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UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2020-2021
12/02/2021

Soluciones

Examen Final Ordinario de Matemáticas-F

Nombre y Apellidos:

Firma y DNI:

P1 [3 pt] Sea $f(x) = (x^2 + 1)\, e^{x-2x^2}$

a) Calcular $\lim_{x\to\pm\infty} f(x)$.

b) Demostrar que $f(x)$ tiene un máximo absoluto para un valor de x dentro del intervalo $[0, 1/2]$.

c) Demostrar que tiene un punto de inflexión en $x = 1$ y otro en el intervalo $[-1/2, 0]$.

d) Dibujar su gráfica.

P2 [3 pt] Sea $F(x) = \int_{-1}^{\tan x} dt\, \frac{t^5}{1+t^2}$. a) Estudiar el crecimiento y decrecimiento de F en el intervalo $[-\pi/4, \pi/3]$. b) Precisar el punto x de dicho intervalo para el que es máximo el valor de F.

P3 [1.5+0.5 pt] a) Determinar los **ocho** primeros términos no nulos del desarrollo en serie de potencias en x de $\cos\left(x\sqrt{1+x}\right)$.

b) ¿A qué valor convergen las series

1) $1 + \frac{\pi^2}{3!} + \frac{\pi^4}{5!} + \frac{\pi^6}{7|} + \cdots$, 2) $1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots$?

P4 [0.5+1+0.5 pt] a) Calcular, si existe, el límite $\lim_{n\to\infty} n^2 e^{\cos n} \sin\frac{1}{n^3}$.

b) Estudiar el carácter convergente o divergente de las series (diga **claramente** qué criterios o teoremas aplica para justificar su respuesta):

1) $\sum_2^\infty \frac{1}{n(\log n)^2}$, 2) $\sum_1^\infty \frac{1}{3^{\log n}}$.

c) Estudiar si converge la integral impropia $\int_1^\infty (2x-1)\, e^{-x}\, dx$.

(Empiece a escribir a la vuelta de esta página)

(P1) To start I calculate $f'(x)$ and $f''(x)$ and check that $f''(1)=0$ as stated in the problem (It is a manner of being sure that my calculations are correct!).

$$(e^{x-2x^2})' = e^{x-2x^2}(1-4x)$$
$$(e^{x-2x^2})'' = e^{x-2x^2}(-3-8x+16x^2)$$

$$f(x) = (x^2+1)e^{x-x^2}$$

$$f'(x) = e^{x-x^2}[1-2x+x^2-4x^3]$$

has 1 real root or 3 real roots (soon to say what the case is)

$$f''(x) = (x^2+1)''e^{x-x^2} + 2(x^2+1)'(e^{x-x^2})' + (x^2+1)(e^{x-x^2})''$$

$$= e^{x-x^2}[2 + 4x(1-4x) + (x^2+1)(-3-8x+16x^2)]$$

$$= e^{x-x^2}[2 + 4x - 16x^2 - 3x^2 - 8x^3 + 16x^4 - 3 - 8x + 16x^2]$$

$$= e^{x-x^2}[-1-4x-3x^2-8x^3+16x^4]$$

The problem says that there is an inflection point at $x=1$. We check that $x=1$ is a solution of

$$-1-4x-3x^2-8x^3+16x^4=0$$

$$-1-4-3-8+16=0$$ good!!

and dividing by $x-1$,

	16	-8	-3	-4	-1
1		16	8	5	1
	16	8	5	1	0

$$16x^3+8x^2+5x+1$$

we factorize

$$f''(x)=e^{x-x^2}(x-1)(16x^3+8x^2+5x+1).$$

We try on

$$16x^3+8x^2+5x+1 \qquad [*]$$

the unique possible entire or rational roots (we saw this in the lectures). These are

$$-\frac{1}{16},\ -\frac{1}{8},\ -\frac{1}{4},\ -\frac{1}{2},\ -1$$

(discarded the positive roots $\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1$ because the polynomial above [*] has no positive roots (all its coefficients are positive).

