Family name ____________________

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UNIVERSIDAD COMPLUTENSE MADRID

BACHELOR IN PHYSICS

Quantum Physics II
Final exam - June 20th, 2022
(Time: 3 hours)

1. [1 point]. The Hilbert space of a system formed by two qubits is $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$.

a) If $\{|0\rangle, |1\rangle\}$ is a basis in each $\mathbb{C}^2$, find a basis for the total Hilbert space and use it to write the state

$$|\psi\rangle = \frac{1}{\sqrt{3}} \big(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle \big)$$

as a density matrix.

b) An observable of the first qubit is is given in the basis $\{|0\rangle, |1\rangle\}$ by $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Compute its expectation value in the state $|\psi\rangle$.

a) A basis of $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ is given by

$$|00\rangle = |0\rangle \otimes |0\rangle, \; |01\rangle = |0\rangle \otimes |1\rangle, \; |10\rangle = |1\rangle \otimes |0\rangle, \; |11\rangle = |1\rangle \otimes |1\rangle.$$

Recall that if $|\psi\rangle = \sum_n c_n \phi_n$ is a pure state, it can be written as adensity matrix ϱ with matrix elements $\varrho_{nm} = c_n c_m^*$. For $|\psi\rangle$ given above we have

$$|\psi\rangle = \frac{1}{\sqrt{3}} \big(|00\rangle + |01\rangle + |11\rangle \big), \qquad \rho = \frac{1}{3} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

b) Using that $\{|0\rangle, |1\rangle\}$ is a basis for the second qubit, we have

$$\langle\psi|A|\psi\rangle = \frac{1}{3} \big(\langle 0| \otimes \langle 0| + \langle 0| \otimes \langle 1| + \langle 1| \otimes \langle 1| \big) A \big(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle \big)$$
$$= \frac{1}{3} \big(A_{00} + A_{00} + A_{01} + A_{10} + A_{11} \big) = \frac{1}{3},$$

where in the last equality we have used that $A_{00} = 1$, $A_{01} = A_{10} = 0$, $A_1 = -1$.

2. [1 points]. Consider the addition $\mathbf{J} = \mathbf{J_1} + \mathbf{J_2}$ of two angular momenta $j_1 = 5$ and $j_2 = 1$. If the system is in a state $|m_1 = 5, m_2 = 0\rangle$ and $\mathbf{J}^2$ is measured, what values can be obtained and with what probabilities?

In terms of states $|J, M\rangle$, the state $|m_1 = 5, m_2 = 0\rangle$ can only be a linear combination

$$\big|m_1 = 5, m_2 = 0\big\rangle = a\,\big|J = 6, M = 5\big\rangle + b\,\big|J = 5, M = 5\big\rangle\,.$$

Hence, a measurement of $\mathbf{J}^2$ may only give

$$\left.\begin{array}{llll} 6(6+1)\hbar^2 = 42\hbar^2 & \text{(corresponds to } J = 6) & \text{with probability } P_{42\hbar^2} = |a|^2 \\ 5(5+1)\hbar^2 = 30\hbar^2 & \text{(corresponds to } J = 5) & \text{with probability } P_{30\hbar^2} = |b|^2 \end{array}\right\} \quad |a|^2 + |b|^2 = 1\,.$$

To determine $|a|^2$ we note that

$$a = \big\langle J = 6, M = 5 \,\big|\, m_1 = 5, m_2 = 0\,\big\rangle$$

and act with $J_- = J_{1-} + J_{2-}$ on the highest M state

$$|J = 6, M = 6\rangle = |m_1 = 5, m_2 = 1\rangle\,.$$

This gives

$$J_-\big|J = 6, M = 6\big\rangle = \hbar\sqrt{12}\,\big|J = 6, M = 5\big\rangle\,,$$

$$\big(J_{1-} + J_{2-}\big)\,\big|m_1 = 5, m_2 = 1\big\rangle = \hbar\sqrt{10}\,\big|m_1 = 4, m_2 = 1\big\rangle + \hbar\sqrt{2}\,\big|m_1 = 5, m_2 = 0\big\rangle\,.$$

