Quantum Physics II
Problem sheets

Mathematical formulation of QM

1.1. [Solution can be found in the discussion of Postulate V in any QM textbook]. Prove using Scrödinger's equation that the norm of a physical state $|\psi(t)\rangle$ does not change in time,

$$\frac{d}{dt}\,\langle\psi(t)|\psi(t)\rangle = 0\,.$$

That is, the Schödinger equation is consistent with conservation of probability.

1.2. [For a solution see e. g. Griffiths, section 3.4.1]. Consider a system in a quantum state $|\psi\rangle$ and two observables A and B, represented by operators A and B. Show that

$$\Delta_\psi A\,\Delta_\psi B \;\geq\; \frac{1}{2}\,\Big|\langle\psi|\,[A,B]\,|\psi\rangle\Big|\,.$$

1.3. Show that for U a unitary operator, i. e. such that $U^+U = UU^+ = 1$, and A self-adjoint, the operator UAU^+ is also self-adjoint and has the same eigenvalues (spectrum) as A.

1.4. Show that

$$\big(\,|\psi\rangle\langle\phi|\,\big)^+ = |\phi\rangle\langle\psi|$$

1.5. [Pauli matrices]. On the Hilbert space $\mathcal{H} = \mathbb{C}^2$, define the operators

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that they are self-adjoint. Find their eigenvalues and eigenstates. Compute the commutators

$$[\sigma_x,\sigma_y], \quad [\sigma_y,\sigma_z], \quad [\sigma_z,\sigma_x]$$

and the operator $\sigma_x^2+\sigma_y^2+\sigma_z^2$. These matrices play an extremely important rôle in QM and are known as Pauli matrices. They are also written as $\boldsymbol{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$.

1.6. Consider a quantum mechanical system with Hamiltonian

$$H = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Find the Hilbert space of the system.
(b) What are its stationary sates?
(c) Explain why the matrix A below can represent an observable for θ real

$$A = \theta \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

(d) The system is prepared at time $t = 0$ in a state

$$|\psi\rangle = \frac{1}{\sqrt{3}} \left(|e_-\rangle + \mathrm{i}\, |e_0\rangle - |e_+\rangle \right)$$

where $|e_-\rangle$, $|e_0\rangle$, $|e_+\rangle$ are the eigenstates of H with eigenvalues $-\hbar\omega$, 0, $\hbar\omega$. Find the possible results and their probabilities for a measurement of H.
(e) If instead of H, a measurement of A is performed, find the possible results and their probabilities.
(f) Is there any physical state on which both H and A can be measured with certainty?
(g) Find the probabilities $\mathrm{Prob}(E_i, a_j)$ of obtaining the values E_i and a_j if H and A are measured in this order.
(h) Find the probabilities $\mathrm{Prob}(a_i, E_j)$ of obtaining the values a_i and E_j if they are measured in the opposite order.
(i) A measurement of A at time $t = 0$ gives $\sqrt{2}\,\theta$. What is the state of the system immediately after the measurement?
(j) Find the expectation value of A at time t after the measurement in (i) is made.
(k) If a second measurement of A is performed at t, what is the probability of obtaining $\sqrt{2}\,\theta$?

1.7. The Hamiltonian of a quantum system with Hilbert space $\mathcal{H} = \mathbb{C}^2$ is

$$H = \epsilon \sigma_z \,, \qquad \epsilon > 0 \,,$$

where $\epsilon > 0$ is a constant with dimensions of energy. The system is initially prepared in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Find the state $|\psi(t)\rangle$ at time t. The observable σ_x is repeatedly measured at times $t = \Delta t,\, 2\Delta t, \ldots, N\Delta t$. What is the probability $P_N(\Delta t)$ of obtaining $+1$ in all performed measurements. Study the limit $\Delta t \to 0$ and $N \to \infty$ of $P_N(\Delta t)$, while keeping the total observation time $T := N\Delta t$ finite.

1.8. The Hamiltonian of a quantum system is

$$H = \epsilon \sigma_z \,, \qquad \epsilon > 0 \,,$$

where σ_z is the z-Pauli matrix. Call $|g\rangle$ to the ground state and $|e\rangle$ to the first excited state. The system is initially prepared in a state

$$|\psi(0)\rangle = \frac{1}{\sqrt{5}} \left(\mathrm{i}\, |g\rangle - 2\, |e\rangle \right) .$$

At time $t = t_c$ the Hamiltonian is **suddenly** modified from H to

$$H' = \epsilon\,(\sigma_z + \sigma_x)\,.$$

This means that the change in the Hamiltonian is so fast that the physical state has no time to adjust, so the state at $t = t_c^+$ is the same as at $t = t_c^-$, but for $t > t_c$ time evolution is governed by H'. Find $|\psi(t)\rangle$ and the expectation value of the energy at $t < t_c$ and $t > t_c$. The results should show that the expectation value at $t > t_c$ is different from that at $t < t_c$. Where does the extra energy come from?

