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Curso 2020-2021

Soluciones. Julio 2021

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2020-2021
05/07/2021

Examen Final Extraordinario de Matemáticas-F

Nombre y Apellidos:

Firma y DNI:

Nota: No se darán puntos por respuestas sin la debida justificación. El examen son 10 puntos.

P1 [3 pt] a) Hallar el valor de la integral

$$\int_0^{\pi/2} \frac{4}{1+\tan x}\, dx.$$

b) Discutir si la integral es impropia o no lo es.

P2 [3 pt] a) Determinar los valores de x para los que converge la serie

$$\sum_{n=1}^{\infty} \frac{4^n\, x^{2n}}{\arctan n}.$$

b) Estudiar si converge para $x = \cos 1$ y para $x = \sin \frac{1}{2}$.

P3 [1 pt] Encontrar todos los valores reales de x que satisfacen $3^{56}\left(\frac{1}{3}\right)^x\left(\frac{1}{3}\right)^{\sqrt{x}} > 1$.

P4 [2 pt] Sea la función

$$f(x) = \frac{|2x-1| - |2x+1|}{x}.$$

a) Indicar si es par, impar o no tiene una paridad definida. b) Calcular $\lim_{x\to 0} f(x)$ c) Hacer una gráfica aproximada de $f(x)$ especificando su dominio. d) Hallar la derivada $f'(x)$. e) Hacer un dibujo de $f'(x)$ señalando los puntos, si los hay, en los que $f'(x)$ es discontinua.

P5 [0.5 pt] Sea $z = x + iy$. Encontrar todos los números reales x, y que cumplen

$$\mathrm{Re}(z^2) + i\,\mathrm{Im}(z^*(1+2i)) = -3, \qquad i \equiv \sqrt{-1}.$$

Nota: Como siempre $\mathrm{Re}\, z$, $\mathrm{Im}\, z$ son la parte real y la parte imaginaria de z, respectivamente; z^* denota el complejo conjugado de z

P6 [0.5 pt] ¿A qué valor convergen las series

i) $1 - \frac{1/\pi}{2} + \frac{(1/\pi)^2}{3} - \frac{(1/\pi)^3}{4} + \frac{(1/\pi)^4}{5} - \frac{(1/\pi)^5}{6} + \cdots,$

ii) $x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \frac{x^5}{4!} - \frac{\pi^6}{5!} + \cdots$?

(1) One of these four possibilities:

1) Direct calculation ($\equiv$ substitution)

2) Change $u = \tan x$, the integrand is

$$R(\sin x, \cos x) = \frac{1}{1+\tan x},$$

therefore even to the change $\sin x \to -\sin x$, $\cos x \to -\cos x$:

$$R(-\sin x, -\cos x) = R(\sin x, \cos x)$$

Es el cambio "standard", el que se debe de siempre

3) $u = \tan \frac{x}{2}$ (siempre)

↑ y este ∀ los casos... no solo R(,) par. Lo que sea.

4) With the property

$$\int_a^b dx\, f(x) = \int_a^b dx\, f(a+b-x),$$

a, b finite.

Here are 1) 2) 4). 3) no lo hago. Ya tienen ustedes suficientes. Lo hacen en casa si quieren [buen ejercicio]

b) The integral is proper, not an improper integral. Why? Because $[0, \pi/2]$ are both finite and because $\frac{1}{1+\tan x}$ is a continuous function on $[0, \pi/2)$.
"not singular"

[The point in which $1+\tan x = 0$ is $x = -\frac{\pi}{4}$, which is not in $[0, \pi/2)$. Because of the periodicity of $\tan x$ we only need to consider to graph $\frac{1}{1+\tan x}$ the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and to find $\int_0^{\pi/2} \frac{1}{1+\tan x}$ only $[0, \pi/2)$, as mentioned. In $[0, \pi/2)$ everything is fine.
]

a) 1)

$$\frac{1}{1+\tan x} = \frac{1}{1+\frac{\sin x}{\cos x}} = \frac{\cos x}{\cos x + \sin x}$$

$$= \frac{2\cos x}{2(\cos x + \sin x)}$$

add $\sin x - \sin x = 0$ to the numerator

$$= \frac{1}{2} \cdot \frac{\cos x + \sin x - \sin x + \cos x}{\cos x + \sin x}$$

