UNIVERSIDAD COMPLUTENSE DE MADRID
DEPARTAMENTO DE FISICA TEORICA I

Física computacional – Curso 2011/12 – Práctica

El péndulo caótico

A pendulum of mass m and length ℓ moves under the action of damping and driving forces $-\gamma\,\dot\theta(t)$ and $F_{\rm D}\sin(\Omega_{\rm D}t)$, with $\theta(t)$ the angle from the vertical axis, and γ, $F_{\rm D}$ and $\Omega_{\rm D}$ constants. In units such that $g=\ell$, the corresponding initial value problem reads

$$\begin{aligned}\ddot\theta(t) &= -\sin\theta(t) - \eta\,\dot\theta(t) + A_{\rm D}\sin(\Omega_{\rm D}t)\\ \theta(0) &= \theta_0\\ \dot\theta(0) &= \omega_0\,,\end{aligned}$$

where η and $A_{\rm D}$ are defined as

$$\eta := \frac{\gamma}{m\ell^2} \qquad A_{\rm D} := \frac{F_{\rm D}}{m\ell^2}\,.$$

We have set $g=\ell$ so that the angular frequency of the free pendulum is $\Omega_0=\sqrt{g/\ell}=1$.

1. Write a Maple code for RK4 that solves the equation of motion. Since the vertical axis has been taken as origin of angles, work in a domain $-\pi\le\theta\le\pi$.

2. Assume that $\eta=A_{\rm D}=0$. Produce one single plot with the trajectories for the three sets of initial data

$$(\theta_0,\omega_0) = (-\pi/4,\,0),\ (-\pi/3,\,\pi/4),\ (3\pi/4,\,\pi/4)\,.$$

Include in the plot a few oscillations.

3. Still with $\eta=A_{\rm D}=0$, make a phase space diagram, velocity $\omega(t)=\dot\theta(t)$ versus position $\theta(t)$, taking as initial data

$$(\theta_0,\omega_0) = (0,\pi/6),\ (0,\pi/3),\ (0,\pi/2),\ (0,\pm2\pi/3),\ (0,\pm3\pi/4),\ (0,\pm11\pi/12)\,.$$

4. Consider $\eta=1/2$ (damping) and $\eta=1$ (overdamping) and go again over question 3. For clarity in the display, in the last thre cases consider only the positive sign in $\pm$.

5. Take from now on $\eta=1/2$, $\Omega_{\rm D}=2/3$ and initial data $(\theta_0,\omega_0)=(0.5,0)$. For zero external force, $A_{\rm D}=0$, the pendulum oscillates with frequency $\Omega_0=1$ and damped amplitude. For small values of $A_{\rm D}$, after some oscillations with frequency $\Omega_0=1$, it begins to oscillate with frequency $\Omega_{\rm D}=2/3$. Finally, for large values of $A_{\rm D}$, the pendulum becomes chaotic. Verify this behaviour by making a graphic of $\theta(t)$ versus t

for values of $A_{\mathrm{D}} = 0.1, 0.2, 0.3, \ldots, 2$. Make an estimate of the ranges of A_{D} for which these regimes occur.

6. Make a phase space diagram for $A_{\mathrm{D}} = 0.2, 0.6$ and a second one for $A_{\mathrm{D}} = 1.2, 1.6$ and briefly explain them.

7. Let the system ocillate approximately 300 times. Warning: depending on the step size $\delta t = h$ this may require a large number of steps! Say that T_{300} is the time after 300 oscillations. Let the pendulum perform about 30 oscillations more, so that $T_{300} \leq t \leq T_{330}$, and make phase diagrams for $A_{\mathrm{D}} = 0.5, 1.070, 1.458, 1.468$. Explain the result.

8. For $T_{300} \leq t \leq T_{330}$, calculate the solution $\theta(t_n)$ at times $t_n = 2\pi n/\Omega_{\mathrm{D}}$, with n an integer, for the values of A_{D} in question 7. Make a graph with the results. The resulting graph for every A_{D} is called a Poincaré section.

9. To study stability and how the driving term affects it, one may solve the initial value problem for new conditions

$$\theta_0' = \theta_0 + \delta\theta_0 \qquad \omega_0' = \omega_0 + \delta\omega_0$$

and look at $\delta\theta(t) = \theta'(t) - \theta(t)$. Take $(\theta_0', \omega_0') = (0.51, 0)$ and explicitly computethe the difference $\delta\theta(t)$ for $A_{\mathrm{D}} = 0.500, 1.070, 1.468$. Study the instability at the Poincaré section for $A_{\mathrm{D}} = 1.468$.