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# Física cuántica II - Examen final - 9 de enero de 2026

(Tiempo: 3 horas)

**1 [2 puntos].** El hamiltoniano de un sistema formado por dos partículas distinguibles de espín 1/2 es

$$H = \hbar\omega(\sigma_z \otimes \sigma_x),$$

con $\omega > 0$ una frecuencia conocida y $\sigma_z$ y $\sigma_x$ las matrices de Pauli. El sistema se prepara en un estado inicial $|\psi(0)\rangle = |z+, z+\rangle$ en el que la tercera componente del espín de ambas partículas es $+\hbar/2$.

a) Escribir el estado del sistema en un tiempo $t$.

b) Calcular la probabilidad de que en $t$ el estado sea el mismo que el inicial.

c) Calcular la probabilidad de que al medir en $t$ las terceras componentes del espín de cada una de las partículas se obtenga $+\hbar/2$ para la primera y $-\hbar/2$ para la segunda.

a) The Hamiltonian is

$$H = \hbar\omega(\sigma_z \otimes \sigma_x) = \hbar\omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hbar\omega \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Its eigenvalues $E$ are the solutions to the equation

$$\det(H - E\boldsymbol{I}) = 0 \quad \Leftrightarrow \quad \det \begin{pmatrix} -E & \hbar\omega \\ \hbar\omega & -E \end{pmatrix} \det \begin{pmatrix} -E & -\hbar\omega \\ -\hbar\omega & -E \end{pmatrix} = 0 \quad \Leftrightarrow \quad (E^2 - \hbar^2\omega^2)^2 = 0$$

$$\Leftrightarrow \quad E_+ = \hbar\omega, \; E_- = -\hbar\omega, \; \text{both with mutiplicty 2.}$$

The orthonormal associated eigenvectors are can be taken as

$$|E_{+1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |E_{+2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad |E_{-1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad |E_{-2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

In terms of them, the initial state is written as (Watch out the factor $\frac{1}{\sqrt{2}}$!)

$$|\psi(0)\rangle = |z+, z+\rangle = |z+\rangle \otimes |z+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|E_{+1}\rangle + |E_{-1}\rangle\right).$$

The state at time $t$ is

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\Big(e^{-\mathrm{i}E_+t/\hbar}\,|E_{+1}\rangle + e^{-\mathrm{i}E_-t/\hbar}\,|E_{-1}\rangle\Big) = \frac{1}{2}\begin{pmatrix} e^{-\mathrm{i}\omega t} + e^{\mathrm{i}\omega t} \\ e^{-\mathrm{i}\omega t} - e^{\mathrm{i}\omega t}0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\omega t) \\ -\mathrm{i}\sin(\omega t) \\ 0 \\ 0 \end{pmatrix}.$$

b) The porbability that $|\psi(t)\rangle$ is the same as the initial state is

$$P(\text{init. state}, t) = \big|\langle\psi(0)|\psi(t)\rangle\big|^2 = \left|\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} \cos(\omega t) \\ -\mathrm{i}\sin(\omega t) \\ 0 \\ 0 \end{pmatrix}\right|^2 = \cos^2(\omega t)$$

c) The probability to obtain for particle 1 spin $\hbar/2$ and for particle 2 spin $-\hbar/2$ in the $z$-direction at time t is

$$P(m_1 = \frac{1}{2},\, m_2 = -\frac{1}{2}, t) = \big|\langle z+, z- |\psi(t)\rangle\big|^2 = \left|\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix} \cos(\omega t) \\ -\mathrm{i}\sin(\omega t) \\ 0 \\ 0 \end{pmatrix}\right|^2 = \sin^2(\omega t)$$

where we have used that

$$|z+, z-\rangle = |z+\rangle \otimes |z-\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