$$= \frac{1}{2}\left[1 + \frac{\cos x - \sin x}{\cos x + \sin x}\right]$$

whose primitive is

$$\frac{1}{2}\left[x + \log\left(\cos x + \sin x\right)\right].$$

Thus

$$\int_0^{\pi/2} \frac{dx}{1+\tan x} = \frac{1}{2}\left[x + \log\left(\cos x + \sin x\right)\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \cancel{\log(0+1)} - 0 - \cancel{\log(1+0)}\right]$$

$$= \frac{\pi}{4}.$$

Result $\boxed{\int_0^{\pi/2} \frac{4}{1+\tan x} = \pi}$

2) $u = \tan x$

$$du = \frac{1}{\cos^2 x}\,dx = \frac{\sin^2 x + \cos^2 x}{\cos^2 x}\,dx = \left(1 + \tan^2 x\right)dx = \left(1+u^2\right)dx$$

$$\int_0^{\pi/2} dx \frac{1}{1+\tan x} = \int_0^{\infty} \frac{du}{(1+u^2)(1+u)}$$

ésta sí es impropia, así que un cambio de variable puede cambiar une integral de proper to improper or improper to proper.

We solve $\int_0^{\infty} \frac{du}{(1+u^2)(1+u)}$ decomposing the integrand into simple fractions, as usual.

$$\frac{1}{(1+u^2)(1+u)} = \frac{Au+B}{1+u^2} + \frac{C}{1+u}, \quad A, B, C \text{ constants}$$

these constants are $A = -1/2,\ B = C_1 = 1/2$

$$\frac{Au+B}{1+u^2} + \frac{C_1}{1+u} = \frac{(Au+B)(1+u) + C_1(1+u^2)}{(1+u^2)(1+u)} = \frac{1}{(1+u^2)(1+u)}$$

$u=-1 \qquad 2C_1 = 1 \quad " \quad C = 1/2$

$u = 0 \qquad B + C = 1 \quad " \quad B = 1/2$

$u = 1 \qquad (A + 1/2)\,2 + 2C = 1$

$\qquad\qquad 2A + \cancel{1} + 2C = \cancel{1},\ A = -C,\ A = -1/2$

Integrand

$$\frac{1}{(1+u^2)(1+u)} = \frac{1/2\,(1-u)}{1+u^2} + \frac{1/2}{1+u}$$

identical to (you can check it)

$$= \frac{1}{2}\left[\frac{1}{1+u^2} - \frac{2u}{2(1+u^2)} + \frac{1}{1+u}\right],$$

and therefore

$$\int \frac{du}{(1+u^2)(1+u)} = \frac{1}{2}\left[\arctan u - \frac{1}{2}\log(1+u^2) + \log(1+u)\right].$$

Consequently,

$$\int_0^{\pi/2} \frac{dx}{1+\tan x} = \frac{1}{2}\left[\arctan u + \log\frac{1+u}{\sqrt{1+u^2}}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\pi/2 - 0 + \log 1 - \log 1\right]$$

$$\lim_{u\to\infty} \frac{1+u}{\sqrt{1+u^2}} = \lim_{u\to\infty} \frac{u}{\sqrt{u^2}} = \lim_{u\to\infty} \frac{u}{u} = 1$$

$\sqrt{u^2} = u$ si $u \Rightarrow \infty$

$= \pi/2.$

nótese: así como hago las substituciones (son límites)

The result is again

$$\boxed{\int_0^{\pi/2} \frac{4}{1+\tan x} = \pi}$$

3) not now.

4) This formula can be checked with

$$\int_a^b dx\, f(x) = -\int_b^a dt\, f(a+b-t) = \int_a^b dt\, f(a+b-t)$$

$t = a+b-x$, a, b finite

t mute variable

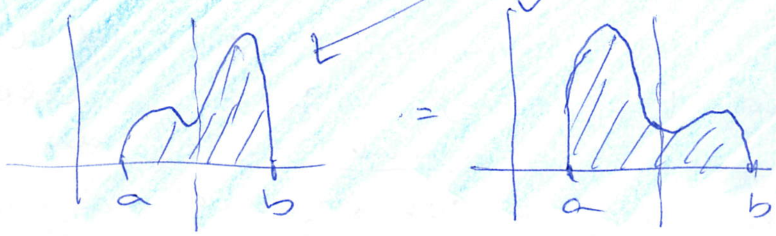
or $x = a+b-t$

$dx = -dt$

No se lo pase en clase.