1.9. A quantum system is formed by two qubits, so its Hilbert space is the tensor product $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. The Hamiltonian of the system is

$$H = \epsilon\left(\sigma_z \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_z\right) + J\left(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y\right)\,,$$

where ϵ and J are constants, both with dimensions of energy, and $\sigma_{x,y,z}$ denote the Pauli matrices. Find the stationary states in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ of eigenstates of $\sigma_z \otimes \sigma_z$.

1.10. A quantum system formed by two qubits, Hilbert space given by $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, is in a state

$$|\psi_{12}\rangle = \frac{1}{\sqrt{2}}\left(\,|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle\right)\,.$$

Show that the expectation value of any observable A of qubit 1, represented by an operator $A^{(1)} = A \otimes 1$, can be recast as

$$\langle A^{(1)}\rangle = \mathrm{Tr}[\rho^{(1)} A]\,,$$

where the matrix $\rho^{(1)}$ is given by

$$\rho^{(1)} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\,.$$

Compare $\langle A^{(1)}\rangle$ with the expectation value

$$\langle A\rangle_{\mathrm{sup}} = \langle \psi_{\mathrm{sup}}|A|\psi_{\mathrm{sup}}\rangle,$$

of A in a superposition state

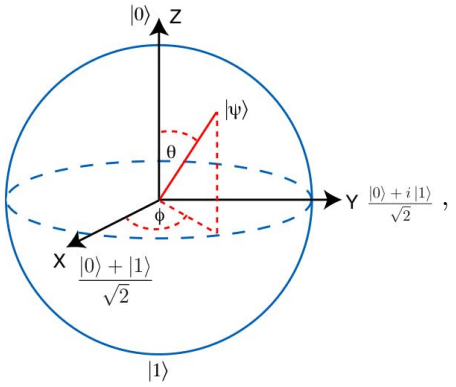
$$|\psi_{\mathrm{sup}}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\,.$$

Find the observables for which both expectation values agree.

1.11. Exercise 1.10 reloaded. Write $|\psi_{12}\rangle$ as a density matrix. Is it a pure state? Compute its two reduced densities matrices.

2.1. Any spin-$\frac{1}{2}$ state can be written as

$$|\psi\rangle = \begin{pmatrix} \cos\dfrac{\theta}{2} \\ e^{\mathrm{i}\phi}\sin\dfrac{\theta}{2} \end{pmatrix}, \qquad (2.1)$$



where $\{\theta, \phi\}$ are coordinates on the two-sphere suitably chosen. The state $|\psi\rangle$ can be rotated by moving the point (θ, ϕ) that represents it over the sphere. For example, a rotation $\mathcal{R}\,(\mathbf{e}_3,\gamma)$ around the z-axis by an angle γ gives, modulo a phase, the state

$$\mathcal{R}(\mathbf{e}_3,\gamma)\,|\psi\rangle = \begin{pmatrix} \cos\dfrac{\theta}{2} \\ e^{\mathrm{i}(\phi+\gamma)}\sin\dfrac{\theta}{2} \end{pmatrix}.$$

(a) Show that the rotation operator $\mathcal{R}\,(\mathbf{e}_3,\gamma)$ can be written as

$$\mathcal{R}(\mathbf{e}_3,\gamma) = e^{-\frac{\mathrm{i}}{\hbar}S_z\gamma}$$

and that it is unitary.

(b) Rotations can be used to align a spin state with an axis. For example, to align $|\psi\rangle$ with the x-axis, one may first rotate it by an angle $\gamma = -\phi$ around the z-axis, and then rotate the resulting state by an angle $\beta = \frac{\pi}{2} - \theta$ around to the y-axis. The first rotation gives

$$|\psi'\rangle = \mathcal{R}(\mathbf{e}_3, -\phi)|\psi\rangle = \begin{pmatrix} \cos\dfrac{\theta}{2} \\ \sin\dfrac{\theta}{2} \end{pmatrix},$$

whilst the second one leads to

$$|\psi''\rangle = \mathcal{R}(\mathbf{e}_2,\beta)\,|\psi'\rangle = \mathcal{R}(\mathbf{e}_2,\beta)\begin{pmatrix} \cos\dfrac{\theta}{2} \\ \sin\dfrac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos\dfrac{\theta+\beta}{2} \\ \sin\dfrac{\theta+\beta}{2} \end{pmatrix} = \left\{ \text{for } \beta = \frac{\pi}{2} - \theta \right\} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Prove that the rotation operator $\mathcal{R}(\mathbf{e}_2,\beta)$ is

$$\mathcal{R}(\mathbf{e}_2,\beta) = e^{-\frac{\mathrm{i}}{\hbar}S_y\beta}.$$

In general, the rotation around an axis $\hat{\mathbf{u}}$ by an angle α is given by