**2 [2 puntos].** Un átomo de hidrǵeno se encuentra en un estado

$$|\psi\rangle = R_{21}(r)\sqrt{\frac{3}{8\pi}}\Big( -\sin\theta\cos\phi\,|m_s = \tfrac{1}{2}\rangle + \cos\theta\,|m_s = -\tfrac{1}{2}\rangle\Big)\,.$$

donde $R_{21}(r)$ es la parte radial de la función de ondas y $(r, \theta, \phi)$ son coordenadas esféricas.

a) Escribir el estado $|\psi\rangle$ en la base desacoplada $|n, l,\, s,\, m_l,\, m_s\rangle$.

b) Determinar los valores que pueden obtenerse y las probabilidades con que se obtienen si se miden $L_z$, $\mathbf{L}^2$, $S_z$, $\mathbf{S}^2$, $J_z$ y $\mathbf{J}^2$, donde $\mathbf{J} = \mathbf{L} + \mathbf{S}$ es el momento angular total.

a) Using

$$Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}e^{\mathrm{i}\phi}\sin\theta, \quad Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}e^{-\mathrm{i}\phi}\sin\theta,$$

the state $|\psi\rangle$ is written in the basis $|(nls)\, m_l\, m_s\rangle$ as

$$|\psi\rangle = R_{21}(r)\left[\frac{1}{2}\,Y_1^1\,|m_s = \tfrac{1}{2}\rangle - \frac{1}{2}\,Y_1^{-1}\,|m_s = \tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}\,Y_1^0\,|m_s = \tfrac{1}{2}\rangle\right].$$

In the notation $|(nls)\, m_l\, m_s\rangle$, the latter becomes

$$\begin{aligned}|\psi\rangle &= \frac{1}{2}\,|(2,1,\tfrac{1}{2})\; m_l = 1,\, m_s = \tfrac{1}{2}\rangle - \frac{1}{2}\,|(2,1,\tfrac{1}{2})\; m_l = -1,\, m_s = \tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}\,|(2,1,\tfrac{1}{2})\; m_l = 0,\, m_s = -\tfrac{1}{2}\rangle \\ &= |(2,1,\tfrac{1}{2})\rangle \otimes \left[\frac{1}{2}\,|m_l = 1,\, m_s = \tfrac{1}{2}\rangle - \frac{1}{2}\,|m_l = -1,\, m_s = \tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}\,|m_l = 0,\, m_s = -\tfrac{1}{2}\rangle\right]\end{aligned}$$

b) The values that can be obtained in a measure of $L_z$ are $\hbar$, $-\hbar$ and 0, with probabilitites

$$P(L_z,\hbar) = \left|\frac{1}{2}\right|^2 = \frac{1}{4}\,, \qquad P(L_z,-\hbar) = \left|-\frac{1}{2}\right|^2 = \frac{1}{4}\,, \qquad P(L_z,0) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}\,.$$

When measuring $\mathbf{L}^2$, result $1(1+1)\hbar^2 = 2\hbar^2$ is obtained with certainty.

The possible results in a measurement of $S_z$ are $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$, with porobabilities

$$P\Big(S_z, \frac{\hbar}{2}\Big) = \left|\frac{1}{2}\right|^2 + \left|-\frac{1}{2}\right|^2 = \frac{1}{2}\,, \qquad P\Big(S_z, -\frac{\hbar}{2}\Big) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}\,.$$

When $\mathbf{S}^2$ is measured, $\frac{1}{2}(1+\frac{1}{2})\hbar^2 = \frac{3}{4}\hbar^2$ is obtained with certainty.