OJO!! No se puede hacer otro punto de inflexión en $x=-1/4$. Lo he concedido yo porque se podía.

Lucky we are! $x=-1/4$ is a root !! of [*]

and

$$\begin{array}{l|l} \cancel{16x^3}+8x^2+5x+1 & 4x+1 \\ \hline -\cancel{16x^3}-4x^2 & 4x^2+x+1 \\ \qquad\quad \cancel{4x^2} & \\ \qquad -\cancel{4x^2}-x & \\ \qquad\qquad 4x+1 & \end{array}$$

Si pide el examen: $16x^3+8x^2+5x+1$ continua y derivable es positiva en $x=0$, con valor 1, y negativa en $x=-1/2$ con valor $-3/2$. Por Bolzano, hay un cero en el interior de $x\in[-1/2, 0]$.

consequently,

$$f''(x)=e^{x-x^2}(x-1)(4x+1)(4x^2+x+1)$$

$$=e^{x-x^2}(x-1)(4x+1)\underbrace{\left(\left(2x+\frac{1}{4}\right)^2+\frac{15}{16}\right)}_{\text{always a positive number (I have completed squares)}}$$

Conclusion:

$x=1$ and $x=-1/4$ are candidate for inflection points (in fact they are inflection points... it is said in the problem)

[The problem affords clues because a polynomial of order 4 is difficult to handle in the exam]

Let us study the first derivative $f'(x)$,

$$f'(x) = e^{x-2x^2}(1-2x+x^2-4x^3),$$

and in particular the polynomial

$$g(x) = 1-2x+x^2-4x^3.$$

1 or 3?

How many **real** roots has the polynomial $g(x)$? The point $x=0$ is not a root, nor it is any negative number x. However $g'(x)$ indicates that $g(x)$ is a decreasing function on $\mathbb{R}$, so $g(x)$ just has one real root. And it is on the interval $[0, 1/2]$ as Bolzano shows.

Proof

$g'(x)$ is negative everywhere:

$$\begin{aligned} g'(x) &= -2+2x-12x^2 \\ &= -2(1-x+6x^2) \\ &= -12\left(\tfrac{1}{6}-\tfrac{x}{6}+x^2\right) \\ &= -12\left[\left(x-\tfrac{1}{12}\right)^2+\tfrac{23}{144}\right] \end{aligned}$$

completing squares

$x=\frac{1}{12}$ is where the inflection point of $g(x)$ is.

= always a negative number,

therefore $g(x)$ is a decreasing function. And continuous, $g(x)$ is continuous on $\mathbb{R}$. Since

$$g(0) = 1, \quad g(1/2) = \cancel{1}-\cancel{1}+\frac{1}{4}-\frac{4}{8} = -\frac{1}{2},$$

+ve number -ve number

Bolzano's theorem indicate that there is one root of $g(x)$ on the interior of $[0, 1/2]$. Also

$$g(1/4) = 1-\frac{1}{2}+\cancel{\frac{1}{16}}-\cancel{\frac{4}{64}} = \frac{1}{2}, \text{ so the root is}$$

on $\left[\frac{1}{4}, \frac{1}{2}\right]$.

Back to our problem, $f(x) = e^{x-x^2}(x^2+1)$. The function $f(x)$ is continuous and derivable on $\mathbb{R}$ (it is product of continuous and derivable functions). It is positive everywhere and the graphic of $f(x)$ never crosses the x-axis. However, it crosses the y-axis at $x=0$ with value

$$f(0)=1.$$

a) $x \to \infty$. Since $e^x < e^{x^2}$ for all x larger than 1

lim

$$x < x^2 \quad \forall x > 1$$

$$e^x < e^{x^2}$$

$$0 < \frac{(x^2+1)e^x}{e^{2x^2}} < \frac{(x^2+1)e^{x^2}}{e^{2x^2}} = \frac{x^2+1}{e^{x^2}} \xrightarrow[x\to\infty]{} 0$$

ahí.