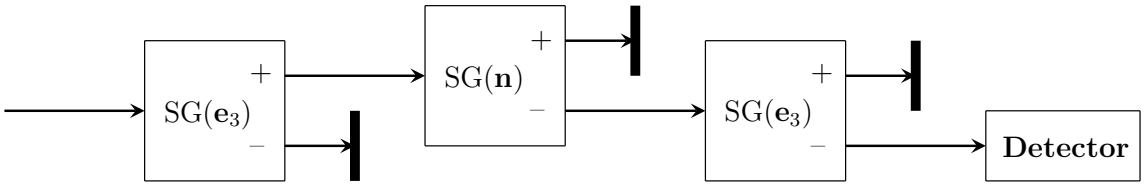
$$\mathcal{R}(\mathbf{u},\alpha) = e^{-\frac{\mathrm{i}}{\hbar}\,\mathbf{S}\cdot\mathbf{u}\,\alpha}\,.$$

(c) Compute the expectation value of $\mathbf{S} = (S_x, S_y, S_z)$ in the state $|\psi\rangle$ in terms of the components of the unit vector $\hat{\mathbf{n}} = (n_1, n_2, n_3)$ representing $|\psi\rangle$ on the sphere.

2.2. Show that the Pauli matrices $(\sigma_1, \sigma_2, \sigma_3)$ satisfy the properties

$$\begin{aligned}
&\text{(i)}\ \ \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{1}\,,\\
&\text{(2i)}\ \ \sigma_i\sigma_j = \delta_{ij}\mathbf{1} + \mathrm{i}\epsilon_{ijk}\,\sigma_k\,,\\
&\text{(3i)}\ \ e^{\mathrm{i}\theta(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})} = \cos\theta\,\mathbf{1} + \mathrm{i}\sin\theta\,(\hat{\mathbf{n}}\cdot\boldsymbol{\sigma})\,.
\end{aligned}$$

2.3. A beam of silver atoms (regarded as particles with spin 1/2) produced in an oven goes through a Stern-Gerlach device oriented in the z-axis. Of the two outcoming beams, that with positive orientation is filtered and sent to a second SG device oriented in the direction of the unit vector $\mathbf{n} = (\sin\theta, 0, \cos\theta)$. On this occasion, the outcoming beam with negative orientation is filtered and sent to a third SG device oriented in the z-axis, as schematically depicted in the figure. Calculate the probability that an atom that leaves the oven comes out of the last SG device with negative orientation.



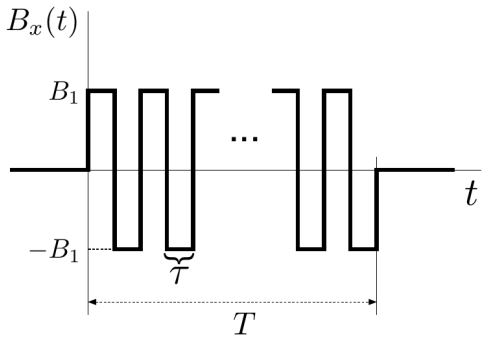
Hint. Recall that positive and negative orientations are always with respect to the axis of the SG device. Because of symmetry reasons, half of the spins that exit the first device are positively oriented and half are negatively oriented.

2.4. A spin-$\frac{1}{2}$ particle is placed in a constant magnetic field $\mathbf{B}_3 = (0, 0, B_3)$, so that it gets oriented along the positive z-axis. Once the alignment has been achieved, and during a time T, a second magnetic field $\mathbf{B}_1 = (B_1, 0, 0)$ is introduced. Take the time t_0 at which the field $\mathbf{B}_1$ is turned on as the initial time, $t_0 = 0$. During the time interval that goes from 0 to τ, the spin will precess about the axis defined by the field $\mathbf{B}_3 + \mathbf{B}_1$.

(a) What is the angular frequency of this precession?

(b) Determine the time T at which the expectation value of the x-component S_x of spin attains its maximum.

2.5. A $\frac{1}{2}$-spin with gyromagnetic ratio γ is in a magnetic field $\mathbf{B} = (B_x(t), 0, B_3)$, with constant z-component B_3, and x-component $B_x(t)$ given by the square wave depicted in the figure below.



Note that $B_x(t)$ vanishes for $t < 0$ and $t > T$, and that for $0 \leq t \leq T$ it takes constant values B_1 and $-B_1$, each one for a time interval of duration τ. Assume that the spin is initially oriented in the positive z-axis, that is, $|\psi(0)\rangle = |z_+\rangle$.

(a) Calculate the axis around which the spin precesses and its precession frequency ω.

(b) For $\tau = \pi/\omega$, determine the expectation value of the magnetic moment at time τ.

(c) Also for $\tau = \pi/\omega$, find the expectation value of the magnetic moment after N oscillations, i. e. at time $t = 2N\tau$. Calculate the total duration T of the pulse for which the angle formed by the magnetic moment expectation value at T and the z-axis is maximal. Particularize the result for $B_1 \ll B_3$.