Using the Clebsch-Gordan table $1 \otimes \frac{1}{2}$, one has

$$\begin{aligned}|m_l = 1,\, m_s = \tfrac{1}{2}\rangle &= |J = \tfrac{3}{2},\, M_J = \tfrac{3}{2}\rangle \\ |m_l = -1,\, m_s = \tfrac{1}{2}\rangle &= \sqrt{\frac{1}{3}}\,|J = \tfrac{3}{2},\, M_J = -\tfrac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|J = \tfrac{1}{2},\, M_J = -\tfrac{1}{2}\rangle \\ |m_l = 0,\, m_s = -\tfrac{1}{2}\rangle &= \sqrt{\frac{2}{3}}\,|J = \tfrac{3}{2},\, M_J = -\tfrac{1}{2}\rangle + \sqrt{\frac{1}{3}}\,|J = \tfrac{1}{2},\, M_J = -\tfrac{1}{2}\rangle\end{aligned}$$

Substituting these in the expression above for $|\psi\rangle$, one obtains

$$
\begin{aligned}
|\psi\rangle = |(2,1,\tfrac{1}{2})\rangle \otimes \Bigg[ & \frac{1}{2}\,|J=\tfrac{3}{2},\, M_J=\tfrac{3}{2}\rangle \\
& -\frac{1}{2}\left(\sqrt{\frac{1}{3}}\,|J=\tfrac{3}{2},\, M_J=-\tfrac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|J=\tfrac{1}{2},\, M_J=-\tfrac{1}{2}\rangle\right) \\
& +\frac{1}{\sqrt{2}}\left(\sqrt{\frac{2}{3}}\,|J=\tfrac{3}{2},\, M_J=-\tfrac{1}{2}\rangle + \sqrt{\frac{1}{3}}\,|J=\tfrac{1}{2},\, M_J=-\tfrac{1}{2}\rangle\right)\Bigg] \\
= |(2,1,\tfrac{1}{2})\rangle \otimes \Bigg[ & \frac{1}{2}\,|J=\tfrac{3}{2},\, M_J=\tfrac{3}{2}\rangle + \sqrt{\frac{1}{12}}\,|J=\tfrac{3}{2},\, M_J=-\tfrac{1}{2}\rangle + \sqrt{\frac{2}{3}}\,|J=\tfrac{1}{2},\, M_J=-\tfrac{1}{2}\rangle\Bigg]
\end{aligned}
$$

A measure of $J_z$ can give $\frac{3}{2}\hbar$ and $-\frac{1}{2}\hbar$ with probabilities

$$
P\Big(J_z, \frac{3}{2}\,\hbar\Big) = \left|\frac{1}{2}\right|^2 = \frac{1}{4}\,, \qquad P\Big(J_z, -\frac{\hbar}{2}\Big) = \left|\sqrt{\frac{1}{12}}\right|^2 + \left|\sqrt{\frac{2}{3}}\right|^2 = \frac{3}{4}\,.
$$

Finally, the possible results for a measure of $\mathbf{J}^2$ are $\frac{3}{2}(\frac{3}{2}+1)\hbar^2 = \frac{15}{4}\hbar^2$ and $\frac{1}{2}(\frac{1}{2}+1)\hbar^2 = \frac{3}{4}\hbar^2$, with probabilities

$$
P\Big(\mathbf{J}^2, \frac{15}{4}\,\hbar^2\Big) = \left|\sqrt{\frac{1}{2}}\right|^2 + \left|\sqrt{\frac{1}{12}}\right|^2 = \frac{1}{3}\,, \qquad P\Big(\mathbf{J}^2, \frac{3}{4}\,\hbar^2\Big) = \left|\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3}\,.
$$

**3 [2 puntos].** Dos electrones (sin interacción entre ellos) en un potencial en tres dimensiones de energía potencial $V(\mathbf{x})$ están sometidos a la acción de un campo magnético externo uniforme de módulo $B$ dirigido según el eje $z$. El hamiltoniano **para cada uno de ellos** es

$$H = H_{\rm orb} - \gamma B S_z\,, \qquad H_{\rm orb} = \frac{\mathbf{P}^2}{2m} + V(\mathbf{x})$$

donde $S_z$ es la tercera componente del operador de espín. Se sabe que el estado fundamental del potencial $V(r)$ tiene energía $\epsilon$ y es doblemente degenerado, con autoestados ortonormales $\phi_1$ y $\phi_2$,

$$H_{\rm orb}\phi_\alpha = \epsilon\phi_\alpha \quad (\alpha = 1,2), \qquad \langle\phi_1|\phi_1\rangle = \langle\phi_2|\phi_2\rangle = 1, \qquad \langle\phi_1|\phi_2\rangle = 0.$$

a) Escribir todos los posibles autoestados del sistema en los que la energía orbital es mínima. ¿Cuál es la energía total de cada uno de estos estados?