It follows that

$$\big|J = 6, M = 5\big\rangle = \sqrt{\frac{5}{6}}\,\big|m_1 = 4, m_2 = 1\big\rangle + \sqrt{\frac{1}{6}}\,\big|m_1 = 5, m_2 = 0\big\rangle \;\Rightarrow\; |a|^2 = \frac{1}{6}$$

and

$$P_{42\hbar^2} = \frac{1}{6}\,, \qquad P_{30\hbar^2} = \frac{5}{6}\,.$$

3. [3 points]. The Hamiltonian of a system formed by two particles with spins $s_1 = 3/2$ and $s_2 = 1$ is

$$H = \frac{\omega}{\hbar}\left[\mathbf{S_1}\cdot \mathbf{S_2} - \hbar\left(S_{1z} + S_{2z}\right)\right],$$

where ω is a constant angular frequency and $\mathbf{S_1}$ and $\mathbf{S_2}$ are the spin operators of the particles. At time $t = 0$ the third component of the spin of each particle is meassured with result $m_1 = -\hbar/2$ and $m_2 = -\hbar$.

a) Write the state $|\psi(0)\rangle$ immediately after the measurement.

b) Find the state $|\psi(t)\rangle$ at time t.

c) If S_{1z} and S_{2z} are measured at $t_1 = \pi/\omega$, what values can be obtained and with what probabilities?

a) In the basis $\{|m_1, m_2\rangle\}$, associated to $\{\mathbf{S}_1^2,\, \mathbf{S}_2^2,\, S_{1z},\, S_{2z}\}$, the state after the measurement is

$$|\psi(0)\rangle = \left|m_1 {=} -\tfrac{1}{2},\, m_2 {=} -1\right\rangle .$$

b) To solve the time-independent Schrödinger equation, it is convenient to work in terms of $\{\mathbf{S}_1^2,\, \mathbf{S}_2^2,\, \mathbf{S}^2,\, S_z\}$, where $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$. The Hamiltonian then becomes

$$H = \frac{\omega}{\hbar}\left[\frac{1}{2}\left(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\right) - \hbar S_z\right].$$

It has eigenstates and eigenvalues

$$H|SM\rangle = E_{SM}|SM\rangle\,, \qquad E_{SM} = \frac{\hbar\omega}{2}\left[S(S+1) - s_1(s_1+1) - s_2(s_2+1) - 2M\right].$$

Using the table of Clebsch-Gordan coefficients for $\frac{3}{2}\otimes 1$, $|\psi(0)\rangle$ can be written as

$$|\psi(0)\rangle = \sqrt{\frac{3}{5}}\,\left|S{=}\tfrac{5}{2}, M{=}-\tfrac{3}{2}\right\rangle + \sqrt{\frac{2}{5}}\,\left|S{=}\tfrac{3}{2}, M{=}-\tfrac{3}{2}\right\rangle .$$

The states on the right hand-side of this equaion have eigenenergies

$$E_{\frac{5}{2},-\frac{3}{2}} = 3\hbar\omega\,, \qquad E_{\frac{3}{2},-\frac{3}{2}} = \frac{\hbar\omega}{2}\,.$$

At time t the state is given by

$$|\psi(t)\rangle = \sqrt{\frac{3}{5}}\; e^{-3\mathrm{i}\omega t}\,\left|S{=}\tfrac{5}{2}, M{=}-\tfrac{3}{2}\right\rangle + \sqrt{\frac{2}{5}}\; e^{-\mathrm{i}\omega t/2}\,\left|S{=}\tfrac{3}{2}, M{=}-\tfrac{3}{2}\right\rangle$$

c) In particular, at $t_1 = \pi/\omega$,

$$|\psi(t_1)\rangle = -\sqrt{\frac{3}{5}}\,\left|S{=}\tfrac{5}{2}, M{=}-\tfrac{3}{2}\right\rangle - \mathrm{i}\sqrt{\frac{2}{5}}\,\left|S{=}\tfrac{3}{2}, M{=}-\tfrac{3}{2}\right\rangle$$