2.6. [Griffiths, 4.34] An electron is at rest in an oscillating magnetic field

$$\mathbf{B} = B\,\cos(\omega t)\,\mathbf{e}_3\,,$$

where B and ω are constants.

(a) Write down the Hamiltonian for this system and the corresponding Schrödinger equation.

(b) The electron is initially in a spin-up state with respect to the x-axis, $|\psi(0)\rangle = |x_+\rangle$. Determine $|\psi(t)\rangle$ at later times.

(c) Find the probability of getting $-\hbar/2$ if S_x is measured a time t.

(d) What is the minimum field B required to force a complete flip in S_x?

Tutorial - Landau levels, a primer

In the lectures the coupling of the orbital magnetic moment of an electron in an Hydrogen atom to an external constant magnetic field (normal Zeeman effect) was discussed. In this tutorial, the case of free electrons is considered.

A free electron in a constant magnetic field $\mathbf{B} = B\hat{\mathbf{e}}_3$ oriented along the x_3-axi has classical Hamiltonian

$$H = \frac{1}{2}\left(\mathbf{p} + \frac{e}{c}\,\mathbf{A}\right)^2, \qquad \mathbf{B} = \boldsymbol{\nabla} \wedge A\,.$$

The convention here is that the electron charge is $-e$, with $e > 0$. Show that the components of the kinetic momentum

$$\boldsymbol{\pi} = \mathbf{p} + \frac{e}{c}\,\mathbf{A}$$

can be written in the Coulomb gauge $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ as

$$\pi_1 = p_1 - \frac{eB}{2c}\,x_2\,, \qquad \pi_2 = p_2 + \frac{eB}{2c}\,x_1\,, \qquad \pi_3 = p_3\,.$$

Upon quantization, $\boldsymbol{\pi}$ becomes an operator that we denote by $\boldsymbol{\Pi}$. Prove that its components satisfy the commutation relations

$$[\Pi_1, \Pi_2] = -\frac{eB}{c}\,\mathrm{i}\hbar\,, \qquad [\Pi_1, \Pi_1] = [\Pi_2, \Pi_2] = 0\,, \qquad [\Pi_1, \Pi_3] = [\Pi_2, \Pi_3] = 0\,.$$

Instead of Π_1 and Π_2, consider for convenience

$$\mathcal{P} = \Pi_1\,, \qquad \mathcal{X} = \frac{c}{eB}\,\Pi_2\,,$$

so that $\mathcal{X}$ and $\mathcal{P}$ have dimensions of position and momentum and have commutators

$$[\mathcal{X}, \mathcal{P}] = \mathrm{i}\hbar\,, \qquad [\mathcal{X}, \mathcal{X}] = [\mathcal{P}, \mathcal{P}] = 0\,.$$

The Hamiltonian of the system can then be written as

$$H = H_\perp + H_\parallel\,, \qquad H_\perp = \frac{1}{2m}\,\mathcal{P}^2 + \frac{1}{2}\,m\omega^2\mathcal{X}^2\,, \qquad H_\parallel = \frac{P_3^2}{2m}\,,$$

where ω is the angular frequency

$$\omega = \frac{eB}{mc}\,.$$

In $H_\perp$ one recognizes the Hamiltonian of a harmonic oscillator with annihilation and creation operators

$$a = \frac{1}{\sqrt{2}}\left(\alpha\mathcal{X} - \frac{1}{\mathrm{i}\hbar\alpha}\,\mathcal{P}\right), \qquad a^+ = \frac{1}{\sqrt{2}}\left(\alpha\mathcal{X} - +\frac{1}{\mathrm{i}\hbar\alpha}\,\mathcal{P}\right), \qquad \alpha = \sqrt{\frac{m\omega}{\hbar}}\,, \qquad [a, a^+] = 1\,.$$

$H_\parallel$ is in turn the Hamiltonian of a free particle in the x_3-direction. To solve the eigenvalue problem of H,

$$H\psi(x_1, x_2, x_3) = E\psi(x_1, x_2, x_3)\,,$$

use separation of variables and write

$$\psi(x_1, x_2, x_3) = \phi_\perp(x_1, x_2)\,\phi_\parallel(x_3)\,, \qquad E = E_\parallel + E_\perp\,.$$

This gives

$$E_{\parallel} = \frac{p_3^2}{2m}\,, \quad \phi_{\parallel}(x_3) = e^{\mathrm{i}p_3 x_3/\hbar}, \quad p_3 \text{ arbitrary}\,,$$

and $(E_\perp, \phi_\perp)$ the solutions of

$$H_\perp \phi(x_1, x_2,) = E_\perp \phi(x_1, x_2)\,.$$

The energies $E_\perp$ are those of the one-dimensional harmonic oscillator,

$$E_{\perp,n} = \left(n + \frac{1}{2}\right)\hbar\omega\,, \qquad n = 0, 1, \ldots$$

The study of their degeneracy and their eigenfunctions $\phi_\perp$, and of the physics behind them, is more involved. Let us nevertheless say a few words on the subject.

