6. Consider a quantum mechanical system with Hamiltonian

$$H = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Find the Hilbert space of the system.
(b) What are its stationary sates?
(c) Explain why the matrix A below can represent an observable for α real

$$A = \theta \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(d) The system is prepared at time $t = 0$ in a state

$$|\psi\rangle = \frac{1}{\sqrt{3}} \left(|e_-\rangle + \mathrm{i}\, |e_0\rangle - |e_+\rangle \right)$$

where $|e_-\rangle$, $|e_0\rangle$, $|e_+\rangle$ are the eigenstates of H. Find the possible results and their probabilities of a measurement of H.

(e) If instead of H, a measurement of A is performed, find the possible results and their probabilities.

(f) Is there any physical state on which both H and A can be measured with certainty?

(g) Find the probabilities $\mathrm{Prob}(h_i, a_j)$ of obtaining the values h_i and a_j if H and A are measured in this order.

(h) Find the probabilities $\mathrm{Prob}(a_i, h_j)$ of obtaining the values a_i and h_j if they are measured in the opposite order.

(i) A measurement of A at time $t = 0$ yields $\sqrt{2}\,\theta$. What is the state of the system immediately after the measurement?

(j) Find the expectation value of A at time t after the measurement in (i).

(k) If a second measurement of A is performed at t, what is the probability of obtaining $\sqrt{2}\,\theta$?

(a) $\mathcal{H} = \mathbb{C}^3$

(b) $|\psi(t)\rangle = e^{-iEt/\hbar} |e\rangle$, with E and $|e\rangle$ solutions of $H|e\rangle = E|e\rangle$

$E_+ = \hbar\omega$, $|e_+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $E_0 = 0$, $|e_0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $E_- = -\hbar\omega$, $|e_-\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(c) Because A is self-adjoint

(d) The only possible results the eigenvalues of H, with probabilities

$$\text{Prob}(H, -\hbar\omega) = |\langle e_- | \psi \rangle|^2 = \frac{1}{3}$$

$$\text{Prob}(H, 0) = |\langle e_0 | \psi \rangle|^2 = \frac{1}{3}$$

$$\text{Prob}(H, \hbar\omega) = |\langle e_+ | \psi \rangle|^2 = \frac{1}{3}$$

(e) The only possible results are the eigenvalues of A. Let us find them:

$$\det(A-\lambda) = 0 \iff \det\begin{pmatrix} -\lambda & \theta & 0 \\ \theta & -\lambda & \theta \\ 0 & \theta & -\lambda \end{pmatrix} = 0 \iff -\lambda^3 + 2\theta^2\lambda = 0$$

$$\iff \lambda(2\theta^2 - \lambda^2) = 0 \implies \lambda = -\sqrt{2}\theta, \ 0, \ \sqrt{2}\theta$$

To find the probabilities of obtaining these result we need the eigenvectors.

$$a_- = -\sqrt{2}\theta \longrightarrow \begin{pmatrix} \sqrt{2}\theta & \theta & 0 \\ \theta & \sqrt{2}\theta & \theta \\ 0 & \theta & \sqrt{2}\theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \implies \left.\begin{aligned} \sqrt{2}v_1 + v_2 = 0 \\ v_1 + \sqrt{2}v_2 + v_3 = 0 \\ v_2 + \sqrt{2}v_3 = 0 \end{aligned}\right\} \implies \begin{aligned} v_2 &= -\sqrt{2}v_1 \\ v_3 &= v_1 \end{aligned}$$

$$\longrightarrow \begin{pmatrix} v_1 \\ -\sqrt{2}v_1 \\ v_1 \end{pmatrix} \xrightarrow[\text{Normalization}]{} |a_-\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$a_0 = 0 \longrightarrow \text{Similar calculations} \longrightarrow |a_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$a_+ = \sqrt{2}\theta \longrightarrow |a_+\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

The probabilities of obtaining each one of the eigenvalues when measuring A on $|\psi\rangle$ are then

$$\text{Prob}(A, -\sqrt{2}\theta) = |\langle a_-|\psi\rangle|^2 = \left| \frac{1}{2}(1, -\sqrt{2}, 1) \underbrace{\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\ i\\ 1\end{pmatrix}}\right|^2 = \left|\frac{1}{2\sqrt{3}}(-\sqrt{2}i)\right|^2 = \frac{1}{6}$$