Using again the Clebsch-Gordan table for $\frac{3}{2}\otimes 1$, we have

$$
\begin{aligned}
|\psi(t_1)\rangle = & -\sqrt{\frac{3}{5}}\left[\sqrt{\frac{3}{5}}\,\big|m_1{=}-\tfrac{1}{2},\,m_2{=}-1\big\rangle+\sqrt{\frac{2}{5}}\,\big|m_1{=}-\tfrac{3}{2},\,m_2{=}0\big\rangle\right] \\
& -\mathrm{i}\sqrt{\frac{2}{5}}\left[\sqrt{\frac{2}{5}}\big|m_1{=}-\tfrac{1}{2},\,m_2{=}-1\big\rangle-\sqrt{\frac{3}{5}}\,\big|m_1{=}-\tfrac{3}{2},\,m_2{=}0\big\rangle\right] \\
= & -\frac{1}{5}\,(3+2\mathrm{i})\,\big|m_1{=}-\tfrac{1}{2},\,m_2{=}-1\big\rangle+\frac{\sqrt{6}}{5}\,(\mathrm{i}-1)\,\big|m_1{=}-\tfrac{3}{2},\,m_2{=}0\big\rangle\,.
\end{aligned}
$$

We conclude that a measurement of S_{1z} may only give $-\hbar/2$, $-3\hbar/2$ with probabilities

$$
P(S_{1z},-\tfrac{\hbar}{2})=\Big|-\frac{1}{5}\,(3+2\mathrm{i})\Big|^2=\frac{13}{25}\,,\qquad P(S_{1z},-\tfrac{3\hbar}{2})=\Big|\frac{\sqrt{6}}{5}\,(\mathrm{i}-1)\Big|^2=\frac{12}{25}\,.
$$

Similarly, a measurement of S_{2z} may only give $-\hbar$, 0 with probabilities

$$
P(S_{2z},-\hbar)=\frac{13}{25}\,,\qquad P(S_{2z},0)=\frac{12}{25}\,.
$$

4. [2 points]. A system is formed by $2N$ non-interacting identical particles in a 1-dimensional harmonic oscillator potential of angular frequency ω. Recalling that the one-particle eigenenergies are $E_n = \hbar\omega(n + \frac{1}{2})$, find the system ground state energy if the particles are

a) bosons with spin $s = 1$.

b) fermions with spin $s = \frac{1}{2}$.

HINT. $\sum_{n=0}^{k}(2n+1) = (k+1)^2$.

a) The lowest energy for the system is achieved by allocating all bosons in the one-particle ground state, whose energy is $E_0 = \hbar\omega/2$. The systen ground state has energy is then $E_{2N} = 2N\,E_0 = N\hbar\omega$.

b) For fermions Pauli's exclusion principle applies and there cannot be two fermions with the smae quantum numbers. This implies that in each energy level only two fermions can be allocated, one with $m_s = 1/2$ and the other one with $m_s = -1/2$. Since we have $2N$ fermions, the ground state energy is

$$E_{2N} = \sum_{n=0}^{N-1} 2E_n = \sum_{n=0}^{N-1} \hbar\omega(2n+1) = \hbar\omega N^2\,.$$

5. [1 point]. Explain which ones of the following expressions are admissible as Hamiltonian of a system formed by two indistinguishable particles

a) $H = a\,\mathbf{S_1}^2$.

b) $H = b\,S_{1z} + c\,S_{2z}$

b) $H = d\,\mathbf{S_1}\cdot\mathbf{S_2}$.

Here $\mathbf{S_1}$ and $\mathbf{S_2}$ are the spin operators of the particles and a, b, c, d are positive real constants.

The Hamiltonian must be invariant under $1 \leftrightarrow 2$.

a) Since both particles have the same spin, $s_1 = s_2 = s$, the action of H on any state is proportional to the identity, $H = \hbar^2 s(s+1)$, hence invariant, hence admissible.

b) In this case m_1 and m_2 may be different, and H is invariant under $1 \leftrightarrow 2$ only for $a = b$, in which case H is admissible.

c) H is trivially invariant under $1 \leftrightarrow 2$ for all d, hence admissible.

