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Física cuántica II - Examen final - 18 de enero de 2024

(Tiempo: 3 horas)

1. [1.5 puntos] Un sistema tiene espacio de Hilbert \mathbb{C}^2 y base ortonormal $\{|1\rangle, |2\rangle\}$. Discutir si la matriz

$$\rho = \begin{pmatrix} \frac{2}{3} & \frac{1-i}{3} \\ \frac{1+i}{3} & \frac{1}{3} \end{pmatrix}$$

describir un estado físico y, en caso de hacerlo, de qué tipo.

The matrix ρ trivially satisfies $\text{tr}(\rho) = 1$ (0.25) and $\rho^\dagger = \rho$ (0.25). Its eigenvalues are the solutions λ to the equation

$$\det \begin{pmatrix} \frac{2}{3} - \lambda & \frac{1-i}{3} \\ \frac{1+i}{3} & \frac{1}{3} - \lambda \end{pmatrix} = 0 \Leftrightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda = 0, 1.$$

Since they are both larger or equal than zero, ρ is positive semi-definite and ρ satisfies all the conditions for it to be a matrix density (0.50). It is also trivial to check that $\rho^2 = \rho$ (0.5), so that ρ describes a pure state. In the given basis, ρ can be written as

$$\rho = |\psi\rangle\langle\psi| \quad \psi = \frac{1-i}{\sqrt{3}} |1\rangle + \frac{1}{\sqrt{3}} |2\rangle.$$

2. [3 points]. El hamiltoniano de una partícula de spin 1/2 es

$$H = \frac{\omega}{\hbar} \mathbf{L} \cdot \mathbf{S},$$

donde ω es una frecuencia angular, \mathbf{L} es el operador momento angular orbital y \mathbf{S} es el operador de spin.

(a) Encontrar las autoenergías y los valores que pueden tomar los números cuánticos de los que éstas dependen.

(b) Determinar el estado $|\psi(t)\rangle$ del sistema en un tiempo t si éste se encuentra inicialmente en

$$|\psi(0)\rangle = |\ell = 2, s = \frac{1}{2}, m_\ell = 2, m_s = -\frac{1}{2}\rangle.$$

(c) Si se mide la tercera componente S_z del spin en un tiempo $t = 2\pi/\omega$, ¿qué valores pueden obtenerse y con qué probabilidades?

(a) In the coupled basis $\{\mathbf{L}^2, \mathbf{S}^2, \mathbf{J}^2, J_z\}$, with $\mathbf{J} = \mathbf{L} + \mathbf{S}$ the total angular momentum, the Hamiltonian reads

$$H = \frac{\omega}{\hbar} (\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2).$$

The Hamiltonian eigenstates and eigenvalues are $|\ell, s, J, M_J\rangle$ and (0.5)

$$E_{\ell, J} = \frac{1}{2} \hbar \omega \left[J(J+1) - \ell(\ell+1) - \frac{3}{4} \right].$$

where we have already set $s = 1/2$. The orbital angular momentum quantum number ℓ may take any nonnegative integer value $\ell = 0, 1, 2, \dots$. The total angular momentum quantum number J is the result of composing ℓ with $s = 1/2$ and may take the values $J = \ell + \frac{3}{2}, \dots, |\ell - \frac{1}{2}|$, equivalently (0.5)

$$j = \begin{cases} \frac{1}{2} & \text{for } \ell = 0, \\ \ell + \frac{1}{2}, \ell - \frac{1}{2} & \text{for } \ell \geq 1. \end{cases}$$