Note that $|\psi\rangle = \frac{1}{\sqrt{3}}(|e_-\rangle + i|e_0\rangle - |e_+\rangle) = \frac{1}{\sqrt{3}}\left[\begin{pmatrix}0\\0\\1\end{pmatrix} + i\begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}\right] = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\ i\\ 1\end{pmatrix}$

For the other two eigenvalues we have

$$\text{Prob}(A, 0) = |\langle a_0|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1, 0, -1)\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\ i\\ 1\end{pmatrix}\right|^2 = \frac{2}{3}$$

$$\text{Prob}(A, \sqrt{2}\theta) = |\langle a_+|\psi\rangle|^2 = \left|\frac{1}{2}(1, \sqrt{2}, 1)\frac{1}{\sqrt{3}}\begin{pmatrix}-1\\ i\\ 1\end{pmatrix}\right|^2 = \frac{1}{6}$$

The three probabilities sum 1.

(f) No, since H and A do not commute

$$[H, A] = \hbar\omega\theta\left[\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} - \begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\right]$$

$$= \hbar\omega\theta\begin{pmatrix}0&1&0\\-1&0&1\\0&-1&0\end{pmatrix}$$

(g) state $|\psi\rangle \xrightarrow{H} E_i$, state $|e_i\rangle \xrightarrow{A} a_j$, state $|a_j\rangle \Rightarrow$

$\text{Prob}(E_i) = |\langle e_i|\psi\rangle|^2$ $\qquad$ $\text{Prob}(a_j) = |\langle a_j|e_i\rangle|^2$

$\Rightarrow \text{Prob}(E_i, a_j) = |\langle e_i|\psi\rangle|^2 \, |\langle a_j|e_i\rangle|^2$ (No sum over i)

We need to compute $|\langle a_j|e_i\rangle|^2$:

$$|\langle a_-|e_-\rangle|^2 = \left|\frac{1}{2}(1, -\sqrt{2}, 1)\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \frac{1}{4}$$

$$|\langle a_-|e_0\rangle|^2 = \left|\frac{1}{2}(1, -\sqrt{2}, 1)\begin{pmatrix}0\\1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$$

$$|\langle a_-|e_+\rangle|^2 = \left|\frac{1}{2}(1,-\sqrt{2},1)\begin{pmatrix}1\\0\\0\end{pmatrix}\right|^2 = \frac{1}{4}$$

$$|\langle a_0|e_-\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1,0,-1)\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \frac{1}{2}$$

$$|\langle a_0|e_0\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1,0,-1)\begin{pmatrix}0\\1\\0\end{pmatrix}\right|^2 = 0$$

$$|\langle a_0|e_+\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1,0,-1)\begin{pmatrix}1\\0\\0\end{pmatrix}\right|^2 = \frac{1}{2}$$

$$|\langle a_+|e_-\rangle|^2 = \left|\frac{1}{2}(1,\sqrt{2},1)\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2 = \frac{1}{4}$$

$$|\langle a_+|e_0\rangle|^2 = \left|\frac{1}{2}(1,\sqrt{2},1)\begin{pmatrix}0\\1\\0\end{pmatrix}\right|^2 = \frac{1}{2}$$

$$|\langle a_+|e_+\rangle|^2 = \left|\frac{1}{2}(1,\sqrt{2},1)\begin{pmatrix}1\\0\\0\end{pmatrix}\right|^2 = \frac{1}{4}$$

This gives

$$P(-\hbar\omega,-\sqrt{2}\theta) = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$$

$$P(-\hbar\omega,0) = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

$$P(-\hbar\omega,\sqrt{2}\theta) = \frac{1}{3}\,\frac{1}{4} = \frac{1}{12}$$

$$P(0,-\sqrt{2}\theta) = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

$$P(0,0) = \frac{1}{3}\cdot 0 = 0$$

$$P(0,\sqrt{2}\theta) = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

$$P(\hbar\omega,-\sqrt{2}\theta) = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$$

$$P(\hbar\omega,0) = \frac{1}{3}\,\frac{1}{2} = \frac{1}{6}$$

$$P(\hbar\omega,\sqrt{2}\theta) = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$$

$\longrightarrow$ They sum 1.