6. [2 points]. The Hamiltonian of a particle with spin 1 is

$$H = \frac{\epsilon}{\hbar} S_z + \frac{\alpha}{\hbar^2}\left(S_+^2 + S_-^2\right),$$

with ϵ and α two constants with dimensions of energy.

a) Assuming that $\alpha \ll \epsilon$, find the energy levels of the system up to second order in α in perturbation theory.

b) Find the exact energy levels.

a) The unperturbed Hamiltonian is $H_0 = \epsilon S_z/\hbar$. Its eigenstates $\{|\phi_n^{(0)}\rangle\}$ are the eigenstates $\{|m\rangle\}$ of the third component of the spin S_z, are non-degenerate and have energies $E_m^{(0)} = m\epsilon$,

$$\{|\phi_n^{(0)}\rangle\} = \{|m\rangle\}\,, \qquad E_m^{(0)} = m\epsilon\,, \qquad m = -1, 0, 1\,.$$

The first-order corrections due to the perturbation $H_I = \alpha\big(S_+^2 + S_-^2\big)/\hbar^2$ are given by

$$E_m^{(1)} = \big\langle m\big|H_I\big|m\big\rangle\,.$$

Using

$$S_\pm\,\big|m\big\rangle = \hbar\,\sqrt{1(1+1) - m(m\pm 1)}\,\big|m\pm 1\big\rangle\,,$$

we have

$$\big\langle 1\big|S_+^2\big|-1\big\rangle = \big\langle -1\big|S_-^2\big|1\big\rangle = 2\hbar^2\,, \qquad \text{any other} \quad \big\langle m\big|S_\pm^2\,\big|k\big\rangle = 0\,. \tag{2}$$

Hence

$$E_m^{(1)} = 0\,.$$

To calculate the second order correction, use the expression

$$E_m^{(2)} = \sum_{k\neq m} \frac{|\langle m|H_I|k\rangle|^2}{E_m^{(0)} - E_k^{(0)}}\,.$$

Eq. (2) implies

$$E_0^{(2)} = 0\,, \qquad E_1^{(2)} = \frac{1}{2\epsilon}\left|\frac{\alpha}{\hbar^2}\langle 1|S_+^2|-1\rangle\right|^2 = \frac{2\alpha^2}{\epsilon}\,, \qquad E_{-1}^{(2)} = -\frac{1}{2\epsilon}\left|\frac{\alpha}{\hbar^2}\langle -1|S_-^2|1\rangle\right|^2 = -\frac{2\alpha^2}{\epsilon}\,.$$

All in all

$$E_0 = O\Big(\frac{\alpha^3}{\epsilon^3}\Big)\,, \qquad E_1 = \epsilon\left[1 + \frac{2\alpha^2}{\epsilon^2} + O\Big(\frac{\alpha^3}{\epsilon^3}\Big)\right], \qquad E_{-1} = -\epsilon\left[1 + \frac{2\alpha^2}{\epsilon^2} + O\Big(\frac{\alpha^3}{\epsilon^3}\Big)\right].$$

b) The matrix elements of H in the basis $\{|m\rangle\}$ are

$$\mathcal{H} = \begin{pmatrix} \langle 0|H|0\rangle & \langle 0|H|1\rangle & \langle 0|H|-1\rangle \\ \langle 1|H|0\rangle & \langle 1|H|1\rangle & \langle 1|H|-1\rangle \\ \langle -1|H|0\rangle & \langle -1|H|1\rangle & \langle -1|H|-1\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon & 2\alpha \\ 0 & 2\alpha & -\epsilon \end{pmatrix}.$$

The exact energies are the eigenvalues of H, namely

$$\det(\mathcal{H} - E_{\text{exact}}) = 0 \quad \Rightarrow \quad E_{\text{exact}} = 0,\, \pm\sqrt{\epsilon^2 + 4\alpha^2}$$

whose expansion in powers of ϵ/α reproduces the perturbative values.