b) Calcular los posibles valores de cuadrado $\mathbf{S}^2$ del espín total $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ y de su tercera componente $S_z$ en cada uno de los estados del apartado anterior.

a) For the system to have minimal orbital energy, both electrons must be in the ground state of $V(\mathbf{x})$. There are various ways to achieve this in a compatible way with Pauli's principle:

(1) Both electrons are in the orbital state $\phi_1$, in which case, in accordane with Pauli's principle, one of them has $m_s = \frac{1}{2}$ and the other one has $m_s = -\frac{1}{2}$. The filled states are then

$$|\phi_1,\, m = +\tfrac{1}{2}\rangle \;\text{ and }\; |\phi_1,\, m = -\tfrac{1}{2}\rangle\,.$$

The system's state is (blue $\to$ quantum numbers; red $\to$ particles) is given by the Slater determinant

$$\Psi(1,2) = \frac{1}{\sqrt{2}}\,\det\begin{pmatrix} \phi_1(1)\;|m_1 = +\frac{1}{2}\rangle & \phi_1(2)\;|m_2 = +\frac{1}{2}\rangle \\ \phi_1(1)\;|m_1 = -\frac{1}{2}\rangle & \phi_1(2)\;|m_2 = -\frac{1}{2}\rangle \end{pmatrix} = \Big[\text{Use notation } |m_1, m_2\rangle \text{ for spin part}\Big]$$

$$= \phi_1(1)\,\phi_1(2)\,\frac{1}{\sqrt{2}}\Big(|+\tfrac{1}{2},-\tfrac{1}{2}\rangle - |-\tfrac{1}{2},+\tfrac{1}{2}\rangle\Big) = \phi_1(1)\,\phi_1(2)\;|S=0,\, M_S=0\rangle$$

and has energy

$$E(1,2) = 2\epsilon - \gamma B\hbar(m_{s_1} + m_{s_2}) = 2\epsilon\,.$$

(2) Both electrons are in the orbital state $\phi_2$, one with $m_s = \frac{1}{2}$ and the other one with $m_s = -\frac{1}{2}$. The filled states are

$$|\phi_2,\, +\tfrac{1}{2}\rangle \;\text{ and }\; |\phi_2,\, -\tfrac{1}{2}\rangle$$

In this case the system's state and its energy are

$$\Psi(1,2) = \phi_2(1)\,\phi_2(2)\;|S=0,\, M_S=0\rangle\,, \qquad E = 2\epsilon\,.$$

(3) One electron is in the orbital state $\phi_1$ and the other one is in $\phi_2$. There are four ways to accomplish this:

(3.1) $|\phi_1,\, +\frac{1}{2}\rangle$ and $|\phi_2,\, +\frac{1}{2}\rangle$ (3.2) $|\phi_1,\, +\frac{1}{2}\rangle$ and $|\phi_2,\, -\frac{1}{2}\rangle$

(3.3) $|\phi_1,\, -\frac{1}{2}\rangle$ and $|\phi_2,\, +\frac{1}{2}\rangle$ (3.4) $|\phi_1,\, -\frac{1}{2}\rangle$ and $|\phi_2\; -\frac{1}{2}\rangle$.

