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# Quantum Physics II – Midterm exam – November 16th, 2022

(Time: 1 hour 30 minutes)

**1 [2 points].** The Hilbert space of a system has basis $\{|1\rangle, |2\rangle, |3\rangle\}$. Given in this basis the matrix

$$\varrho = \begin{pmatrix} \frac{1}{2} & 0 & \frac{\mathrm{i}}{2} \\ 0 & \frac{1}{2} & 0 \\ -\frac{\mathrm{i}}{2} & 0 & 0 \end{pmatrix},$$

does it describe a physical state? If so, of which type?

**2 [2 points].** An electron at rest with gyromagnetic ratio $\gamma > 0$ and spin initially oriented positively in the direction of $\mathbf{n} = \frac{\sqrt{3}}{2}\mathbf{e}_1 - \frac{1}{2}\mathbf{e}_2$ is under the action of a constant magnetic field $\mathbf{B} = B\,(\mathbf{e}_1 - \mathbf{e}_2)$, with $B > 0$. The Hamiltonian is $H = \gamma\,\mathbf{B}\cdot\mathbf{S}$, with $\mathbf{S}$ the electron's spin operator. Find the precession axis, the precession frequency and the angle formed by the expectation value of the electron's spin with the magnetic field.

**3 [1 point].** A system formed by two particles with spins $s_1 = 3/2$ and $s_2 = 1/2$ is in a state whose total spin squared is $6\hbar^2$ and whose total spin third component is $\hbar$. If the spin third component of each particle is measured, what values can be obtained and with what probabilities?

**4 [2 points].** A system is formed by three electrons. The one-particle states and energies are given by $\phi_{kp}$ and $E_{kp} = \epsilon(k+p)$, with $\epsilon > 0$ a constant with dimension of energy and $k, p = 1, 2, \ldots$ Note that the one-particle energies are degenerate. Find the system's ground state energy, its degenaracy and write the corresponding wave functions as Slater determinants.

## Problem 1

$$\rho = \begin{pmatrix} 1/2 & 0 & i/2 \\ 0 & 1/2 & 0 \\ -i/2 & 0 & 0 \end{pmatrix}$$

The matrix $\rho$ satisfies

i) $\mathrm{tr}\,\rho = 1$

2i) $\rho^\dagger = \rho$

3i) Its eigenvalues are the solutions $\lambda$ to the equation

$$0 = \det \begin{pmatrix} 1/2-\lambda & 0 & i/2 \\ 0 & 1/2-\lambda & 0 \\ -i/2 & 0 & -\lambda \end{pmatrix} = -\lambda\left(\frac{1}{2}-\lambda\right)^2 - \frac{1}{4}\left(\frac{1}{2}-\lambda\right) = -\left(\frac{1}{2}-\lambda\right)\left[\lambda\left(\frac{1}{2}-\lambda\right)+\frac{1}{4}\right]$$

$$0 = \left(\frac{1}{2}-\lambda\right)\left(\lambda^2 - \frac{\lambda}{2} - \frac{1}{4}\right) \Rightarrow \lambda_0 = 0\,,\ \lambda_\pm = \frac{1}{4}\left(1 \pm \sqrt{5}\right)$$

Since $\lambda_- = \frac{1}{4}(1-\sqrt{5})$ is negative, $\rho$ is not positive semidefinite and hence $\rho$ does not describe a physical state.

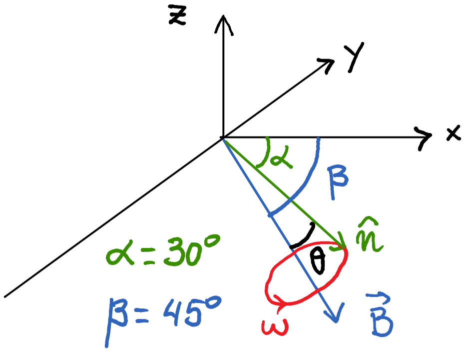
## Problem 2

As discussed in the lectures, the expectation value $\langle \vec{S} \rangle_\psi$ of the spin in a state $|\psi\rangle$ precesses about the magnetic field with angle $\theta$. To compute $\theta$, we calculate $\langle \vec{S} \rangle_\psi$ at time $t=0$ and the angle that the latter forms with $\vec{B}$. Now

$$\langle \vec{S} \rangle_{\psi(0)} = \langle \hat{n}_+ | \vec{S} | \hat{n}_+ \rangle = [\text{part c) problem 2.1}] = \frac{\hbar}{2}\,\hat{n}\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0\right)$$

$|\hat{n}_+\rangle$ = spin state positively oriented along $\hat{n}$

Hence




Axis: $\hat{B} = \frac{1}{\sqrt{2}}(1,-1,0)$

Frequency: $\omega = \gamma|\vec{B}| = \sqrt{2}\gamma B$

angle $= \theta = \beta - \alpha = 15°$

## Problem 3

The system is in a state $|S=2, M=1\rangle := |s_1=3/2, s_2=1/2, S=2, M=1\rangle$.
Use the table of Clebsch-Gordan coefficients to write it in the basis $|m_1, m_2\rangle := |s_1=3/2, s_2=1/2, m_1, m_2\rangle$ as

$$|S=2, M=1\rangle = \frac{1}{2}\,|m_1=3/2, m_2=-1/2\rangle + \frac{\sqrt{3}}{2}\,|m_1=1/2, m_2=1/2\rangle$$

If $S_{1z}$ and $S_{2z}$ are measured the following results may be obtained

$$\mathrm{Prob}(S_{1z}, \tfrac{3}{2}\hbar) = \frac{1}{4}, \quad \mathrm{Prob}(S_{1z}, \tfrac{\hbar}{2}) = \frac{3}{4}$$

$$\mathrm{Prob}(S_{2z}, -\tfrac{\hbar}{2}) = \frac{1}{4}, \quad \mathrm{Prob}(S_{2z}, \tfrac{\hbar}{2}) = \frac{3}{4}$$

## Problem 4

The lowest one-particle energies are

$E_{11} = 2\varepsilon$ eigenstate: $\phi_{11}$

$E_{12} = E_{21} = 3\varepsilon$ eigenstates: $\phi_{12}$ and $\phi_{21}$.

$\vdots$

Two electrons go into $\phi_{11}$ one with $m_s = 1/2$ and the other one with $m_s = -1/2$:

2 electrons go to $\phi_{\alpha_1} = \phi_{11+}$, $\phi_{\alpha_2} = \phi_{11-}$

The other electron goes to a level with energy $3\varepsilon$. There are four different ways to do that

3rd electron goes to $\phi_{\alpha_3} = \phi_{12+}, \phi_{12-}, \phi_{21+}, \phi_{21-}$ (*)

Hence

$$E(1,2,3) = 2\varepsilon + 2\varepsilon + 3\varepsilon = 7\varepsilon, \quad \text{degeneracy} = 4$$

$$\psi(1,2,3) = \frac{1}{\sqrt{3!}} \det \begin{pmatrix} \phi_{\alpha_1}(1) & \phi_{\alpha_1}(2) & \phi_{\alpha_1}(3) \\ \phi_{\alpha_2}(1) & \phi_{\alpha_2}(2) & \phi_{\alpha_2}(3) \\ \phi_{\alpha_3}(1) & \phi_{\alpha_3}(2) & \phi_{\alpha}(3) \end{pmatrix}$$