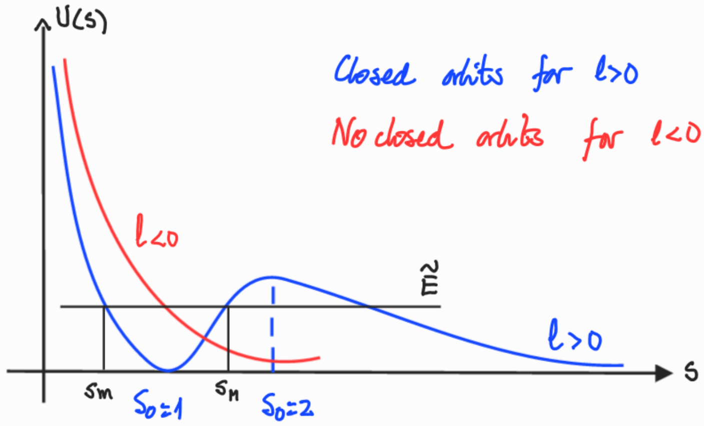
$$U''(s_0 = 1) = 2k > 0 \quad \Rightarrow \quad s_0 = 1 \text{ minimum}$$

$$U''(s_0 = 2) = -\frac{k}{16} < 0 \quad \Rightarrow \quad s_0 = 2 \text{ maximum}$$

Case $\ell < 0$: $\quad \frac{\mu q}{|\ell| r} = \frac{1}{s} \quad \Rightarrow \quad U(r) = \frac{\ell^2}{2mr^2}\left(1 + \frac{\mu q}{|\ell| r}\right) = k \frac{1}{s^2}\left(1 + \frac{1}{s}\right)^2$

$$U'(s) = -\frac{2k}{s^5}(s^2 + 3s + 2), \qquad U''(s) = \frac{2k}{s^6}(3s^2 + 12s + 10)$$

No equilibrium points for $s > 0$



For an energy $0 < \tilde{E} < U(s_0 = 2) = \dfrac{3\ell^4}{32\, m^2 \mu^2 q^2}$ there is a closed orbit with periapsis r_m and apoapsis r_M.

$$s_0 = 2 \Rightarrow r_0 = \frac{\mu q}{\ell} s_0 = \frac{2\mu q}{\ell} \quad , \quad U(s_0=2) = \frac{3\ell^4}{32 m^2 \mu^2 q^2}$$