UNIVERSIDAD COMPLUTENSE DE MADRID
DEPARTAMENTO DE FISICA TEORICA I
MÁSTER EN FÍSICA FUNDAMENTAL

Campos y cuerdas
Parte de cuerdas - Problemas

Problem 1. Show that under an arbitrary variation $\delta g^{\mu\nu}$ of the (inverse) metric the following identities hold:

$$
\begin{aligned}
\delta\sqrt{g} &= -\frac{1}{2}\sqrt{g}\, g_{\mu\nu}\, \delta g^{\mu\nu} = \frac{1}{2}\sqrt{g}\, g^{\mu\nu}\, \delta g_{\mu\nu} \\
\sqrt{g}\, g^{\mu\nu}\, \delta R_{\mu\nu} &= \text{divergence of a vector}\,.
\end{aligned}
$$

Problem 2. Recalling that tensors transform under general coordinate transformations $\sigma^a \to \tilde{\sigma}^a = \sigma^a - \xi^a(\sigma)$ as

$$
\tilde{T}^{a_1\cdots a_n}{}_{b_1\cdots b_m}(\tilde{\sigma}) = \frac{\partial\tilde{\sigma}^{a_1}}{\partial\sigma^{c_1}} \cdots \frac{\partial\tilde{\sigma}^{a_n}}{\partial\sigma^{c_n}} \frac{\partial\sigma^{d_1}}{\partial\tilde{\sigma}^{b_1}} \cdots \frac{\partial\sigma^{d_m}}{\partial\tilde{\sigma}^{b_m}}\, T^{c_1\cdots c_n}{}_{d_1\cdots d_m}(\sigma)\,,
$$

prove that for a scalar field $\psi(\sigma)$ one has

$$
\delta\psi = \xi^c \partial_c \psi\,,
$$

where δT denotes $\delta T := \tilde{T}(\sigma) - T(\sigma)$ for an arbitrary tensor field T. Show that under such transformations the metric inverse h^{ab} and the metric determinant $h = \det(h_{ab})$ change as

$$
\begin{aligned}
\delta h^{ab} &= \xi^c \left(\partial_c h^{ab}\right) - \left(\partial_c \xi^a\right) h^{cb} - \left(\partial_c \xi^b\right) h^{ac} \\
\delta(\sqrt{h}) &= \partial_a \left(\xi^a \sqrt{h}\right)\,.
\end{aligned}
$$

Problem 3. Consider the string classical action terms

$$
\begin{aligned}
S_G &= -\frac{1}{4\pi\alpha'} \int d^2\sigma\, \sqrt{h}\, h^{ab}\, G_{\mu\nu}(X)\, \partial_a X^\mu\, \partial_b X^\nu \\
S_B &= -\frac{1}{4\pi\alpha'} \int d^2\sigma\, \epsilon^{ab}\, B_{\mu\nu}(X)\, \partial_a X^\mu\, \partial_b X^\nu \\
S_\Phi &= \frac{1}{4\pi} \int d^2\sigma\, \sqrt{h}\, R^{(2)}\, \Phi(X) \\
S_\lambda &= \lambda \int d^2\sigma\, \sqrt{h}\,,
\end{aligned}
$$

where, as usual, $G_{\mu\nu}(X)$ denotes the metric on the target spacetime, $B_{\mu\nu}(X)$ the components of the Kalb-Ramond 2-form and $\Phi(X)$ the dilaton.

(a) S_λ is not invariant under Weyl transformations $h_{ab} \to \omega^2\, h_{ab}$. Based on this, it is discarded as a term in the classical string action. Another argument is provided by the following exercise. Consider the action $S = S_G + S_\lambda$ and obtain the corresponding field equation for h_{ab}. Prove that it implies $h_{ab} = 0$.

(b) Derive the field equations for the action $S = S_G + S_B + S_\Phi$. Particularize them for the spacetime metric

$$ds^2 = -dx^-\, dx^+ - m^2 \boldsymbol{x}^2_{D-2} (dx^+)^2 + d\boldsymbol{x}^2_{D-2}\,, \tag{1}$$

where m is a parameter with dimensions of mass and $\boldsymbol{x} = (x^1, \cdots, x^{D-2})$ are Cartesian coordinates.

