Some general definitions (and results)

Local error = error comitted in one step =: e_{n+1}

$$e_{n+1} = \underset{\substack{\uparrow \\ \text{numerical} \\ \text{solution}}}{y_{n+1}} - \underset{\substack{\uparrow \\ \text{exact} \\ \text{solution}}}{y(t_n+h)} \quad \text{for} \quad y_k = y(t_k) \quad k \leq n$$

Global error = difference between numerical and exact solutions

$$E_{n+1} = y_{n+1} - y(t_n+h)$$

Consistency. A method is said to be consistent if

$$\lim_{h \to 0} \frac{e_n}{h} = 0$$

The method has order p if $e_n = O(h^{p+1})$ as $h \to 0$

Convergence. A method is said to be convergent if

$$\lim_{h \to 0} |E_n| = 0 \quad \text{for all } n.$$

Note. Consistency is a necessary condition for convergence, but not sufficient. See below

Stability. Let y_n be the numerical solution at $n>0$ for an initial condition y_0. Let z_n the numerical solution for an initial condition $y_0+\delta_0$. The method is stable if there exists an $\hat{h}>0$ such that $|z_n-y_n|$ is bounded from above for all $h\in(0,\hat{h})$ and all $t_n\in[0,T]$

→ This definition involves the limit $h\to 0$ since $|z_n-y_n|$ must be bounded for all $h\in(0,\hat{h})$

→ In practice this definition involves $\frac{\partial f}{\partial y}(t_n,y_n)$, which makes it difficult to use.

Absolute stability A numerical method is absolutely stable for a mesh h if $|z_n-y_n|\leq\delta_0$

→ This definition takes a specific value of h.

→ It still depends on $\frac{\partial f}{\partial y}(t_n,y_n)$

→ In practice, absolute stability is discussed almost exclusively for $f(t,y)=\lambda y$, with λ constant.

Results for one-step methods having the form

$$y_{n+1} = y_n + h\Phi(t_n, y_n, h) \qquad (M1)$$

Stability. If $\Phi(t, y, h)$ is Lipschitz on y then the method (M1) is stable.

Proof: $y_0 \longrightarrow \quad y_{n+1} = y_n + h\Phi(t_n, y_n, h)$

$z_0 \longrightarrow \quad z_{n+1} = z_n + h\Phi(t_n, z_n, h)$

$$|z_{n+1} - y_{n+1}| \leq |z_n - y_n| + h\,|\Phi(t_n, z_n, h) - \Phi(t_n, y_n, h)|$$

Φ Lipschitz on y means $|\Phi(t,z,h) - \Phi(t,y,h)| \leq \underset{\substack{\uparrow \\ \text{constant.}}}{L}|z-y| \Rightarrow$

$$\Rightarrow |z_{n+1} - y_{n+1}| \leq (1+hL)\,|z_n - y_n|$$

$$\leq (1+hL)(1+hL)\,|z_{n-1} - y_{n-1}|$$

$$\leq \dots \leq (1+hL)^{n+1}\,|y_0 - z_0|$$

$1+x \leq e^x$ for all $x \in \mathbb{R} \Rightarrow |z_{n+1} - y_{n+1}| \leq e^{(n+1)hL}\,|\delta_0|$

Since $t_{n+1} = (n+1)h \leq T$, $\quad |z_{n+1} - y_{n+1}| \leq e^{LT}\,|\delta_0| \Rightarrow$ Stability

$[0, T = Nh]$ = time interval

Bound from above that does not depend on h

→ This is a result for stability (not for absolute stability)

Consistency. The method (M1) is consistent if $\Phi(t, y, h) = f(t, y)$

Proof : $\frac{1}{h} e_n = \frac{1}{h}[y_n - y(t_n)]$

$$= \frac{1}{h}[\cancel{y_{n-1}} + h\Phi(t_{n-1}, y_{n-1}, h) - y(t_n - h)]$$

$$\cancel{y_n(t_{n-1})} - h\underbrace{y''(t_{n-1})}_{\substack{\parallel \\ f(t_{n-1}, y_{n-1})}} + O(h^2)$$