Geometrical meaning

these two areas coincide



[Échele imaginación a ese dibujo]

midpoint

midpoint reflection

$$\int_0^{\pi/2} \frac{1}{1+\tan x} = \int_0^{\pi/2} \frac{1}{1+\tan\left(\frac{\pi}{2}-x\right)}$$

$$\int_a^b dx\, f(x) = \int_a^b dx\, f(a+b-x)$$

but

$$\tan(A-x) = \frac{\tan A - \tan x}{1+\tan A \tan x}$$

$$= \frac{\cancel{\tan A}\left(1 - \tan x/\tan A\right)}{\cancel{\tan A}\left(\frac{1}{\tan A} + \tan x\right)}$$

$$= \frac{1}{\tan x}.$$

now use $A = \pi/2$

$\tan \pi/2 = \infty$

o sea

$$\tan\left(\frac{\pi}{2}-x\right) = \frac{1}{\tan x}$$

Thus

$$I \equiv \int_0^{\pi/2} \frac{1}{1+\tan x} = \int_0^{\pi/2} \frac{1}{1+\frac{1}{\tan x}} = \frac{\tan x}{1+\tan x}$$

$I = \frac{1}{2} I + \frac{1}{2} I$

$$= \frac{1}{2}\left[\int_0^{\pi/2} dx \, \frac{1+\tan x}{1+\tan x}\right]$$

$$= \frac{1}{2}\int_0^{\pi/2} dx$$

$$= \pi/4 .$$

(Y no he hecho la primitiva, no la he creado/ encontrado/ buscado). Sólo he creado que si dos integrales con distinto integrando en $[0, \pi/2]$ son iguales también son iguales a la semisuma de ambas)

$$\int_0^{\pi/2} \frac{1}{1+\tan x} = \frac{\pi}{4} .$$

Siempre se obtiene lo mismo, se haga cómo se haga...

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(2) The series

$$\sum_1^{\infty} \frac{4^n x^{2n}}{\arctan n} = \frac{4x^2}{\arctan 1} + \frac{4^2 x^4}{\arctan 2} + \frac{4^3 x^6}{\arctan 3} + \dots$$

is a power series. The simplest manner is using the "ratio test" assuming x is arbitrary but fixed, then

$$b_n \equiv \frac{4^n x^{2n}}{\arctan n}$$

only depends on n and and $\sum_1^{\infty} b_n$ is a numerical series x is a parameter

Calculate

absolute value

$$\rho = \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right|$$

and if

$$\rho = \begin{cases} <1 & \text{the series is (absolutely) convergent} \\ =1 & \text{not known yet} \\ >1 & \text{divergent} \end{cases}$$

$$\left| \frac{b_{n+1}}{b_n} \right| = \left| \frac{\dfrac{4^{n+1} x^{2n+2}}{\arctan(n+1)}}{\dfrac{4^n x^{2n}}{\arctan n}} \right| = \frac{4 \arctan n}{\arctan(n+1)} |x|^2$$

$$\rho = |x|^2 \lim_{n\to\infty} \frac{4 \arctan n}{\arctan(n+1)}$$

not depending on n (out of the limit)

$$= |x|^2 \lim_{n\to\infty} 4 \frac{\cancel{\arctan n}}{\cancel{\arctan n}}$$

$$= 4|x|^2$$

Conclusion: The series is absolutely convergent therefore convergent if $|x| < 1/2$. It is divergent if $|x| > 1/2$ and for $x = 1/2, -1/2$ we test it now.

Notice that the series

$$\sum_1^\infty \frac{4^n x^{2n}}{\arctan n},$$

because of the x^{2n}, does not distinguish between $x = 1/2, -1/2$. Inserting $x = \pm 1/2$ in this series we have to study the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{\arctan n} = \frac{1}{\arctan 1} + \frac{1}{\arctan 2} + \frac{1}{\arctan 3} + \dots$$

This series is **divergent** by several reasons:

i) By the preliminary test

$$\lim_{n\to\infty} \frac{1}{\arctan n} = \frac{2}{\pi} \neq 0$$

ii) By comparison with the harmonic series $\sum_1^{\infty} \frac{1}{n}$ since

$$\sum_1^{\infty} \frac{1}{\arctan n} > \sum_1^{\infty} \frac{1}{n} = \text{divergent.}$$

Geometry:

$\sin x < x < \tan x$

red blue green

$x \in (0, \frac{\pi}{2})$

circumference of radius 1 ↗

x: angle in **radians**

Red: | : is the "sin x"

blue:)x : is the "length of the arc"

Green | : is the tangent.