The same arguments as for the conventional harmonic oscillator show that the ground state $\varphi := \phi_{\perp,0}$ satisfies an equation

$$a\varphi = 0\,. \tag{3.2}$$

Show that, in terms of $X_1 = x_1$, $X_2 = x_2$, $P_1 = -\mathrm{i}\hbar\partial_1$ and $P_2 = -\mathrm{i}\hbar\partial_2$, the annihilation operator reads

$$a = \frac{1}{2}\left[\sqrt{\frac{2\hbar c}{eB}}\,(\,\partial_1 - \mathrm{i}\partial_2) + \sqrt{\frac{eB}{2\hbar c}}\,(x_1 - \mathrm{i}x_2)\right].$$

Introduce now dimensionless complex coordinates z and $\bar{z}$, given by

$$z = \sqrt{\frac{eB}{2\hbar c}}\,(x_1 - \mathrm{i}x_2)\,, \qquad \bar{z} = \sqrt{\frac{eB}{2\hbar c}}\,(x_1 + \mathrm{i}x_2)\,,$$

and prove that the operator a becomes

$$a = \frac{\partial}{\partial\bar{z}} + \frac{z}{2}\,.$$

Eq. (3.2) then takes the form

$$\left(\frac{\partial}{\partial\bar{z}} + \frac{z}{2}\right)\varphi(z, \bar{z}) = 0\,. \tag{3.3}$$

To solve the latter make the ansatz

$$\varphi(z, \bar{z}) = \varphi(z\bar{z})\,.$$

Show, upon substitution in eq. (3.3), that the function φ must satisfy

$$\varphi' + \frac{1}{2}\varphi = 0\,,$$

where the prime in φ' denotes differentiation with respect to its argument. Its solution is

$$\varphi = c\,e^{-z\bar{z}/2}\,,$$

with c an arbitrary integration constant. The key observation now is that one may replace c with an arbitrary function $c(z)$ of z and still have a solution. Hence

$$\varphi(z, \bar{z}) = c(z)\,e^{-z\bar{z}/2}\,. \tag{3.4}$$

A different derivation of this solution goes as follows. To ensure square integrability on the z-plane, one makes the ansatz

$$\varphi = c(z, \bar{z})\,e^{-z\bar{z}/2}\,,$$

Substitution in eq. (3.3) then gives

$$\frac{\partial c}{\partial \bar{z}} = 0\,,$$

which implies solutions of the form (3.4).

Since $c(z)$ is arbitrary, there are infinitely many ground state wave functions. It is customary to choose for them monomials z^n, thus having

$$\varphi_n(z,\bar{z}) = N_n\, z^n\, e^{-z\bar{z}/4}\,, \tag{3.5}$$

with N_n a normalization constant. To determine N_n, we proceed as usual and require

$$1 = \int d^2x\, |\varphi_n|^2 = |N_n|^2\, \frac{eB}{2\hbar c}\, 2\pi \int_0^\infty dr\, r^{2n+1}\, e^{-eBr^2/4\hbar c} = |N_n|^2\, \pi \left(\frac{eB}{2\hbar c}\right)^{n+1} n!$$

Choosing N_n real and positive, we have

$$N_n = \frac{1}{\sqrt{\pi n!}} \left(\frac{2\hbar c}{eB}\right)^{(n+1)/2}$$

The probability of finding the electron at a distance r from the origin is

$$P(r) = 2\pi r |\varphi_n|^2 = \frac{2}{n!} \left(\frac{2\hbar c}{eB}\right)^n r^{2n+1}\, e^{-eBr^2/4\hbar c}\,,$$

which is a smeared ring around the origin.

Coming back to the choice of $c(z)$, note that there are certain restrictions. Functions that grow at $|z| \to \infty$ as $e^{|z|^2}$ or faster are not allowed. Functions that involve rational powers of z are multivalued, hence not permitted. Negative powers of z spoil square integrability, thus are also excluded. One is basically left with polynomials of arbitrary orders, which correspond to the choice made above.

Excited states are obtained by recurrently acting with the creation operator a^+ on the ground states.

Addition of angular momenta

4.1. [Griffiths 4.37] A particle of spin 1 and a particle of spin 2 are at rest in a configuration whose total spin is 3 and its z-component is 1, that is, it is an eigenstate of $\mathbf{S}^2$ and S_z with eigenvalues $3(3+1)\hbar^2$ and $\hbar$. If the z-component of the angular momentum of the spin-2 particle is measured, what values can one get and what is the probability of each one?

An electron with spin down is in the state ψ_{510} of an hydrogen atom. Assuming that the total angular momentum squared of the electron alone (i, e. excluding the proton spin) can be measured, what values might one get and what is the probability of each one?

4.2. In the composition of two angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$ calculate the commutator $[\mathbf{J}^2, J_{1i}]$, with $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$ the total angular momentum.