(b) The initial state is given in the decoupled basis $\{\mathbf{L}^2, \mathbf{S}^2, L_z, S_z\}$. Using the Clebsch-Gordan table for $2 \times \frac{1}{2}$, it is written in the coupled basis as

$$|\psi(0)\rangle = \left| m_\ell = 2, m_s = -\frac{1}{2} \right\rangle = \sqrt{\frac{1}{5}} \left| J = \frac{5}{2}, M_J = \frac{3}{2} \right\rangle + \sqrt{\frac{4}{5}} \left| J = \frac{3}{2}, M_J = \frac{3}{2} \right\rangle,$$

where for simplicity we have omitted the numbers $\ell = 2$ and $s = \frac{1}{2}$ from the notation. Since

$$\begin{aligned} \left| J = \frac{5}{2}, M_J = \frac{3}{2} \right\rangle & \text{ is stationary with energy } E_{\ell=2, J=5/2} = \hbar \omega, \text{ and} \\ \left| J = \frac{3}{2}, M_J = \frac{3}{2} \right\rangle & \text{ is stationary with energy } E_{\ell=2, J=3/2} = -\frac{3}{2} \hbar \omega, \end{aligned}$$

the state at time t is (0.5 for change of basis, 0.5 for time evolution)

$$|\psi(t)\rangle = \sqrt{\frac{1}{5}} e^{-i\omega t} \left| J = \frac{5}{2}, \ell = 2 \right\rangle + \sqrt{\frac{4}{5}} e^{3i\omega t/2} \left| J = \frac{3}{2}, \ell = 2 \right\rangle.$$

(c) In particular, at time $t = 2\pi/\omega$,

$$|\psi(2\pi/\omega)\rangle = \sqrt{\frac{1}{5}} \left| J = \frac{5}{2}, M_J = \frac{3}{2} \right\rangle - \sqrt{\frac{4}{5}} \left| J = \frac{3}{2}, M_J = \frac{3}{2} \right\rangle.$$

We are asked about probabilities for measurements of S_z , so. we go back to the coupled basis $\{\mathbf{L}^2, \mathbf{S}^2, L_z, S_z\}$. Using again the $2 \times \frac{1}{2}$ Clebsch-Gordan table, it follows that

$$\begin{aligned} \left| J = \frac{5}{2}, M_J = \frac{3}{2} \right\rangle & = \sqrt{\frac{1}{5}} \left| m_\ell = 2, m_s = -\frac{1}{2} \right\rangle + \sqrt{\frac{4}{5}} \left| m_\ell = 1, m_s = \frac{1}{2} \right\rangle, \\ \left| J = \frac{3}{2}, M_J = \frac{3}{2} \right\rangle & = \sqrt{\frac{4}{5}} \left| m_\ell = 2, m_s = -\frac{1}{2} \right\rangle - \sqrt{\frac{1}{5}} \left| m_\ell = 1, m_s = -\frac{1}{2} \right\rangle. \end{aligned}$$

Upon substitution in the expression of $|\psi(t = 2\pi/\omega)\rangle$, this yields (0.5 for change backwards)

$$|\psi(t = 2\pi/\omega)\rangle = -\frac{3}{5} \left| m_\ell = 2, m_s = -\frac{1}{2} \right\rangle + \frac{4}{5} \left| m_\ell = 1, m_s = \frac{1}{2} \right\rangle.$$

If S_z is measured, it is only possible to obtain $-\hbar/2$ and $\hbar/2$ with probabilities (0.5)

$$\text{Prob}\left(m_s = -\frac{1}{2}, t = \frac{2\pi}{\omega}\right) = \frac{9}{25}, \quad \text{Prob}\left(m_s = \frac{1}{2}, t = \frac{2\pi}{\omega}\right) = \frac{16}{25}.$$

3 [2 puntos]. Un sistema está formado por tres partículas idénticas de masa m y spin $s = 1/2$ en un pozo infinito unidimensional de anchura a con interacción dada por el potencial

$$V = -\frac{A}{\hbar^2} (\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2 \cdot \mathbf{S}_3 + \mathbf{S}_3 \cdot \mathbf{S}_1).$$

donde $A > 0$ es una constante con dimensiones de energía y \mathbf{S}_i es el operador de spin de la partícula i . Se sabe que el sistema se encuentra en un estado con spin total en la dirección del eje z igual a $+\frac{3}{2}\hbar$. Escribir los posibles estados en lo que puede encontrarse el sistema y sus energías.

The interaction term in the Hamiltonian can be written as

$$V = -\frac{A}{2} (\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2 - \mathbf{S}_3^2),$$

with $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$ the total spin operator of the system.