For these fillings, the system's states and their energies are given by

$$(3.1)\quad \Psi(1,2) = \frac{1}{\sqrt{2}}\ \det\begin{pmatrix} |\phi_1,\, +\frac{1}{2}\rangle_1 & |\phi_1,\, +\frac{1}{2}\rangle_2 \\ |\phi_2,\, +\frac{1}{2}\rangle_1 & |\phi_2,\, +\frac{1}{2}\rangle_2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left[\phi_1(1)\,\phi_2(2) - \phi_2(1)\,\phi_1(2)\right]\ |+\tfrac{1}{2},\, +\tfrac{1}{2}\rangle$$

$$= \frac{1}{\sqrt{2}}\left[\phi_1(1)\,\phi_2(2) - \phi_2(1)\,\phi_1(2)\right]\ |S=1,\, M_S\!=1\rangle\,, \qquad E(1,2) = 2\epsilon - \gamma B\hbar\,,$$

$$(3.2)\quad \Psi(1,2) = \frac{1}{\sqrt{2}}\ \det\begin{pmatrix} |\phi_1,\, +\frac{1}{2}\rangle_1 & |\phi_1,\, +\frac{1}{2}\rangle_2 \\ |\phi_2,\, -\frac{1}{2}\rangle_1 & |\phi_2,\, -\frac{1}{2}\rangle_2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\Big[\phi_1(1)\,\phi_2(2)\ |+\tfrac{1}{2},\, -\tfrac{1}{2}\rangle - \phi_2(1)\,\phi_1(2)\ |-\tfrac{1}{2},\, +\tfrac{1}{2}\rangle\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[\,\phi_1(1)\,\phi_2(2)\,\frac{1}{\sqrt{2}}\Big(\,|S=1,\, M_S\!=1\rangle + |S=0,\, M_S\!=0\rangle\Big)$$

$$-\ \phi_2(1)\,\phi_1(2)\ \frac{1}{\sqrt{2}}\Big(\,|S=1,\, M_S\!=1\rangle - |S=0,\, M_S\!=0\rangle\Big)\Big]$$

$$= \frac{1}{2}\Big[\,\phi_1(1)\,\phi_2(2) - \phi_2(1)\,\phi_1(2)\Big]\ |S=1,\, M_S\!=1\rangle$$

$$+\ \frac{1}{2}\Big[\phi_1(1)\,\phi_2(2) + \phi_2(1)\,\phi_1(2)\Big]\ |S=0,\, M_S\!=0\rangle\,, \qquad E(1,2) = 2\epsilon\,.$$

$$(3.3)\quad \Psi(1,2) = \frac{1}{\sqrt{2}}\ \det\begin{pmatrix} |\phi_1,\, -\frac{1}{2}\rangle_1 & |\phi_1,\, -\frac{1}{2}\rangle_2 \\ |\phi_2,\, +\frac{1}{2}\rangle_1 & |\phi_2,\, +\frac{1}{2}\rangle_2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\Big[\phi_1(1)\,\phi_2(2)\ |-\tfrac{1}{2},\, +\tfrac{1}{2}\rangle - \phi_2(1)\,\phi_1(2)\ |+\tfrac{1}{2},\, -\tfrac{1}{2}\rangle\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[\,\phi_1(1)\,\phi_2(2)\,\frac{1}{\sqrt{2}}\Big(\,|S=1,\, M_S\!=1\rangle - |S=0,\, M_S\!=0\rangle\Big)$$

$$-\ \phi_2(1)\,\phi_1(2)\ \frac{1}{\sqrt{2}}\Big(\,|S=1,\, M_S\!=1\rangle + |S=0,\, M_S\!=0\rangle\Big)\Big]$$

$$= \frac{1}{2}\Big[\,\phi_1(1)\,\phi_2(2) - \phi_2(1)\,\phi_1(2)\Big]\ |S=1,\, M_S\!=1\rangle$$

$$-\ \frac{1}{2}\Big[\phi_1(1)\,\phi_2(2) + \phi_2(1)\,\phi_1(2)\Big]\ |S=0,\, M_S\!=0\rangle\,, \qquad E(1,2) = 2\epsilon\,.$$