4.3. Two spin-$\frac{1}{2}$ particles at rest interact with each other through the Hamiltonian

$$H = K\,\frac{(\mathbf{S_1}\cdot\mathbf{S_2})\,\mathbf{x}^2 - (\mathbf{S_1}\cdot\mathbf{x})\,(\mathbf{S_2}\cdot\mathbf{x})}{|\mathbf{x}|^5}\,,$$

where $\mathbf{x}$ is the position of particle 2 relative to particle 1, and K is a constant with suitable units. If initially one spin is parallel to $\mathbf{x}$ and the other one is antiparallel, find the time after which the parallel spin is antiparallel and the antiparallel spin is parallel.

Identical particles

5.1. In a system of N identical particles consider an observable B. Show that if B is invariant under permutations, and $|\psi_S\rangle$ is a completely symmetric state, so is $B|\psi_S\rangle$. Show that the same holds true for completely antisymmetric states $|\psi_A\rangle$.

5.2. In a system of non-interacting particles, the one-particle energies are nondegenerate, do not depend on spin and are given by $E_n = n^2\epsilon$, with $n = 1, 2, \ldots$ Find the ground state, its energy and its degeneracy in each one of the following cases:

(1) Three identical spin zero particles.

(2) Three identical spin-$\frac{1}{2}$ particles.

(3) Three distinguishable particles of arbitrary spin.

5.3. A system is formed by two electrons in a central potential. Assuming that the interaction between them is negligible, find their wave function if each one of them is in an orbital $2p$ with orbital angular momentum third component $\hbar$. Calculate the total orbital angular momentum, the total spin and the total angular momentum (orbital plus spin) of the system.

5.4. Two identical fermions of mass m and spin 3/2 are in a three-dimensional isotropic harmonic potential of angular frequency ω under the influence of a weak constant magnetic field B oriented in the z-axis. The Hamiltonian is

$$H = H_1 + H_2 = H_{\rm op}\otimes 1 + 1\otimes H_{\rm op}\,,$$

where the one-particle Hamiltonian $H_{\rm op}$ reads

$$H_{\rm op} = \frac{\mathbf{P}^2}{2m} + \frac{1}{2}\,\omega^2\mathbf{x}^2 - \gamma B S_z\,,$$

and γ stands for the fermion's gyromagnetic ratio.

(1) Determine the one-particle energy levels and draw a figure with the lowest four. Note that since

the field B is weak, $\gamma B << \omega$.

(2) Find the energy and the wave function of the ground state of the system.

(3) What results can be obtained if $\mathbf{S}^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2$ is measured and with what probabilities?

5.5. The spatial part of the wave function of a system formed by two identical spin-$\frac{3}{2}$ particles is antisymmetric. Find the spin part in the basis $\{\mathbf{S}_1^2, \mathbf{S}_2^2, \mathbf{S}^2, S_z\}$ if the z-component of the total spin is $2\hbar$. What values can be obtained in a measurement of $\mathbf{S}^2$ and with which probabilities?

5.6. Let $\psi_i(\mathbf{x})$ be two one-particle orthonormal wave functions, $i = 1, 2$, defined on $\mathbb{R}^3$,

$$\langle \psi_i | \psi_j \rangle = \int_{\mathbb{R}^3} \mathrm{d}^3x \; \psi_i^*(\mathbf{x}) \; \psi_j(\mathbf{x}) = \delta_{ij} \, .$$

(1) Find the wave function $\Psi(\mathbf{x}_1, \mathbf{x}_2)$ for two fermions, two bosons and two distinguishable particles assuming that each particle is in a different one-particle state $\psi_i(\mathbf{x})$.

(2) Write for the three cases above the expectation value $\langle r^2 \rangle$ of the distance squared between the particles, $r^2 = |\mathbf{x}_1 - \mathbf{x}_2|^2$, in terms of

$$\mathbf{X}_{ij} := \langle \psi_i | \mathbf{X} | \psi_j \rangle = \int_{\mathbb{R}^3} \mathrm{d}^3x \; \psi_i^*(\mathbf{x}) \; \mathbf{x} \; \psi_j(\mathbf{x}) \, ,$$

$$X^2_{ij} := \langle \psi_i | \mathbf{X}^2 | \psi_j \rangle = \int_{\mathbb{R}^3} \mathrm{d}^3x \; \psi_i^*(\mathbf{x}) \; \mathbf{x}^2 \; \psi_j(\mathbf{x}) \, .$$

(3) Show that $\langle r^2 \rangle_{\text{Bose}} \leq \langle r^2 \rangle_{\text{distinguishable}} \leq \langle r^2 \rangle_{\text{Fermi}}$.

Time-independent perturbation theory

6.1. A charged particle with charge q is harmonically oscillating with angular frequency ω; Hamiltonian given by

$$H_0 = \frac{P^2}{2m} + \frac{1}{2} \, m\omega^2 x^2 .$$

If a constant electric field $\mathcal{E}$ in the positive direction is introduced, calculate the first nontrivial corrections in perturbation theory to the harmonic oscillator energies. Calculate the first non-trivial correction to the ground state wave function. Compare the results thus obtained with the exact expressions.

