Family name ____________________

Given name ____________________

Signature ____________________

UNIVERSIDAD COMPLUTENSE MADRID

BACHELOR IN PHYSICS

Quantum Physics II

Mid-term exam – November 17th, 2021

(Time: 1 hour, 25 minutes)

In exercises 1 and 2, only fill in the boxes (calculations will not be collected).

1 [2 points]. In a Hilbert space, $|1\rangle$ and $|2\rangle$ are pure states, orthogonal to each other. Consider the following three operators with matrix elementes given by

$$\varrho_1 = \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \varrho_2 = \begin{pmatrix} \frac{1}{2} & \frac{\mathrm{i}}{2} \\ \frac{\mathrm{i}}{2} & \frac{\mathrm{i}}{2} \end{pmatrix}, \qquad \varrho_3 = \begin{pmatrix} \frac{1}{4} & -\frac{\mathrm{i}\sqrt{3}}{2} \\ \frac{\mathrm{i}\sqrt{3}}{2} & \frac{3}{4} \end{pmatrix},$$

Which ones of them are admissible matrix densities?

Density matrices: None

Which ones describe mixed states?

Mixed states: None

Which ones describe pure states?

Pure states: None

In case there is any pure state, write it as a linear superposition of $|1\rangle$ and $|2\rangle$.

Does not apply

- ϱ_1 is not a density matrix since $\mathrm{tr}\varrho_1^2 = \mathrm{tr}\begin{pmatrix} 2 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} = 3$ is not less or equal than 1.
- ϱ_2 is not a density matrix because it is not selfadjoint.
- ϱ_3 is not a density matrix because it is not postive semidefinite. Indeed, for any complex 2-vector $v = (v_1, v_2)$,

$$\langle v|\varrho_3|v\rangle = \frac{1}{4}\,(v_1^*, v_2^*)\begin{pmatrix} 1 & -2\mathrm{i}\sqrt{3} \\ 2\mathrm{i}\sqrt{3} & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{4}\left(|v_1 - 2\mathrm{i}\sqrt{3}v_2|^2 - 9|v_2|^2\right)$$

is not larger or equal than zero. For example, for $v_1 = 2\mathrm{i}\sqrt{3}v_2$, the right hand side is negative.

2 [0.5+0.5 points]. Two angular momenta $j_1 = 3/2$ and $j_2 = 1$ are composed. What are the posibble values of the resulting total angular momentum

$$\frac{3}{2} \otimes 1 = \frac{5}{2},\ \frac{3}{2},\ \frac{1}{2}$$

Write the state $|m_1 = \frac{3}{2},\, m_2 = -1\rangle$ in terms of the states $|J,\, M_J\rangle$. (Provide the expression without argumentation).

$$|m_1 = \tfrac{3}{2},\, m_2 = -1\rangle = \sqrt{\tfrac{1}{10}}\,|J = \tfrac{5}{2},\, M_J = \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{5}}\,|J = \tfrac{3}{2},\, M_J = \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{2}}\,|J = \tfrac{1}{2},\, M_J = \tfrac{1}{2}\rangle$$

3 [3 points]. Three electrons (spin 1/2) are in a potential whose eigenstates and eigenenergies are

$$|\phi_n\rangle\,, \qquad E_n = [n(n-1)+1]\,\epsilon, \qquad n = 0, 1, 2, \ldots,$$

where ϵ is a constant with dimensions of energy. Find the ground state energy and its degeneracy. Write the ground states wave functions.

The lowest one-particle energy is $E = \epsilon$, which is attained for $n = 0, 1$. It has degeneracy two and its eigenstates are $|\phi_0\rangle$ and $|\phi_1\rangle$. If spin is included, degeneracy is four, and the eigenstates are $|\phi_{0,m_s=\pm 1/2}\rangle$ and $|\phi_{1,m_s=\pm 1/2}\rangle$.