$$(3.4)\quad \Psi(1,2) = \frac{1}{\sqrt{2}}\ \det\begin{pmatrix} |\phi_1,\, -\frac{1}{2}\rangle_1 & |\phi_1,\, -\frac{1}{2}\rangle_2 \\ |\phi_2,\, -\frac{1}{2}\rangle_1 & |\phi_2,\, -\frac{1}{2}\rangle_2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\Big[\,\phi_1(1)\,\phi_2(2) - \phi_2(1)\,\phi_1(2)\Big]\ |-\tfrac{1}{2},\, -\tfrac{1}{2}\rangle$$

$$= \frac{1}{\sqrt{2}}\Big[\,\phi_1(1)\,\phi_2(2) - \phi_2(1)\,\phi_1(2)\Big]\ |S=1,\, M_S\!=-1\rangle\,, \qquad E(1,2) = 2\epsilon - \gamma B\hbar\,.$$

b) In obvious notation, one has

$$\begin{aligned}
\Psi_{(1)}: &\quad S=0, \quad M_s=0 \\
\Psi_{(2)}: &\quad S=0, \quad M_s=0 \\
\Psi_{(3.1)}: &\quad S=0, \quad M_s=1 \\
\Psi_{(3.4)}: &\quad S=1, \quad M_s=-1 \\
\Psi_{(3.2)} \text{ and } \Psi_{(3.3)}: &\quad \text{do not have definite } S, \text{ nor } M_S.
\end{aligned}$$

**4 [2 puntos].** Un átomo de hidrógeno en un estado $|2\,1\,0\rangle$ interacciona con un campo eléctrico $\boldsymbol{\mathcal{E}}$ y un campo magnético $\mathbf{B}$ externos constantes. El eléctrico está dirigido según el eje $z$ y el magnético tiene dirección arbitraria, de forma que los hamiltonianos de interacción con los campos son en unidades internacionales

$$H_E = e\,\mathcal{E}z\,, \qquad H_B = \frac{e}{2m}\,\mathbf{B}{\cdot}\mathbf{L}\,.$$

Encontrar, sin realizar los cálculos, para qué valores de $\ell$ y $m$ los elementos de matriz $\langle 2\,\ell\,m|H_B|2\,1\,0\rangle$ y $\langle 2\,\ell\,m|H_E|2\,1\,0\rangle$ son distintos de cero.

Para $\langle 2\,\ell\,m|H_E|2\,1\,0\rangle$ se tiene que

$$\langle 2\,\ell\,m|H_E|2\,1\,0\rangle = e\,\mathcal{E}\,\langle 2\,\ell\,m|z|2\,1\,0\rangle\,.$$

El elemento de matriz $\langle 2\,\ell\,m|z|2\,1\,0\rangle$ es una integral definida sobre todo $\mathbf{R}^3$ (intervalo simétrico) de una función que bajo $\mathbf{x}\to-\mathbf{x}$ se comporta como $(-1)^{1+1+\ell}$, por lo que, para que $\langle 2\,\ell\,m|z|2\,1\,0\rangle$ sea distinto de cero, se necsita que $\ell$ sea par. Ahora bien, en $\langle 2\,\ell\,m|$, $\ell$ sólo puede tomar los valores 0 y 1, lo que descarta $\ell=1$, quedando $\ell=0$, que a su vez, implica que $m=0$. Así pues,

$$\langle 2\,\ell\,m|H_E|2\,1\,0\rangle = \delta_{\ell,0}\,\delta_{m,0}\,e\,\mathcal{E}\,\langle 2\,0\,0|z|2\,1\,0\rangle$$

Esto es consistente con la composición de $z$ ($\ell_1=1$, $m_1=0$) con $|210\rangle$ ($\ell_2=1$, $m_2=0$).