The metric (2) is a *pp*-wave metric and is a Penrose limit of $AdS_n \times S^m$, with $D = n + m$. Penrose proved in the mid 70's that all solutions to Einstein's equations approach a *pp*-wave metric in the neighborhood of a null geodesic. The limiting procedure is known as Penrose limit. $\big[$See R. Penrose, in "Differential geometry and relativity", edited by M. Cahen and M. Flato, Reidel (Dordrecht 1976), pp. 271$\big]$. Later in 2000, Güeven established the same result for various string theories. $\big[$See R. Güven, Phys. Lett. **B482** (2000) 255 [arXiv:hep-th/0005061]$\big]$.

Problem 4. Take $G_{\mu\nu}$ as in eq. (2) and $B_{\mu\nu} = \Phi = 0$.

(a) Write the field equations and the Virasoro constraints for $h_{ab} = \mathrm{diag}(-1, +1)$. [Either by using the variational principle for h_{ab}, $G_{\mu\nu}$ and $B_{\mu\nu}$ as above or by substituting in the general expressions of the field equations derived in the lectures for arbitrary h and a general background (G, B)].

(b) Go back to the classical action S_G with a general h_{ab} and impose $\det(h_{ab}) = -1$. Vary the classical action with respecto to of $h^{\tau\tau}$ and $h^{\tau\sigma}$. In doing this, note that $h_{\tau\tau}\, h_{\sigma\sigma} - (h_{\tau\sigma})^2 = -1$, so that $h^{\sigma\sigma}$ can be written in terms of $h^{\tau\tau}$ and $h^{\tau\sigma}$. Show that the resulting two equations agree with the Virasoro constraints.

(c) Consider the **closed string**. Show that light-cone gauge $X^+ = \tau$ is admissible. From now on work in this gauge.

(d) Prove that that the field equation for X^- follows from the Virasoro constraints and the field equations for X^i.

(e) Find the classical hamiltonian in terms of the string coordinates and their conjugate momenta.

(f) Solve the equations of motion and write the mode expansion for the hamiltonian.

(g) Compute the symplectic form and derive the commutation relations for the mode annihilation and creation operators.

(h) Use the commutation relations derived in (g) to find the commutation relation for $X^\mu(\tau, \sigma)$ and $P_\nu(\tau, \sigma')$.

Hint. The $G_{\mu\nu}$ considered here is one of the two known Penrose limits of IIB supergravity, the other one being Minkowski spacetime. Its quantization can be found in Section 5.4 of R. R. Metsaev, Nucl. Phys, **625** (2002) 70 [arXiv:hep-th/112044].

Problem 5. Take Minkowski's metric for $G_{\mu\nu}$ and assume that the only nonvanishing components of the field $B_{\mu\nu}$ are

$$B_{12} = B = \text{const.}$$

Consider the action $S = S_G + S_B$ for the **open string** and work in gauge $h_{ab} = \text{diag}(-1, +1)$. Assume $D = 4$.

(a) Write the field equations and the Virasoro constraints.

(b) Show that light-cone gauge is admissible.

(c) Solve the equations of motion for X^1 and X^2. Integrate the Virasoro equations to find X^-.

(d) Compute the symplectic form. In doing this, recall that only the physical degrees of freedom (transverse directions in the case at hand) must be considered.

(e) Find the commutation relations for the annihilation and creation operators in the string modes.

(f) Compute $[X^1(\tau, \sigma)\,,\, X^2(\tau, \sigma')]$.

Hint. The boundary conditions are not $\partial_\sigma X^i\big|_{\sigma=0,\pi} = 0$ any more, due to the presence of a non-trivial B-field. This problem and many other issues of the quantized string in this background are studied in C. S. Chu, P. M. Ho, Nucl. Phys. **B550** (1999) 151 [arXiv:hep-th/9812219].