There are four different ways to accomadate three electrons in the four different one-particle ground states:

i) Allocate two electrons in $|\phi_{0,1/2}\rangle$ and $|\phi_{0,-1/2}\rangle$, and the third electron in $|\phi_{1,m_s}\rangle$, with m_s either $\frac{1}{2}$ or $-\frac{1}{2}$. The corresponding wave functions are

$$|\Psi(1,2,3)\rangle = \frac{1}{3!}\det\begin{pmatrix} |\phi_{0,1/2}(1)\rangle & |\phi_{0,1/2}(2)\rangle & |\phi_{0,1/2}(3)\rangle \\ |\phi_{0,-1/2}(1)\rangle & |\phi_{0,-1/2}(2)\rangle & |\phi_{0,-1/2}(3)\rangle \\ |\phi_{1,\pm 1/2}(1)\rangle & |\phi_{1,\pm 1/2}(2)\rangle & |\phi_{1,\pm 1/2}(3)\rangle \end{pmatrix}.$$

2i) Two electrons in $|\phi_{1,1/2}\rangle$, $|\phi_{1,-1/2}\rangle$, and the third one in $|\phi_{0,\pm 1/2}\rangle$, with wave functions

$$|\Psi(1,2,3)\rangle = \frac{1}{3!}\det\begin{pmatrix} |\phi_{1,1/2}(1)\rangle & |\phi_{1,1/2}(2)\rangle & |\phi_{1,1/2}(3)\rangle \\ |\phi_{1,-1/2}(1)\rangle & |\phi_{1,-1/2}(2)\rangle & |\phi_{1,-1/2}(3)\rangle \\ |\phi_{0,\pm 1/2}(1)\rangle & |\phi_{0,\pm 1/2}(2)\rangle & |\phi_{0,\pm 1/2}(3)\rangle \end{pmatrix}.$$

All in all, the ground state has energy $E = 3\epsilon$, degeneracy 4, and eigenfunctions as above.

4 [4 points]. Consider an electron in a Hydrogen atom . The electron interacts with a uniform external magnetic field B oriented in the z-axis through the Hamiltonian term

$$\omega(L_z + 2S_z)\,, \qquad \omega = \frac{eB}{2mc}\,.$$

a) Find the system eigenenergies. Note that the total Hamiltonian is $H_{\text{Hydrogen}} + \omega(L_z + 2S_z)$.

b) Initially the electron is in a state with $n = 2$, $\ell = 1$, total angular momentum (orbital+spin) $J = 1/2$ and total angular momentum third-component $M_J = \frac{1}{2}$. If the energy is measured at time zero, what values can be obtained and with what probabilities?

c) Find the state at time t. If at time $t = \pi/\omega$ the square of the orbital angular momentum is measured, what values can be obtained and with what probabilities?

a) The eigenenergies and eigenstates are

$$E_{nm_\ell m_s} = \tilde{E}_n + \hbar\omega\,(m_\ell + 2m_s)\,, \qquad \tilde{E}_n = -\frac{Z^2e^2}{2a_0n^2} \qquad |n,\ell,s=\tfrac{1}{2},m_\ell,m_s\rangle\,.$$

where $\tilde{E}_n$ are the usual Hydrogen atom energies in Gaussian units.

b) Using the Clebsch-Gordan table for $1\otimes\frac{1}{2}$, the initial state $|n=2,\ell=1,s=\frac{1}{2},j=\frac{1}{2},m_j=\frac{1}{2}\rangle$ can be written in terms of $|n=2,\ell=1,s=\frac{1}{2},m_\ell,m_s\rangle$ as

$$|\psi(0)\rangle = |2,1,\tfrac{1}{2},j=\tfrac{1}{2},m_j=\tfrac{1}{2}\rangle \;=\sqrt{\tfrac{2}{3}}\,|2,1,\tfrac{1}{2},m_\ell=1,m_s=-\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,|2,1,\tfrac{1}{2},m_\ell=0,m_s=\tfrac{1}{2}\rangle\,.$$

The only possible results for a measurement of the the energy at $t = 0$ are then

$$E_{2,1,-\frac{1}{2}} = \tilde{E}_2 \qquad \text{with probability } \frac{2}{3}\,,$$

$$E_{2,0,\frac{1}{2}} = \tilde{E}_2 + \hbar\omega \qquad \text{with probability } \frac{1}{3}\,.$$

c) The state at time t is

$$|\psi(0)\rangle = e^{-\mathrm{i}tE_2/\hbar}\left[\sqrt{\tfrac{2}{3}}\,|2,1,\tfrac{1}{2},m_\ell=1,m_s=-\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,e^{-\mathrm{i}\omega t}\,|2,1,\tfrac{1}{2},m_\ell=0,m_s=\tfrac{1}{2}\rangle\right].$$

It has $\ell = 1$. Hence the only possible result for a measument of $\mathbf{L}^2$ is $2\hbar^2$, with probability 1.