Take $x < \tan x$, $x \in (0, \pi/2)$

and call $\tan x = s$, $s \in (0, \infty)$

$x = \arctan s$

and then $\frac{1}{\arctan s} > \frac{1}{s}$ (use n instead of s, n in $(0, \infty)$

$$\frac{1}{\arctan n} > \frac{1}{n} > 0 \qquad n \in (0, \infty)$$

$$\Rightarrow \sum_1^{\infty} \frac{1}{\arctan n} > \sum_1^{\infty} \frac{1}{n} = \text{divergent}$$

conclusión:

$$\sum_{1}^{\infty} \frac{4^n x^{2n}}{\arctan n} \text{ is } \begin{cases} \text{convergent if } |x| < \frac{1}{2} \\ \text{divergent if } |x| \geq \frac{1}{2}. \end{cases}$$

pero muy lentona en ambos casos.

a) $\cos x = \cos 1$. What is the value of $\cos 1$ in decimal arithmetic?

Since

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad \text{convergent for all } x \text{ finite},$$

$$\cos 1 = \underbrace{1 - \frac{1}{2!}}_{=\frac{1}{2}} + \underbrace{\frac{1}{4!} - \frac{1}{6!}}_{=\frac{6\cdot 5 - 1}{6!} = \frac{29}{6!} = \frac{29}{30}\cdot\frac{1}{24}} + \frac{1}{8!} - \frac{1}{10!} + \dots$$

positive small terms neglected.

$\approx \frac{1}{2} + \frac{29}{30}\cdot\frac{1}{24}$. [error if we stop here is smaller than $\frac{1}{8!}$]

= ~~greater~~ than than 1/2

$\underbrace{\frac{1}{8!} - \frac{1}{10!}}_{\text{positive small term}}$

$= 0.5 + 0.0429$

$\underbrace{\frac{1}{12!} - \frac{1}{14!}}_{\text{positive small term}}$

$\simeq 0.5403$

29 | 24
50 1.2083
200
080

1.208 | 3
008 4026
20
20

$\frac{1.2083}{30} = 0.04026$

b) Estimate $\sin 1/2$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$ convergent for all x finite

$\sin\frac{1}{2} < \frac{1}{2}$ (because $\sin x < x$, $x \in (0, \pi/2]$)

(así ya se puede parar uno. Sigo... no obstante)

$$\sin\frac{1}{2} = \underbrace{\frac{1}{2} - \frac{(1/2)^3}{3!}} + \underbrace{\frac{(1/2)^5}{5!} - \frac{(1/2)^7}{7!}}_{\text{positive small term}} + \underbrace{\cdots}_{\text{positive small term.}}$$

$$\frac{1}{2} - \frac{1}{8 \cdot 3!}$$
$$= \frac{1}{2}\left[1 - \frac{1}{4 \cdot 3!}\right]$$
$$= \frac{1}{2}\left[1 - \frac{1}{4!}\right] = \frac{1}{2}\left[\frac{4!-1}{4!}\right] \simeq \frac{1}{2}\,\frac{23}{24}$$

23 | 24
140 0.958
200 × 0.5
0.4790

$\simeq 0.479$

Since $x = \cos 1$ is greater than $1/2$ and $x = \sin 1/2$ is smaller than $1/2$:

$x = \cos 1$ series is divergent

$x = \sin 1/2$ " " convergent.

Matemáticas
Examen Final Extraordinario de Julio
Soluciones

(P3) $3^{56}\left(\frac{1}{3}\right)^{x}\left(\frac{1}{3}\right)^{\sqrt{x}} > 1.$

$$\frac{3^{56}}{3^{x+\sqrt{x}}} = 3^{56-x-\sqrt{x}} > 3^{0} = 1.$$

Then x has to satisfy

$$56 - x - \sqrt{x} > 0$$

and of course x is non-negative for $\sqrt{x}$ to exist. The previous equation is also

(*) $x + \sqrt{x} - 56 < 0$ with $x \geq 0$,

that corresponds to

$$\underbrace{(\sqrt{x}+8)}_{\text{this is always greater than } 0} \cdot (\sqrt{x}-7) < 0,$$

so we just find where

$$\sqrt{x} - 7 < 0$$

If a, b are positive numbers, $\sqrt{a} < \sqrt{b}$ implies $a < b$

clase

$0 \leq \sqrt{x} < 7$ recuerde, $x \geq 0$

is $0 \leq x < 49$

solución: $x \in [0,49)$

(*) si $\sqrt{x} = u$, $x = u^2$:

$u^2 + u - 56 = 0$ has roots -8, 7

and $u^2 - u - 56 = (u+8)(u-7)$

(P4) $f(x)= \dfrac{|2x-1|-|2x+1|}{x}$.

