

Particle on the magnetic field of a magnetic dipole $\vec{\mu}$ - Orbits in $z=0$ plane

$$\vec{A} = \frac{\vec{\mu} \wedge \vec{x}}{|\vec{x}|^3} \quad A_1 = \frac{\mu_2 x_3 - \mu_3 x_2}{|\vec{x}|^3} \quad A_2 = \frac{\mu_3 x_1 - \mu_1 x_3}{|\vec{x}|^3} \quad A_3 = \frac{\mu_1 x_2 - \mu_2 x_1}{|\vec{x}|^3}$$

Take $\vec{\mu} = \mu \hat{e}_3$ and use cylindrical coordinates

$$\left. \begin{aligned} x_1 &= r \cos \phi \\ x_2 &= r \sin \phi \\ x_3 &= z \end{aligned} \right\} \Rightarrow \begin{aligned} \dot{x}_1 &= \dot{r} \cos \phi - r \dot{\phi} \sin \phi \\ \dot{x}_2 &= \dot{r} \sin \phi + r \dot{\phi} \cos \phi \\ \dot{x}_3 &= 0 \end{aligned}$$

$$A_1 = -\frac{\mu r \sin \phi}{(r^2 + z^2)^{3/2}} \quad A_2 = \frac{\mu r \cos \phi}{(r^2 + z^2)^{3/2}} \quad A_3 = 0$$

The Lagrangian is

$$\begin{aligned} L &= \frac{1}{2} m \dot{\vec{x}}^2 + q \dot{\vec{x}} \cdot \vec{A} = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) + q (\dot{x}_1 A_1 + \dot{x}_2 A_2 + \dot{x}_3 A_3) \\ &= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2) + \frac{\mu q r^2 \dot{\phi}}{(r^2 + z^2)^{3/2}} \end{aligned}$$

The Euler-Lagrange equations are

$$\begin{aligned} m \ddot{r} - m r \dot{\phi}^2 + \frac{2\mu q r \dot{\phi}}{(r^2 + z^2)^{3/2}} - 3 \frac{\mu q r^3 \dot{\phi}}{(r^2 + z^2)^{5/2}} &= 0 \\ \frac{d}{dt} \left[m r^2 \dot{\phi} + \frac{\mu q r^2}{(r^2 + z^2)^{3/2}} \right] &= 0 \Rightarrow m r^2 \dot{\phi} + \frac{\mu q r^2}{(r^2 + z^2)^{3/2}} = \text{const.} = l \\ m \ddot{z} - 3 \frac{\mu q r^2 z \dot{\phi}}{(r^2 + z^2)^{5/2}} &= 0 \end{aligned}$$

Solutions with $z=0$ have

$$\begin{aligned} m r^2 \dot{\phi} + \frac{\mu q}{r} &= l \Rightarrow \dot{\phi} = \frac{1}{m r^2} \left(l - \frac{\mu q}{r} \right) \\ m \ddot{r} - m r \dot{\phi}^2 - \frac{\mu q \dot{\phi}}{r^2} &= 0 \Leftrightarrow m \ddot{r} - m r \frac{1}{m^2 r^4} \left(l - \frac{\mu q}{r} \right)^2 - \frac{\mu q}{r^2} \frac{1}{m r^2} \left(l - \frac{\mu q}{r} \right) = 0 \\ m \ddot{r} - \frac{1}{m r^4} \left(r l^2 - 3 \mu q l + \frac{\mu q^2}{r} \right) &= 0 \end{aligned}$$

Recall from lesson 1 that for a particle in an EM field, the energy was conserved and given by

$$E = \frac{1}{2} m \dot{\vec{x}}^2 + V(\vec{x})$$

Alternatively

$$H = \frac{1}{2m} (\vec{p} - q\vec{A})^2 = \frac{1}{2} m \dot{\vec{x}}^2 \text{ is conserved}$$

In our case $V(\vec{x})=0$, so that for $z=0$

$$E = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) = \text{conserved}$$

$$E = \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{1}{2} m r^2 \left[\frac{1}{m r^2} \left(l - \frac{\mu q}{r} \right) \right]^2}_{U(r)} \quad \frac{l^2}{2m} \quad \frac{l^2}{\mu^2 q^2}$$

Case $l > 0$: $\frac{\mu q}{l r} = \frac{1}{s} \Rightarrow U(r) = \frac{l^2}{2m r^2} \left(1 - \frac{\mu q}{l r} \right)^2 = \frac{l^4}{2m^2 \mu^2 q^2} \frac{1}{s^2} \left(1 - \frac{1}{s} \right)^2$

$$U'(s) = -\frac{2k}{s^3} \left(1 - \frac{1}{s} \right)^2 + \frac{2k}{s^2} \left(1 - \frac{1}{s} \right) \frac{1}{s^2} = \frac{2k}{s^5} (-s^2 + 3s - 2)$$

$$U''(s) = -\frac{10k}{s^6} (-s^2 + 3s - 2) + \frac{2k}{s^5} (-2s + 3) = \frac{2k}{s^6} (3s^2 - 12s + 10)$$

$$U'(s_0) = 0 \Rightarrow -s_0^2 + 3s_0 - 2 = 0 \Rightarrow s_0 = 1, 2$$

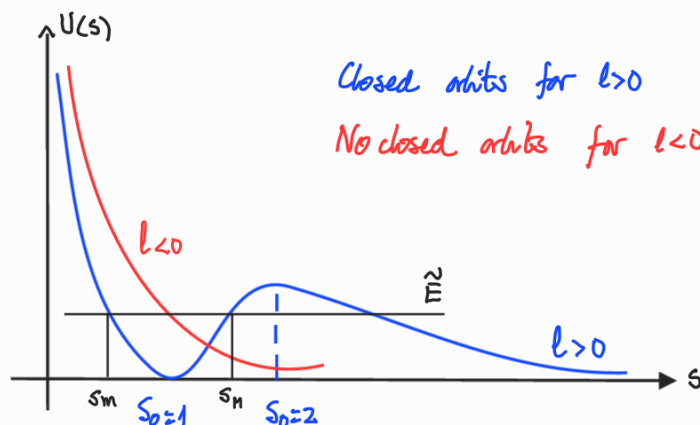
$$U''(s_0=1) = 2k > 0 \Rightarrow s_0=1 \text{ minimum}$$

$$U''(s_0=2) = -\frac{k}{16} < 0 \Rightarrow s_0=2 \text{ maximum}$$

Case $l < 0$: $\frac{\mu q}{|l| r} = \frac{1}{s} \Rightarrow U(r) = \frac{l^2}{2m r^2} \left(1 + \frac{\mu q}{|l| r} \right) = k \frac{1}{s^2} \left(1 + \frac{1}{s} \right)^2$

$$U'(s) = -\frac{2k}{s^5} (s^2 + 3s + 2), \quad U''(s) = \frac{2k}{s^6} (3s^2 + 12s + 10)$$

No equilibrium points for $s > 0$



For an energy
apocapsis r_M .

$0 < \tilde{E} < U(s_0=2) = \frac{3\ell^4}{32m^2\mu^2g^2}$ there is a closed orbit with periapsis r_m and

$$s_0=2 \Rightarrow r_0 = \frac{\mu g}{\ell} s_0 = \frac{2\mu g}{\ell}, \quad U(s_0=2) = \frac{3\ell^4}{32m^2\mu^2g^2}$$