6.2. A system with Hilbert space $\mathbb{C}^3$ has Hamiltonian given by

$$H = H_0 + H_{\text{I}} \, , \quad H_0 = \epsilon \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad H_{\text{I}} = a \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad 0 < a \ll \epsilon \, .$$

Find the exact eigenvalues and eigenvectors. Use first-order perturbation theory to determine the energy levels. Compare the approximate results with the exact expressions.

6.3. A charged particle, charge q, in an infinite square well of width a centered at the origin is placed in a constant electric field $\mathcal{E}$ oriented in the positive direction. Calculate the first order correction to the ground state energy.

6.4. The Hamiltonian of a spin-2 particle is

$$H = H_0 + H_{\rm I}\,, \quad H_0 = \epsilon\,(S_z^2 - 3\hbar S_z)\,, \quad H_{\rm I} = a\hbar S_x\,, \quad 0 < a \ll \epsilon\,.$$

Solve the unperturbed problem. Find the first order corrections in perturbation theory to the energies of the ground state and of the first excited state.

6.5. [Griffiths 6.3] Two identical particles of spin 0 are in an infinite square well centered at the origin of width a. They interact through the potential

$$V(x_1, x_2) = -\,\frac{a}{2}\,V_0\,\delta(x_1 - x_2)\,,$$

where V_0 is a constant with dimensions of energy. Considering this interaction as a perturbation, calculate the first-order correction to the energies of the ground and first excited states.

6.6. The effect of the finite size of the nucleus on the Hydrogen atom energy levels can be studied using perturbation theory. This can be done by assuming that the nucleus is a uniformly charged sphere of radius $r_{\rm n}$ with total charge Ze. This changes the potential energy to

$$V(r) = \begin{cases} \dfrac{Ze^2}{8\pi\epsilon_0\, r_{\rm n}^3}\,(r^2 - 3r_{\rm n}^2) & \text{if} \quad r \le r_{\rm n} \\ -\,\dfrac{Ze^2}{4\pi\epsilon_0\, r} & \text{if} \quad r \ge r_{\rm n} \end{cases}$$

The ground state wave function of the Hydrogen is

$$\psi_{100}(r) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_{\rm B}}\right)^{3/2} e^{-Zr/a_{\rm B}}\,.$$

Noting that $a_{\rm B} \sim 10^{-10}\,{\rm m}$ and $r_{\rm n} \sim 10^{-15}\,{\rm m}$, one may take $\exp\big(-Zr/a_{\rm B}\big) \simeq 1$ inside the nucleus, provided $Z \lesssim 100$. Compute under this assumption the first-order correction to the ground state energy.

6.7. The Hamiltonian of a particle of mass m in two dimensions is

$$H = \frac{1}{2m}\left(P_x^2 + P_y^2\right) + \frac{1}{2}\,m\omega^2\,(x^2 + y^2)\,(1 + \lambda\alpha^2 xy)\,, \quad \alpha^2 = \frac{m\omega}{\hbar}\,,$$

where $\lambda \ll 1$ is a dimensionless parameter. Use perturbation theory to calculate the first-order correction to the lowest three energies.

6.8. A one-dimensional harmonic oscillator of angular frequency ω is under the action of a perturbation given by $H_{\rm I} = \lambda\hbar\omega\sin(\alpha x)$, with λ a dimensionless parameter much smaller than 1. Calculate the first-order correction to the ground state energy and its wave function (calculations a bit longer).

Hint. The first part is very simple; it reduces to computing the expectation value of $H_{\rm I}$ in the unperturbed gound state $\phi_0^{(0)}(x)$, which is zero, for this is the ingeral of an odd function over a symmetric domain. The calculation of the first order correction to the gound state function involves the computation of the matrix elements $\langle\phi_n^{(0)}|H_{\rm I}|\phi_0^{(0)}\rangle$. This can be easily done by (i) using for $\phi_n^{(0)}(x)$ its expressions in terms of the Hermite polynomial $H_n(\alpha x)$ and the Rodrigues generating formula for $H_n(\alpha x)$, (2i) writing $\sin(\alpha x) = (e^{{\rm i}\alpha x} - e^{-{\rm i}\alpha x})/2{\rm i}$, and (3i) integrating by parts. Alternatively, one may write $\sin(\alpha x) = (e^{{\rm i}\alpha x} - e^{-{\rm i}\alpha x})/2{\rm i}$, express x in terms of a^+ and a and use that $e^{A+B} = e^A e^B e^{-[A,B]/2}$ for operators A and B such that $[A,[A,B]] = [B,[A,B]] = 0$.

Variational method

7.1. [Griffiths 7.11] Using the variational method find the ground state energy of the Hydrogen atom for a trial function of the form $\psi(\lambda,\mathbf{x}) = Ne^{-\lambda r^2}$, with N a normalization constant.