Remember : $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$

a) Par o impar : calculate $f(-x)$

$$f(-x) = -\frac{|-2x-1|-|-2x+1|}{x}$$

$|-2x-1| = |-1(2x+1)| = |2x+1|$, and $|-2x+1| = |-(2x-1)| = |2x-1|$

$$\to = -\frac{|2x+1|-|2x-1|}{x} = \frac{|2x-1|-|2x+1|}{x} = f(x).$$

Obviamente, $f(x)$ es par (lo que facilita su gráfica).

b) $\lim_{x\to 0} f(x) = -4$ as we justify in c)

c) We write $f(x)$ in a most suitable form for calculations. Notice that important points in the Dom f are $x=-1/2$ and $x=1/2$:

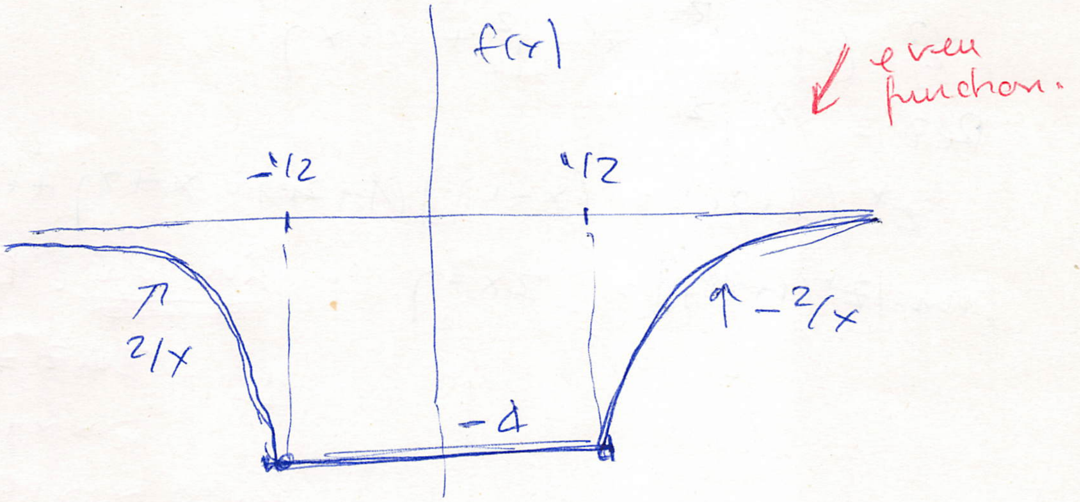
x		$-1/2$		$1/2$	
$\lvert 2x-1\rvert$:	$-2x+1$		$-2x+1$		$2x-1$
$\lvert 2x+1\rvert$:	$-2x-1$		$2x+1$		$2x+1$
$\lvert 2x-1\rvert - \lvert 2x+1\rvert$:	2		$-4x$		-2
$\dfrac{\lvert 2x-1\rvert-\lvert 2x+1\rvert}{x}$:	$\dfrac{2}{x}$		-4		$-\dfrac{2}{x}$

So $f(x)$ is also

$$f(x) = \begin{cases} 2/x & x \leq -1/2, \\ -4 & -\frac{1}{2} \leq x \leq 1/2, \\ -\frac{2}{x} & \frac{1}{2} \leq x. \end{cases}$$

esta manera de escribir $f(x)$ es mucho mejor para cálculos!

The graphic of $f(x)$ is



and

$$\text{Dom } f = \mathbb{R}.$$

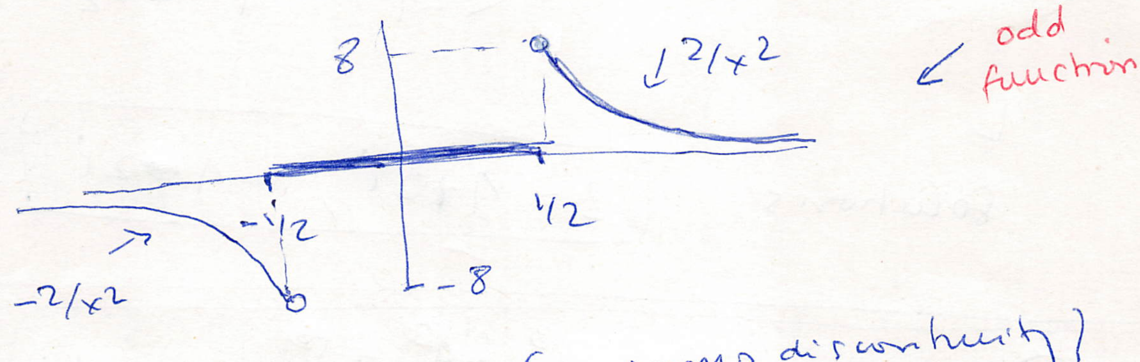
ya se ve en el dibujo que $f'(x)$ no va a existir en los "picos" $x = -1/2,\ x = 1/2$.