A su vez, para $\langle 2\,\ell\,m|H_B|2\,1\,0\rangle$, usando

$$\begin{aligned}
\mathbf{B}{\cdot}\mathbf{L} &= B_x\,L_x + B_y\,L_z + B_z\,L_z \\
&= \frac{1}{2}\,B_x\,(L_+ + L_-) + \frac{1}{2\mathrm{i}}\,B_y\,(L_+ - \mathrm{i}L_-) + B_z\,L_z \\
&= \frac{1}{2}\,(B_x - \mathrm{i}B_y)\,L_+ + \frac{1}{2}\,(B_x + \mathrm{i}B_y)\,L_- + B_z\,L_z =: \frac{1}{2}\,B_-L_+ + \frac{1}{2}\,B_+L_- + B_z\,L_z\,,
\end{aligned}$$

se tiene que

$$\langle 2\,\ell\,m|H_E|2\,1\,0\rangle = \frac{e}{2m}\left[\frac{B_-}{2}\,\langle 2\,\ell\,m|L_+|2\,1\,0\rangle + \frac{B_+}{2}\,\langle 2\,\ell\,m|L_-|2\,1\,0\rangle + B_z\,\langle 2\,\ell\,m|L_z|2\,1\,0\rangle\right]$$

De

$$L_\pm|\ell\,m\rangle = \hbar\sqrt{\ell(\ell+1) - m_\ell(m\pm1)}\;|\ell\,m\pm1\rangle\,, \qquad L_z|\ell\,m\rangle = \hbar m|\ell\,m\rangle\,,$$

se sique que

$$\begin{aligned}
\langle 2\,\ell\,m|H_E|2\,1\,0\rangle &= \frac{e}{2m}\left[\frac{B_-}{2}\,\sqrt{2}\,\hbar\,\langle 2\,\ell\,m|2\,1\,1\rangle + \frac{B_+}{2}\,\sqrt{2}\,\hbar\,\langle 2\,\ell\,m|2\,1\;-1\rangle + 0\right] \\
&= \frac{\hbar e}{2\sqrt{2}m}\,\delta_{\ell,1}\,\big(B_-\,\delta_{m,1} + B_+\,\delta_{m,-1}\big)\,.
\end{aligned}$$

Por tanto, $\langle 2\,\ell\,m|H_E|2\,1\,0\rangle$ es distinto de cero para $\ell=1$ y, o bien $m=1$, o bien $m=-1$, que nunca solapan con $\ell=0$ y $m=0$ para el caso de campo eléctrico.

**5 [2 puntos].** Un oscilador armónico unidimensional, con hamiltoniano

$$H_0 = \frac{P^2}{2m} + \frac{1}{2}\,m\omega x^2,$$

se encuentra inicialmente en el estado fundamental $|0\rangle$. En un tiempo $t = 0$ se activa una perturbación

$$V = \lambda\,\frac{P^4}{m^3c^2}\,,$$

donde $\lambda \ll 1$ es una constante sin dimensiones.

a) Calcular a primer orden en teoría de perturbaciones la probabilidad de transición a un estado $|n\rangle$ distinto del fundamental en un tiempo $t$.

b) Si $\lambda = 2mc^2/\hbar\omega$, estimar los tiempos $t$ para los que el resultado perturbativo anterior tiene sentido.

a) At first order in perturbation theory (Born approximation) the transition probability from the ground state to a state $|n\rangle$ in a time $T$ is give by

$$P_{0\to n}(0,t) = \frac{1}{\hbar^2}\left|\int_0^t dt' e^{\mathrm{i}\omega_{n0}t'}\,V_{n0}(t')\right|^2,$$

where

$$\omega_{n0} = \frac{E_n - E_0}{\hbar} = n\omega\,,\,, \quad V_{n0}(t) = \frac{\lambda}{m^3\,c^2}\,\langle n|P^4|0\rangle\,.$$

Since $V_{n0}(t)$ does not depend on time (let us postpone its calculation, for the time being), we perform the time integral in $P_{0\to n}(0,t)$,

$$P_{0\to n}(0,t) = \frac{\lambda^2}{m^6\,c^4\,\hbar^2}\left|\int_0^t dt' e^{\mathrm{i}n\omega t'}\right|^2\left|\langle n|P^4|0\rangle\right|^2 = \frac{\lambda^2}{m^6\,c^4\,\hbar^2}\,\frac{4\,\sin^2\left(\frac{n\omega t}{2}\right)}{n^2\omega^2}\,\left|\langle n|P^4|0\rangle\right|^2.$$

