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Solutions: | Parcial Nov 2020

Matemáticas

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2020-2021

Parcial de Matemáticas: grupo F

Nombre y Apellidos:

Firma y DNI:

Nota: En esta prueba no se permiten libros ni apuntes ni **calculadora**. La nota total de este examen son 10 puntos.

P1 [10/3 pt] Sea $f(x) = |e^x - 3|$.

a) Esbozar su gráfica.

b) Determinar los puntos en los que $f(x) \geq 1$

c) Determinar, si existe, la función inversa en el intervalo $[2, 4]$

d) Probar que f es inyectiva en el intervalo $[4, \infty)$ y calcular $g'(60)$ siendo $g(x) = f^{-1}(x)$.

e) Determinar la ecuación de un recta tangente a $f(x)$ en el punto $x = -1$.

P2 [10/3 pt] Hallar, si existen, los siguientes límites:

a) $\lim_{n\to\infty}(\sqrt[3]{n^4 - n^2} - n\, e^{\arctan n})$, b) $\lim_{n\to\infty} \dfrac{2n - \sqrt{n^3}}{3n + \log n}$,

c) $\lim_{x\to 0} \dfrac{6x - \sin 2x}{2x + 3\sin 4x}$, d) $\lim_{x\to\infty} \tanh(\cosh x - \cos x)$.

P3 [5/3+5/3 pt] Elegir dos opciones (ab, ac ó bc) de las tres siguientes:

a) Probar por inducción si es cierto que para todo n entero positivo con $n \geq 3$ se cumple que

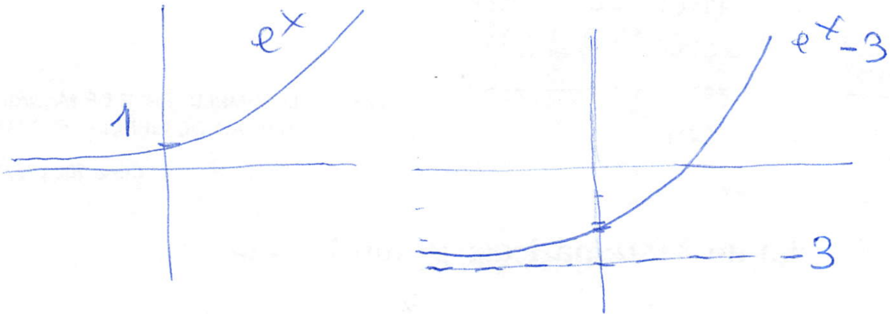
$$\binom{n+2}{3} - \binom{n}{3} = n^2.$$

b) Derminar si las siguiente funciones son pares, impares o no tienen una paridad definida: i) $f(x) = \log \dfrac{1+x}{1-x}$, ii) $f(x) = \sqrt{1 + x + x^2} - \sqrt{1 - x + x^2}$, iii) $f(x) = \log(x + \sqrt{1 + x^2})$.

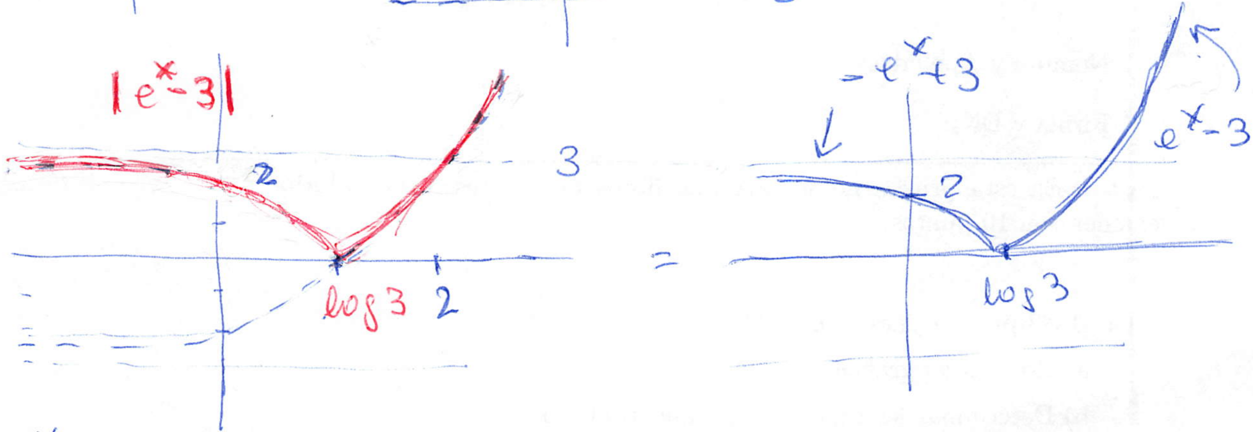
c) Utilizando Delta-Epsilon probar que el límite de la función $f(x) = x^4$ en $x = 2$ es 16.