Solution. One first normalizes the trial wave function and obtains

$$\psi(\lambda,x) = \sqrt{\frac{2\lambda}{\pi}}\, e^{-\lambda r^2}\,.$$

One next computes $E(\lambda) = \langle\psi(\lambda)|H|\psi(\lambda)\rangle$, with H the Hydrogen atom Hamiltonian. This gives

$$E(\lambda) = \frac{2\hbar^2}{m}\left[-\frac{3\lambda}{4} + \frac{e^2 m}{(4\pi\epsilon_0)\,\hbar^2}\sqrt{\frac{2\lambda}{\pi}}\right],$$

where as usual $(4\pi\epsilon_0) = 1$ in Gaussian units. Finally one studies the minima of $E(\lambda)$ and obtains a minumum at

$$\lambda_{\rm min} = \frac{1}{(4\pi\epsilon_0)^2}\,\frac{8m^2e^4}{9\pi\hbar^2} \quad\Rightarrow\quad E(\lambda_{\rm min}) = \frac{8\pi}{3}\underbrace{\left[-\frac{mc^2\alpha^2}{2\,(4\pi\epsilon_0)}\right]}_{E_1} \simeq 0.85\,E_1 = -11.5\,{\rm eV}\,,$$

with $\alpha \simeq 1/137$ the fine structure constant.

7.2. [Griffiths 7.12] Consider a particle in a Yukawa potential $V(r) = -ge^{-\mu r}/r$, where μ is a constant with dimensions of (length)$^{-1}$ and g is a constant with dimensions of length times energy, so the Hamiltonian of the particle is

$$H = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}^2 - g\,\frac{e^{-\mu r}}{r}$$

Find a sensible trial function and calculate the ground state energy using the variational method. Assume $gm/\hbar^2\mu \gg 1$.

Note. The Coulomb force has infinite range, meaning that no matter how far away from each other the two charges are, they feel the electric field produced by the other. This has to do with the fact that the photon (particle that in QFT mediates the electromagnetic interaction) has zero mass. A potential accounting for an effective finite range interaction mediated by a particle of mass $\mu c/\hbar$ is the Yukawa potential. Because of the negative sign in the exponetial, the Yukawa potential approaches zero at $r \to \infty$ much faster that the Coulomb potential. With this in mind, one may consider as trial function the Hydrogen atom wave function with the Bohr radius replaced with a parameter a. One can also argue that to ensure square integrability of the wave function and all its derivatives one makes an ansatz of the form $Ne^{-r/a}$, with N a normalization constant and a a parameter with dimensions of length.

7.3. A quantum system has Hamiltonian H and Hilbert space $\mathcal{H}$. Assume that H does not depend on time and let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two normalized arbitrary states, ortogonal to each other. To find an upper bound of the ground state energy, we use the variational method and take as trial state a linear combination $|\psi\rangle$ of $|\psi_1\rangle$ and $|\psi_2\rangle$. Show that the extrema of

$$E = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

are the eigenvalues of the restriction of the Hamiltonian to the subspace spanned by $|\psi_1\rangle$ and $|\psi_2\rangle$.

7.4. Consider a particle in a one-dimensional potential

$$V(x) = \begin{cases} \infty & \text{if } x \leq 0\,, \\ bx & \text{if } x > 0\,. \end{cases}$$

Make a sensible choice for a trial function and calculate an upper bound to the ground state energy using the variational method.

Note. Convince yourself that such an ansatz is provided by

$$\psi_\lambda(x) = \begin{cases} 0 & \text{if } x \leq 0\,, \\ Nx\, e^{\lambda x} & \text{if } x > 0\,, \end{cases}$$

with N a normalization constant and λ a parameter with dimensions of length.

Time-dependent perturbation theroy

8.1. An electric field

$$\mathcal{E}(t) = \frac{\mathcal{E}_0}{\sqrt{\pi}\,\tau}\; e^{-t^2/\tau^2}\;,$$

with $\mathcal{E}_0$ and τ constant, atcs on a one-dimensional harmonic oscillator with angular frquency ω. If the system is at $t = -\infty$ in its ground state, compute the probability in Born's approximation that it be

in its first excited state at $t = +\infty$. Consider the particular cases $\omega\tau \ll 1$ and $\omega\tau \gg 1$. Study the validity of the Born approximation.

8.2. A one-dimensional harmonic oscillator is in its ground state. Its oscillation center stats moving at $t = 0$ with constant velocity v_0 and keeps moving until time T, at which it stops, so that the complete Hamiltonian reads

$$H = \frac{p^2}{2m} + \frac{1}{2}\, m\omega^2 \left[x - a(t)\right]^2 ,$$

with

$$a(t) = \begin{cases} 0 & \text{if } \; t < 0 \\ v_0 t & \text{if } \; 0 < t < T \\ 0 & \text{if } \; T < t \end{cases} .$$

Find in Born's approximation the probability that at the end of the process the oscillator be in its first excited state.