(h) The same arguments as for the previous case give

$$\text{Prob}(a_i, E_j) = |\langle a_i|\psi\rangle|^2 \, |\langle E_j|a_i\rangle|^2$$

$$\text{Prob}(a_-, e_-) = \frac{1}{6}\cdot\frac{1}{4} = \frac{1}{24}$$

$$\text{Prob}(a_-, e_0) = \frac{1}{6}\cdot\frac{1}{2} = \frac{1}{12}$$

$$\text{Prob}(a_-, e_+) = \frac{1}{6}\cdot\frac{1}{4} = \frac{1}{24}$$

$$\text{Prob}(a_0, e_-) = \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3}$$

$$\text{Prob}(a_0, e_0) = \frac{2}{3}\cdot 0 = 0$$

$$\text{Prob}(a_0, e_+) = \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3}$$

$$\text{Prob}(a_+, e_-) = \frac{1}{6}\cdot\frac{1}{4} = \frac{1}{24}$$

$$\text{Prob}(a_+, e_0) = \frac{1}{6}\cdot\frac{1}{2} = \frac{1}{12}$$

$$\text{Prob}(a_+, e_+) = \frac{1}{6}\cdot\frac{1}{4} = \frac{1}{24}$$

$\longrightarrow$ They sum 1

The order in which the measurements are made matters:

$$\left.\begin{array}{l} \text{Prob}(E_i, a_j) = |\langle e_i|\psi\rangle|^2 |\langle a_j|e_i\rangle|^2 \\ \text{Prob}(a_j, E_i) = |\langle a_j|\psi\rangle|^2 \, |\langle e_i|a_j\rangle|^2 \end{array}\right\} \text{ different}$$

(i) State after measure is $|a_+\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

j) $|a_+\rangle$ can be written as

$$|a_+\rangle = \frac{1}{2}\left(|e_+\rangle + \sqrt{2}\,|e_0\rangle + |e_-\rangle\right)$$

It evolves in times with the Hamiltonian, so at time t we have

$$|a_+(t)\rangle = \frac{1}{2}\left(e^{-i\omega t}|e_+\rangle + \sqrt{2}|e_0\rangle + e^{i\omega t}|e_-\rangle\right) = \frac{1}{2}\begin{pmatrix} e^{-i\omega t} \\ \sqrt{2} \\ e^{i\omega t} \end{pmatrix}$$

One then has

$$\langle a_+(t)|A|a_+(t)\rangle = \frac{1}{2}\left(e^{+i\omega t}, \sqrt{2}, e^{-i\omega t}\right)\theta\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} e^{-i\omega t} \\ \sqrt{2} \\ e^{i\omega t} \end{pmatrix}$$

$$= \frac{\theta}{4}\left(e^{i\omega t}, \sqrt{2}, e^{-i\omega t}\right)\begin{pmatrix} \sqrt{2} \\ e^{-i\omega t} + e^{i\omega t} \\ \sqrt{2} \end{pmatrix}$$

$$= \frac{\theta}{4}\left(\sqrt{2}e^{i\omega t} + \sqrt{2}(e^{-i\omega t} + e^{i\omega t}) + \sqrt{2}e^{-i\omega t}\right)$$

$$= \sqrt{2}\theta\cos\omega t$$

(k) $\mathrm{Prob}(A, \sqrt{2}\theta)_{|a_+(t)\rangle} = |\langle a_+|a_+(t)\rangle|^2$

$$= \left|\frac{1}{2}(1, \sqrt{2}, 1)\,\frac{1}{2}\begin{pmatrix} e^{-i\omega t} \\ \sqrt{2} \\ e^{i\omega t} \end{pmatrix}\right|^2$$

$$= \left|\frac{1}{4}\left(e^{-i\omega t} + 2 + e^{i\omega t}\right)\right|^2$$

$$= \left|\frac{1}{2}(1 + \cos\omega t)\right|^2 = \cos^4\frac{\omega t}{2}$$