$$\boxed{\lim_{x\to\infty} f(x) = 0}$$

b) $x \to -\infty$

$$\boxed{\lim_{x\to -\infty} f(x) = 0}$$

$$(x^2+1)e^{x-x^2} = (x^2+1)\, e^{-|x|-2x^2}$$

x negative $= -|x|$

$$= \frac{(x^2+1)}{e^{|x|+2x^2}} \xrightarrow[x\to\infty]{} 0$$

[Las exponenciales dominan a los polinomios en el ∞]

Nota del Profesor: El límite $x\to\infty$ es diferente al límite $x\to-\infty$. El primero es una "lucha intestina" de exponenciales ¿quién puede más: e^x cuando $x\to\infty$ ó e^{-x^2} cuando $x\to\infty$? Obviamente e^{-x^2}. El segundo es una lucha de un pequeño (x^2+1) con un gigante $e^{\cdots}$. Gana $e^{\cdots}$. Es una lucha de potencias con exponenciales

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The graphic of f(x) is → obviamente un máximo.

x	$-\infty$			0	x_{max}	$1/2$	1	∞
$f'(x)$		$+$	$+$	$+$	0	$-$	$-$	$-$
$f(x)$	0	$\nearrow$	$\nearrow$	$\nearrow$	$1 \nearrow$		$5/4 \searrow$	$\searrow$	0

This table summarizes the following facts:

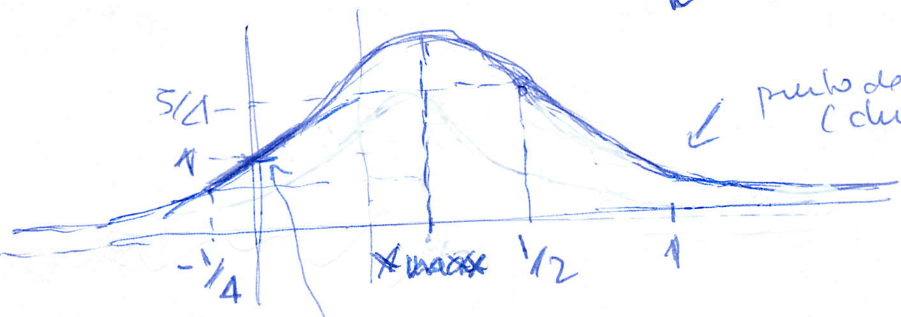
1) $\lim_{x\to\pm\infty} f(x) = 0$

2) $f(0) = 1$ ↗ flecha ↗ flecha ↘

3) $f'(0) = 1$ (positive), $f'(1/2) = -1/4$ (negativa)

4) $f(1/2) = 1 + 1/4 = 5/4$

↙ no a escala



punto de inflexion (cambio genera un dibujo)

Comment: with care!

a) At $x=0$
$(x^2+1)e^{x-x^2} \simeq 1+x$. (a 45°) : (la función se comporte como una recta a 45° que pase por 1.

b) At $x=1/2$
$$(x^2+1)\,e^{x-x^2} = \underset{5/4}{f(1/2)} + \underset{-1/4}{f'(\tfrac{0}{2})}(x-\tfrac12) + \frac{f''(\tfrac{0}{2})}{2!}(x-\tfrac12)^2 + \dots$$
with $\frac{f''(0/2)}{2!} = -15/8$

$$\simeq \frac54 - \frac14\left(x-\frac12\right) - \frac{15}{8}\left(x-\frac12\right)^2 + \dots$$

Substitute $x = 0.4$ and obtain that $f(0.4) \simeq$
$\simeq \frac54 - \frac14(-0.1) - \frac{15}{8}(0.01) = \frac54 + \frac{0.1}{4} - 1.875\times 0.01$
$= 1.25 + 0.025 - 0.01875 \simeq 1.25625$. Then
$f(x_{max})$ is a global maximum larger than $f(1/2) = \frac54 = 1.25$