Being identical particles with spin $\frac{1}{2}$, the system's total wave function $\Psi_A(1, 2, 3)$ must be completely antisymmetric. The only spin state with the total spin's z -component $M_S = m_1 + m_2 + m_3 = \frac{3}{2}$ is (0.5 spin part)

$$\left| s_1 = 1/2, s_2 = 1/2, s_3 = 1/2, m_1 = 1/2, m_2 = 1/2, m_3 = 1/2 \right\rangle = \left| S = 3/2, M_S = 3/2 \right\rangle,$$

where in the last equality we have used that in the composition of three angular momenta $s = 1/2$ the only state with $M_S = 3/2$ has $S = 3/2$. The state $\left| S = 3/2, M_S = 3/2 \right\rangle$ above is symmetric under $(s_i, m_i) \leftrightarrow (s_j, m_j)$. For the total wave function to be antisymmetric, its spatial part must then be antisymmetric under $x_i \leftrightarrow x_j$. Since the eigenstates $\phi_n(x)$ of the infinite well are nondegenerate with energies

$$\phi_n(x) \rightarrow E_n = \frac{\pi^2 \hbar^2}{2ma^2} n^2 \quad n = 1, 2, \dots,$$

the total wave function is (0.5 spatial part)

$$\Psi_A(1, 2, 3) = \frac{1}{\sqrt{3!}} \det \begin{pmatrix} \phi_{n_1}(x_1) & \phi_{n_1}(x_2) & \phi_{n_1}(x_3) \\ \phi_{n_2}(x_1) & \phi_{n_2}(x_2) & \phi_{n_2}(x_3) \\ \phi_{n_3}(x_1) & \phi_{n_3}(x_2) & \phi_{n_3}(x_3) \end{pmatrix} \otimes \left| S = 3/2, M_S = 3/2 \right\rangle$$

Using that $\left| S = 3/2, M_S = 3/2 \right\rangle$ is an eigenstate of V with eigenvalue $-3A\hbar^2/4$,

$$V \left| S = 3/2, M_S = 3/2 \right\rangle = -\frac{3A\hbar^2}{4} \left| S = 3/2, M_S = 3/2 \right\rangle,$$

the state $\Psi_A(1, 2, 3)$ is an eigenstate of H with eigenenergy (0.5 space contribution, 0.5 spin contribution)

$$E = \frac{\hbar^2 \pi^2}{2ma^2} (n_1^2 + n_2^2 + n_3^2) - \frac{3A\hbar^2}{4}.$$

4 [2 puntos]. Una partícula de masa m oscila armónicamente en dos dimensiones, siendo su hamiltoniano

$$H_0 = \frac{\hbar^2}{2m} (P_x^2 + P_y^2) + \frac{1}{2} m\omega^2 (x^2 + y^2).$$

¿Cuál es la energía del primer estado excitado y su degeneración? Usando teoría de perturbaciones calcular la energía de dicho estado tras introducir una perturbación $H_I = bxy$, con b una constante con dimensiones de masa/(tiempo)².

The Hamiltonian H_0 is the sum of two harmonic oscillators, one in the x -direction and one in the y -direction, both with angular frequency ω . If $|n\rangle$ ($n = 0, 1, 2, \dots$) denotes the eigenstate of the one-dimensional harmonic oscillator with eigenenergy $E_n = \hbar\omega(n + \frac{1}{2})$, the eigenstates and eigenvalues of H_0 are

$$E_{n_x+n_y}^{(0)} = \hbar\omega(n_x + n_y + 1), \quad |n_x, n_y\rangle, \quad n_x, n_y = 0, 1, 2, \dots$$

The ground state is $|00\rangle$, has energy $E_0^{(0)} = \hbar\omega$ and is nondegenerate. The first excited state has degeneracy 2, for both $|10\rangle$ and $|01\rangle$ are eigenstates of H_0 with eigenvalue $E_1^{(0)} = 2\hbar\omega$ (0.5).