To compute $\langle n|P^4|0\rangle$, write

$$P = \frac{\mathrm{i}\hbar\alpha}{\sqrt{2}}\left(a^+ - a\right)$$

and use

$$\begin{aligned}
(a^+ - a)\,|0\rangle &= |1\rangle,\\
(a^+ - a)^2\,|0\rangle &= (a^+ - a)\,|1\rangle = \sqrt{2}\,|2\rangle - |0\rangle,\\
(a^+ - a)^3\,|0\rangle &= (a^+ - a)\left(\sqrt{2}\,|2\rangle - |0\rangle\right) = \sqrt{3{\cdot}2}\,|3\rangle - |1\rangle - 2\,|1\rangle\\
(a^+ - a)^4\,|0\rangle &= (a^+ - a)\left(\sqrt{6}\,|3\rangle - 3\,|1\rangle\right) = \sqrt{4{\cdot}6}\,|4\rangle - 3\sqrt{2}\,|2\rangle - \sqrt{6{\cdot}3}\,|2\rangle - 3\,|0\rangle,\\
\langle n|\left(a^+ - a\right)^4|0\rangle &= \langle n|\Big(2\sqrt{6}\,|4\rangle - 6\sqrt{2}\,|2\rangle - 3\,|0\rangle\Big) = 2\sqrt{6}\,\delta_{n4} - 6\sqrt{2}\,\delta_{n2} - 3\,\delta_{n0}.
\end{aligned}$$

This gives (for $|n\rangle \neq |0\rangle$)

$$\langle n|P^4|0\rangle = \frac{\hbar^4\alpha^4}{4}\,\langle n|(a^+ - a)^4|0\rangle = \frac{\hbar^4\alpha^4}{2}\left(\sqrt{6}\,\delta_{n,4} - 3\sqrt{2}\,\delta_{n,2}\right).$$

Putting everything together, we have

$$P_{0\to n}(0,t) = \frac{3}{2}\,\frac{\lambda^2\hbar^6\alpha^8}{m^6\,c^4\,\omega^2}\,\frac{1}{n^2}\left(\,\delta_{n,4} + 3\,\delta_{n,2}\,\right)\sin^2\left(\frac{n\omega t}{2}\right).$$

Using $\alpha^2 = m\omega/\hbar$,

$$P_{0\to n}(0,t) = \frac{3}{2}\left(\frac{\lambda\hbar\omega}{mc^2}\right)^2\frac{1}{n^2}\left(\,\delta_{n,4} + 3\,\delta_{n,2}\right)\sin^2\left(\frac{n\omega t}{2}\right).$$

There are only two possible transitions at this order,

$$|0\rangle \to |2\rangle:\;\; P_{0\to 2}(0,t) = \frac{9}{2}\left(\frac{\lambda\hbar\omega}{2mc^2}\right)^2\sin^2(\omega t),$$

$$|0\rangle \to |4\rangle:\;\; P_{0\to 4}(0,t) = \frac{3}{8}\left(\frac{\lambda\hbar\omega}{2mc^2}\right)^2\sin^2(2\omega t).$$

b) For $\lambda = 2mc^2/\hbar\omega$,

$$P_{0\to 2}(0,t) = \frac{9}{2}\,\sin^2(\omega t), \qquad P_{0\to 4}(0,t) = \frac{3}{8}\,\sin^2(2\omega t).$$

Since we are using perturbation theory, the probabilities should be much smaller than 1. Hence

$$P_{0\to 2}(0,t) \ll 1 \;\;\text{for}\;\; \omega t \ll \arcsin(\sqrt{2}/3)$$

$$P_{0\to 2}(0,t) \ll 1 \;\;\text{for}\;\; \omega t \ll \arcsin(\sqrt{3}/2\sqrt{2})\,.$$