(P3) a) $\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \dots \qquad R = \infty$

Substitute u by $x\sqrt{1+x}$ which is continuous and derivable at $x=0$.

$$\cos x\sqrt{1+x} = 1 - \frac{x^2}{2!}(1+x) + \frac{x^4}{4!}(1+x)^2 - \frac{x^6}{6!}(1+x)^3 + \frac{x^8}{8!}(1+x)^4 + \dots$$

$$= 1 - \frac{x^2}{2!} - \frac{x^3}{2!} + \frac{x^4}{4!} + x^5\cdot\frac{2}{4!} + x^6\left[\frac{1}{4!} - \frac{1}{6!}\right]$$

$$- x^7\frac{3}{6!} + x^8\left[\frac{1}{8!} - \frac{3}{6!}\right] + \dots$$

$$\frac{1}{4!} - \frac{1}{6!} = \frac{30-1}{6!} = \frac{29}{6!} = \frac{29}{720} \qquad 6! = 30\cdot 4! = 30\cdot 24 = 720$$

$$\frac{3}{6!} = \frac{1}{6\cdot5\cdot4\cdot2} = \frac{1}{20\cdot12} = \frac{1}{240} \qquad 8! = 56\cdot 720 = 40320$$

$$\frac{1}{8!} - \frac{3}{6!} = \frac{1 - 8\cdot7\cdot6}{8!} = \frac{-167}{8!} = \frac{-167}{40320}$$

Thus,

$$\cos x\sqrt{1+x} = 1 - \frac{x^2}{2} - \frac{x^3}{2} + \frac{x^4}{24} + \frac{x^5}{12} + \frac{29}{720}x^6 - \frac{x^7}{240} - \frac{167}{40320}x^8 \dots, \quad R = 1.$$

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b) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

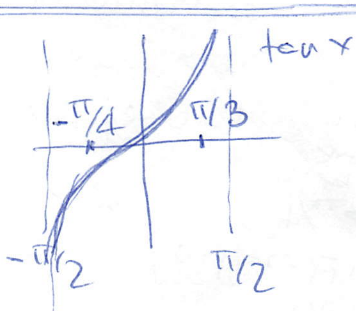
1) $1 + \frac{\pi^2}{3!} + \frac{\pi^4}{5!} + \frac{\pi^6}{7!} + \dots = \frac{\sinh \pi}{\pi}$

2) $J_0(x)$.

(P2) If $x = -\pi/4$, $F(-\pi/4) = 0 \quad \left[\int_{-1}^{-1} = 0\right]$

If $x = \pi/3$,

$$F(\pi/3) = \int_{-1}^{\sqrt{3}} dt\, \frac{t^5}{1+t^2}.$$



The function

$$H(u)=\int_{-1}^{u} dt\,\frac{t^5}{1+t^2} \quad H'(u)=\frac{u^5}{1+u^2}$$

with $u\in[-1,\sqrt{3}]$ is continuous and derivable on $[-1,\sqrt{3}]$ because $\frac{t^5}{1+t^2}$ is continuous on that interval, therefore by the FTIC (Teorema fundamental del cálculo integral)

lo es en $(-\infty,\infty)$ pero no nos hace falta tanto..., especialmente con $\tan x$ que tiene asíntotas en $x=\pm\pi/2, \pm 3\pi/2\ldots$ no, me quedo en $[-\pi/4,\pi/3]$.