Problem 6. Consider a closed string in the four-dimensional background

$$\begin{aligned} ds^2 &= 2\, dx^- dx^+ + (dx^1)^2 + (dx^2)^2 + (x^2\, dx^1 - x^1\, dx^2)\, dx^+ \\ B_{12} &= x^+ \qquad \text{any other } B_{\mu\nu} = 0 \qquad \Phi = 0\,. \end{aligned}$$

(a) Write the equations of motion in $h_{ab} = \eta_{ab} = \text{diag}(-1, +1)$ gauge.

(b) Show that light-cone gauge is a good gauge choice and assume it from now on.

(c) Write the Virasoro constraints in this gauge.

(d) Show that the field equation for X^- follows from the Virasoro constraints and the field equations for X^1 and X^2.

(e) Find the classical hamiltonian in terms of the string coordinates and their conjugate momenta.

(f) Solve the equations of motions and write the mode expansion for the hamiltonian.

(g) Quantize the string.

Hint. This string background was proposed and proved to be non-anomalous in C. Nappi and E. Witten, Phys. Rev. Lett. **71** (1993) 3751 [arXiv:hep-th/9310112]. It is a particular case of a *pp*-wave metric. The answer to the posed questions can be found in P. Forgacs, P. A. Horvathy, Z. Horvath, L. Palla, Heavy Ion Phys. **1** (1995) 65 [arXiv:hep-th/9503222].

Problem 7. Consider an **open string** in a background $G_{\mu\nu} = \eta_{\mu\nu}$, $\Phi = 0$ and

$$B_{0,D-1} = E = \text{const}$$

as the only nonvanishing component of $B_{\mu\nu}$.

(a) Discuss if light-cone gauge $X^+ = \tau$ is admissible. If not,

(b) Propose a condition for X^+ admissible as a gauge condition.

Problem 8. Consider the action for the superstring in the notation of Green-Schwarz-Witten, vol. 1. That is,

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\left(\partial_a X^\mu\,\partial^a X_\mu - \mathrm{i}\,\bar{\psi}^\mu\rho^a\partial_a\psi_\mu\right),$$

where ψ^μ are D Majorana spinors and ρ^a are the 2-dimensional Dirac matrices $\{\rho^a,\rho^b\} = -2\eta^{ab}$. The latter can be taken as as

$$\rho^0 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \rho^1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

In this basis the components of a Majorana spinor ψ can be taken real. We will denote htem by

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}.$$

(a) Derive the field equations and the boundary conditions for $\psi^\mu_\pm$. Consider the open string and closed string separately.

(b) Consider now the supersymmetry transformations

$$\delta X^\mu = \bar{\epsilon}\psi^\mu \qquad \delta\psi^\mu = -\mathrm{i}\,\rho^a\partial_a X^\nu\epsilon\,,$$

where ϵ is a constant (anticommuting) Majorana spinor. Show (do and display the calculations explicitly) that

$$\left[\delta_1,\delta_2\right]_{\text{on shell}} = 2\left(\bar{\epsilon}_2\rho^a\epsilon_1\right)\partial_a$$

(c) Show that the action above is invariant under such transformations.

(d) Show explicitly that the field equations are also invariant under supersymmetry.

(e) Find the supersymmetric current and check its conservation.

Problem 9. Prove that the Virasoro amplitudes (coefficients in the classical mode expansions for T_{ab}) satisfy the Poisson brackets

$$[L_m, L_n]_{\text{PB}} = (m-n)\,L_{m+n}\,.$$

Problem 10. Consider light-cone quantization of the open string in Minkowski spacetime, $G_{\mu\nu} = \eta_{\mu\nu}$. Using the commutation relations for creation and annihilation operators

$$[x^i, p^j] = \delta^{ij} \qquad [\alpha^i_m, \alpha^j_n] = m\,\delta^{ij}\,\delta_{m+n,0}\,,$$

compute the commutator $[X^i(\tau,\sigma)\,,\,P^j(\tau,\sigma')]$.

Problem 11. Show that Lorentz invariance for the quantized open string in Minkowski spacetime (light-cone quantization) requires $D = 26$.

Problem 12. Consider the NS sector of the superstring in Minkowski spacetime and quantize it in light-cone gauge $X^+ = \tau$. Show that for the generators of the Lorentz group M^{i-} and M^{j-} to satisfy the usual commutation relation one needs $D = 10$.