They cancel each other since e_n is local error $e_n = y_n - y(t_n)$ for $e_{k<n} = 0$

$$\Rightarrow \lim_{h \to 0} \frac{e_n}{h} = \lim_{h \to 0}[\Phi(t_{n-1}, y_{n-1}, h) - f(t_{n-1}, y_{n-1})]$$

$$\Rightarrow \lim \frac{e_n}{h} = 0 \quad \text{if} \quad \Phi(t, y, 0) = f(t, y)$$

Convergence. Consistency is not enough for convergence. Stability (which in turn follows from a Lipschitz condition for the one-step methods that we are considering here) is also required. This is what the following theorem states

Theorem. Let $\Phi(t, y, h)$ be a continuous function of t, y and h on $0 \leq t \leq T$, $-\infty < y < \infty$ and $0 \leq h \leq \hat{h}$, and let it satisfy a Lipschitz condition on y. Then the one-step method (M1) converges to a solution of $y' = f(t, y)$, $y(0) = y_0$ if and only if it is consistent.

Proof Let $z(t)$ satisfy

$$\left.\begin{aligned} z' &= \Phi(t, z, 0) \\ z(0) &= y_0 \end{aligned}\right\}$$

and let z_n be defined by

$$\left.\begin{aligned} z_{n+1} &= z_n + h\Phi(t_n, z_n, h) \\ z_0 &= y_0 \end{aligned}\right\}$$

Using the mean value theorem, one has

$$\begin{aligned} z(t_{n+1}) - z(t_n) &= h z'(t_n + h\theta_n) \qquad 0 < \theta_n < 1 \\ &= h\Phi(t_n + h\theta_n, z(t_n + h\theta_n), 0) \end{aligned}$$

It follows that

$$\begin{aligned} z(t_{n+1}) - z_{n+1} &= z(t_n) - z_n + h\left[\Phi(t_n + h\theta_n, z(t_n + h\theta_n), 0) - \Phi(t_n, z_n, h)\right] \\ &= z(t_n) - z_n + h\Big\{\Phi(t_n + h\theta_n, z(t_n + h\theta_n), 0) - \Phi(t_n, z(t_n), 0) \\ &\qquad + \Phi(t_n, z(t_n), h) - \Phi(t_n, z_n, h) \\ &\qquad + \Phi(t_n, z(t_n), 0) - \Phi(t_n, z(t_n), h)\Big\} \end{aligned}$$

The Lipschitz condition implies

second line -> $|\Phi(t_n, z(t_n), h) - \Phi(t_n, z_n, h)| \leq L\,|z(t_n) - z_n|$

Continuity of Φ implies

third line -> $\max\limits_{0 \leq t \leq T} |\Phi(t_n, z(t_n), 0) - \Phi(t_n, z(t_n), h)| = o(h)$

first line -> $\max\limits_{0 \leq t \leq T} |\Phi(t_n + h\theta_n, z(t_n + h\theta_n), 0) - \Phi(t_n, z(t_n), 0)| = o(h)$

All in all

$$|z(t_{n+1}) - z_{n+1}| \leq (1+hL)\,|z(t_n) - z_n| + h\,o(h)$$

$$\leq (1+hL)\left[(1+hL)\,|z(t_{n-1}) - z_{n-1}| + h\,o(h)\right] + h\,o(h)$$

$$\leq \dots \leq (1+hL)^{n+1}\,|z(0) - z_0|$$

$$+ h\,|\underbrace{1 + (1+hL) + (1+hL)^2 + \dots + (1+hL)^n}_{\overset{\shortparallel}{\frac{1-(1+hL)^{n+1}}{1-(1+hL)}}}|\,o(h)$$

Noting that $z(0) - z_0 = 0$, one has

$$|z(t_{n+1}) - z_{n+1}| \leq \frac{(1+hL)^{n+1} - 1}{L}\,o(h) \leq \frac{e^{(n+1)hL} - 1}{L}\,o(h)$$

$$\leq \frac{e^{LT} - 1}{L}\,o(h) \xrightarrow[h \to 0]{} 0 \quad \text{for all } n$$

Global error. Assume Φ satisfies the conditions of the previous theorem and let the one-step method be of order p. Then the global error is bounded by

$$|E_n| \leq \frac{Ch^p}{L}\,(e^{LT} - 1)$$

with C a constant.

[Proof not included]