$$F(x)=\int_{-1}^{\tan x} dt\,\frac{t^5}{1+t^2}=H(\tan x)$$

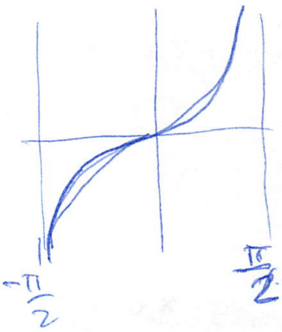
is derivable on $[-\pi/4,\pi/3]$ and

$$F'(x)=H'(\tan x)\cdot\frac{1}{\cos^2 x}$$

$$=\frac{\tan^5 x}{1+\tan^2 x}\cdot\frac{1}{\cos^2 x} \qquad \text{“}\ \frac{1}{\cos^2}=\frac{\cos^2+\sin^2}{1}=1+\tan^2$$

$$=\frac{\tan^5 x\cdot\cancel{(1+\tan^2 x)}}{\cancel{1+\tan^2 x}}$$

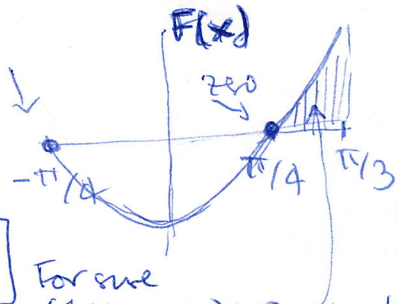
$\tan^5 x = F'(x)$



thus, according to the sign of $F'(x)$: $-$ if x is on $[-\frac{\pi}{4},0]$, $+$ if x is on $[0,\pi/3]$, the function $F(x)$ decreases on $[-\frac{\pi}{4},0]$ and increases on $[0,\pi/3]$. It has a minimum when $x=0$.

[Observe that $F(x)$ differentiable then $F(x)$ is continuous on $[-\pi/4,\pi/3]$ and has maximum ~~and minimum~~ on $[-\pi/4,\pi/3]$ as the Extreme Value theorem states].

then the maximum has to be at $\pi/4$ [but $F(-\pi/4)=0$] or at $\pi/3$ [$F(\pi/3)$. what is the value of $F(\pi/3)$?]



For sure that $F(\pi/3)>0$: that area.

[next page...]

$$F(0)=\int_{-1}^{0} dt\,\frac{t^5}{1+t^2}=\text{negative}$$

minimus inventeas

$-3\cdot 2\cdot 1=-6$ (la integral) area.

the absolute minimum is at $x=0$ but the absolute maximum is at the edge $x=\pi/3$, and it is equal to $F(\pi/3)$ where

$$F(\pi/3) = \int_{-1}^{\sqrt{3}} = \underbrace{\int_{-1}^{1}}_{\substack{0 \text{ by parity} \\ \equiv F(\pi/4) \equiv \int_{-1}^{1} dt \frac{t^5}{1+t^2}}} + \int_{1}^{\sqrt{3}} \frac{t^5}{1+t^2}\,dt$$

$$= \int_{1}^{\sqrt{3}} dt \underbrace{\frac{t^5}{1+t^2}}_{+\text{ve on } [1,\sqrt{3}]}.$$

observe that [*] indicates that $1 < F(\pi/3) < 2$

We are **not** asked explicitly in the problem for the value of $F(\pi/3)$, but an estimate is mandatory. The true value is $1+\frac{\log 2}{2}$.

$$\frac{t^5}{1+t^2} = \frac{t^5+t^3-t^3}{1+t^2} = t^3 - \frac{t^3}{1+t^2} = t^3 - \frac{(t^3+t)-t}{1+t^2}$$

$$= t^3 - t + \frac{t}{1+t^2}$$

$$\int_1^{\sqrt{3}} \frac{t^5}{1+t^2} = \left[\frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2}\log(1+t^2)\right]_1^{\sqrt{3}} = \frac{9}{4} - \frac{1}{4} - \frac{3}{2} + \frac{1}{2} + \frac{1}{2}\log 4 - \frac{1}{2}\log 2$$

$$= 1 + \frac{1}{2}\log 2 > 0 \qquad \log 2 \simeq 0.69.$$

A la vista del resultado, es mejor hacer la integral con el cambio $u=t^2$.