obviously

$$\lim_{x \to 0} f(x) = -4$$

(the function is continuous at $x=0$).

d) $f'(x) = \begin{cases} -2/x^2 & x < -1/2 \\ 0 & -\frac{1}{2} < x < \frac{1}{2} \\ 2/x^2 & x > 1/2 \end{cases}$, with graphic.



$f'(x)$ is discontinuous (a jump discontinuity) at $x = -1/2, 1/2$.

$$\text{Dom } f' = \mathbb{R} - \{\tfrac{1}{2}, -1/2\}.$$

P5

$$z = x+iy$$

$$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$

$$\operatorname{Re} z^2 = x^2 - y^2$$

$$z^*(1+2i) = (x-iy)(1+2i) = x+2y + i(2x-y)$$

$$\operatorname{Im}\left[z^*(1+2i)\right] = 2x-y$$: sin la i. Es definición.

Def : $\operatorname{Im} z = y$ if $z = x+iy$ with x, y real.

Thus

$$\operatorname{Re}(z^2) + i \operatorname{Im}\left(z^*(1+2i)\right) = -3$$

is

$$x^2 - y^2 + i(2x-y) = -3,$$

corresponding to

$$x^2 - y^2 = -3$$
$$2x - y = 0$$

$y = 2x$; $x^2 - y^2 = x^2 - 4x^2 = -3x^2 = -3 \Rightarrow x^2 = 1$

$$x \begin{cases} 1 \\ -1 \end{cases} \qquad y \begin{cases} +2 \\ -2 \end{cases}$$

Solutions: $\boxed{z = 1+2i,\ -1-2i}$

I dont check them but you can check in the exam.

P6 i) $1 - \frac{1/\pi}{2} + \frac{(1/\pi)^2}{3} - \frac{\left(\frac{1}{\pi}\right)^3}{4} + \frac{\left(\frac{1}{\pi}\right)^4}{5} - \frac{\left(\frac{1}{\pi}\right)^5}{6} + \cdots = \pi \log\left(1+\frac{1}{\pi}\right)$

why? $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \ldots \quad , |x|<1$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \quad |x|<1$$

$$\frac{1}{x}\log(1+x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \cdots \quad |x|<1$$

(i) $x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \frac{x^5}{4!} - \frac{x^6}{5!} + \cdots = xe^{-x}$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots,$$

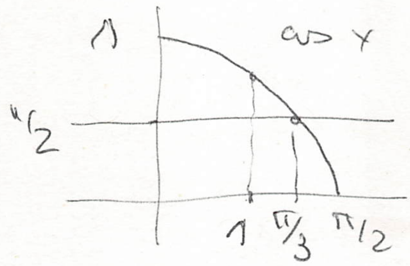
$$xe^{-x} = x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \frac{x^5}{4!} - \frac{x^6}{5!} + \cdots.$$

Los alumnos hacen esto para saber si cos 1, sin 1/2 son menores o mayores que 1/2.

cos x is a decreasing function in $[0, \pi/2]$

$\cos 60^\circ = 1/2$

$60^\circ = \frac{\pi}{3} > 1$



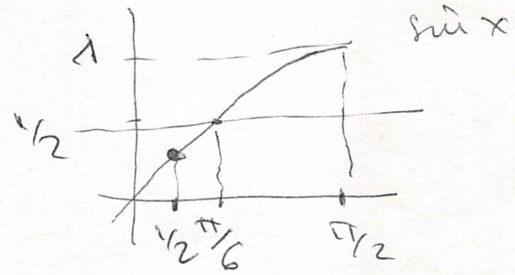
Then $\cos 1 > 1/2$.

—— 0 ——

sin x is an increasing function in $[0, \pi/2]$

$\sin 30^\circ = 1/2$

$30^\circ = \frac{\pi}{6} > \frac{1}{2}$



Then $\sin 1/2 < \frac{1}{2}$.

—— 0 ——