The first order corrections $E_1^{(1)}$ in perturbation theory to the unperturbed energy $E_1^{(0)}$ are the eigenvalues of the matrix

$$\mathbf{H}_I = \begin{pmatrix} \langle 10|H_I|10\rangle & \langle 10|H_I|01\rangle \\ \langle 01|H_I|10\rangle & \langle 01|H_I|01\rangle \end{pmatrix}$$

Since the perturbation $H_I = bxy$ is symmetric under $x \leftrightarrow y$, one has

$$\langle 10|H_I|10\rangle = \langle 01|H_I|01\rangle, \quad \langle 10|H_I|01\rangle = \langle 01|H_I|10\rangle.$$

To compute the $\langle 10|H_I|10\rangle$ and $\langle 10|H_I|01\rangle$ use

$$x = \frac{1}{\alpha\sqrt{2}} (a + a^+),$$

and

$$\begin{aligned} a|n\rangle &= \sqrt{n}|n-1\rangle & \Rightarrow & \quad a|0\rangle = 0, \quad a|1\rangle = |0\rangle, \\ a^+|n\rangle &= \sqrt{n+1}|n+1\rangle & \Rightarrow & \quad a^+|0\rangle = |1\rangle, \quad a^+|1\rangle = \sqrt{2}|2\rangle. \end{aligned}$$

This gives

$$\begin{aligned} \langle 10|H_I|10\rangle &= b \langle 1|x|1\rangle \langle 0|y|0\rangle = \frac{b}{2\alpha^2} \langle 1|(a + a^+)|1\rangle \langle 0|(a + a^+)|0\rangle = 0 \\ \langle 10|H_I|01\rangle &= b \langle 1|x|0\rangle \langle 0|y|1\rangle = \frac{b}{2\alpha^2} \langle 1|(a + a^+)|0\rangle \langle 0|(a + a^+)|1\rangle = \frac{b}{2\alpha^2}. \end{aligned}$$

and (1.0)

$$\mathbf{H}_I = \frac{b}{2\alpha^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The eigenvalues of \mathbf{H}_I are $\pm \frac{b}{2\alpha^2}$, so the first excited state splits in two levels with energies (0.5)

$$E_{1,\pm} = 2\hbar\omega \pm \frac{b}{2\alpha^2}.$$

5 [1.5 puntos]. Un átomo de hidrógeno se encuentra en su estado fundamental. Se somete a un campo eléctrico exterior

$$\mathcal{E} = \begin{cases} 0 & \text{si } t \leq 0, \\ \mathcal{E}_0 e^{-t/\tau} & \text{si } t > 0, \end{cases}$$

dirigido según el eje z , con \mathcal{E}_0 una constante. Discutir si es posible que el átomo transite a un estado con $n = 2$, y de serlo identificar los posible estados finales.

The sytem's Hamiltonian is

$$H = H_0 + V(t), \quad H_0 = \text{Hydrogen atom Hamiltonian}, \\ V(t) = ze\mathcal{E}(t).$$

The transition probability from an initial state $|i\rangle$ at time $t = 0$ to a final state $|f\rangle$ after a time t is at first order in perturbation theory

$$P_{if}(0, \infty) = \frac{1}{\hbar^2} \left| \int_0^\infty dt e^{i\omega_{fi}t} V_{fi}(t) \right|^2, \quad \omega_{fi} = \frac{E_f - E_i}{\hbar}, \quad V_{fi}(t) = \langle f|V(t)|i\rangle.$$

In our case,

$$|i\rangle = |1, 0, 0\rangle, \quad E_i = E_1; \quad |f\rangle = |2, \ell, m\rangle, \quad E_f = E_2; \quad E_n = -\frac{Z^2 e^2}{(4\pi\epsilon_0)2a_0 n^2}.$$

To have a nonzero probability one must have (0.5)

$$V_{fi}(t) \neq 0 \Leftrightarrow \langle 2, \ell, m|z|1, 0, 0\rangle \neq 0.$$

Since the ket $|1, 0, 0\rangle$ has angular momentum ($\ell_1 = 0, m_1 = 0$) and the operator z has ($\ell_2 = 1, m_2 = 0$), their composition has $L = 1$ and $M = 0$. The only possible final state with $n = 2$ is thus $|2, 1, 0\rangle$ (1.0).