(P4) a) $\sin x \simeq x$ when $x \to 0$ and, consequently, $\sin\frac{1}{n^3} \simeq \frac{1}{n^3}$ when $n\to\infty$.

then, $$n^2 e^{\cos n} \sin\frac{1}{n^3} \underset{n \text{ large}}{\simeq} \frac{n^2}{n^3} e^{\cos n} = \frac{e^{\cos n}}{n}$$

that tends to zero when $n\to\infty$:

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[*] $t^3 - t < \frac{t^5}{1+t^2} < \frac{t^5}{t^2} = t^3$; $\int_1^{\sqrt{3}} t^3 = \frac{9}{4} - \frac{1}{4} = 2$, $\int_1^{\sqrt{3}} t = \left[\frac{t^2}{2}\right]_1^{\sqrt{3}} = \frac{3}{2} - \frac{1}{2} = 1$

$2 - 1 < F(\pi/3) < 2$ $\quad \boxed{1 < F(\pi/3) < 2}$

$$\lim_{n\to\infty} n^2 e^{\cos n} \cdot \sin\frac{1}{n^3} = 0 \qquad \left(\frac{\text{acotada}}{\infty}=0 \quad \frac{\text{bounded}}{\infty}=0\right)$$

b) 1) $\sum_2^\infty \frac{1}{n(\log n)^2}$: Convergent by the integral test.

$f(x)=\frac{1}{x(\log x)^2}$: positive and decreasing function for $x\geq 2$. The series has the same character as

$$\int^\infty \frac{dx}{x(\log x)^2} = -\left[\frac{1}{\log x}\right]^\infty = 0,,$$

then, convergent. El límite inferior no importa.

2) $\sum_1^\infty \frac{1}{3^{\log n}} = \frac{1}{1}+\frac{1}{3^{\log 2}}+\frac{1}{3^{\log 3}}+\dots$

$$3^{\log n} = (e^{\log 3})^{\log n} = (e^{\log n})^{\log 3}$$
$$= n^{\log 3} \simeq n^{1.1}$$

,, $\log 3 \simeq 1.1$

$\log 3 = x$

$e^x = 3$

e is 2.7.

$\boxed{3>e \quad \log 3 > \log e = 1.}$ → mismo razonamiento (conclusión)

Conclusion: the series is

$$\sum_1^\infty \frac{1}{3^{\log n}} = \sum_1^\infty \frac{1}{n^{\log 3}},$$

which is a p-series with $p=\log 3>1$, therefore, convergent too.

Por desigualdades: $e^2>3 \quad n^2 = e^{2\log n} > 3^{\log n}$

c) It is a very simple integral, so just find the primitive of $(2x-1)e^{-x}$ [integrating by parts or using unknown coefficients] and after take the limit

$$\lim_{c\to\infty}\int_1^c (2x-1)e^{-x}\,dx \qquad \frac{1}{n^2} < \frac{1}{3^{\log n}}$$

Integrering by parts:

$$\int x e^{-x} dx = -\int x \, d(e^{-x}) = -xe^{-x} - \int e^{-x}$$
$$= -xe^{-x} + e^{-x} = e^{-x}(-\cancel{x}+1),$$

Then

$$\int (2x-1) e^{-x} = (-2x-1) e^{-x} + C_1$$

Unknown coefficients: determine constants a, b

$$\int (2x-1) e^{-x} = e^{-x} (ax+b)$$

such that

$$\left[e^{-x}(ax+b)\right]' = (2x-1) e^{-x}.$$
$$\parallel$$
$$e^{-x}(-ax+a-b) = (2x-1) e^{-x} \quad \parallel \quad a=-2, \; b=-1$$

conclusion:

$$\int_1^{\infty} (2x-1) e^{-x} = \lim_{c\to\infty} \int_1^c (2x-1) e^{-x}$$
$$= \lim_{c\to\infty} \left[(-2x-1) e^{-x}\right]_1^c$$
$$= \underbrace{\lim_{c\to\infty}\left[(-2c-1) e^{-c}\right]}_{= 0} + 3e^{-1}$$
$$= \frac{3}{e}.$$

The integral is convergent.