(Empiece a escribir a la vuelta de la página, por favor)

P1



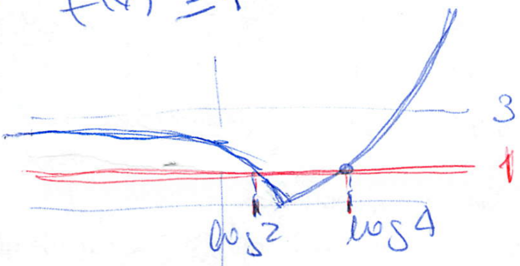
a)



$e^x=3$,

$x=\log 3 \simeq 1.103$ (without pocket calculator) (ver abajo)

b) $f(x)\geq 1$



$e^x-3\geq 1$, $\quad e^x\geq 4$, $\quad x\geq \log 4$

$-e^x+3\geq 1$ $\quad 2\geq e^x$ $\quad \log 2\geq x$

$f(x)\geq 1$ when $\boxed{x\leq \log 2 \text{ or } \log 4\leq x}$

$\log 2\simeq 0.69$ (de memoria: es uno famoso)

$\log 4 = 2\log 2 \simeq 1.38$

$e^x=3$, $\quad e^x-3=0$

$x-\dfrac{e^x-3}{e^x}=x-1+\dfrac{3}{e^x}$. Take $x=1$: $\log 3 \simeq \dfrac{3}{e}=\dfrac{3}{2.72}=\dfrac{300}{272}$

$\simeq 1.103$

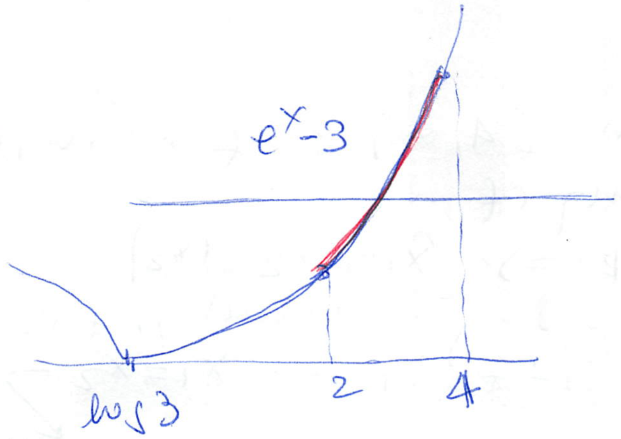
$\simeq 1.10$

300 | 272

280   1.103

  800

c)



On the interval [2,4] the inverse function of $|e^x-3| = e^x-3$ does exist because e^x-3 is one-to-one (injective) and onto, in particular:

restricted to this interval

$$\text{Dom}\,(e^x-3) = [2,4]$$
$$R\,(e^x-3) = [e^2-3,\ e^4-3]$$

and the inverse $g(x)$ of $f(x)$ has

$$\text{Dom}\ g(x) = [e^2-3,\ e^4-3]$$
$$R\ g(x) = [2,4].$$

Besides,

$$y = e^x-3$$
$$x = e^y-3$$
$$x+3 = e^y$$
$$\log(x+3) = y \qquad // \qquad g(x) = \log(x+3).$$

Checking:

$$g(f(x)) = g(e^x-3) = \log(e^x-\cancel{3}+\cancel{3}) = x$$
$$f(g(x)) = f(\log(x+3)) = e^{\log(x+3)}-3 = x$$

$$e^2-3 \simeq 2.72^2-3 = 7.3984-3 = 4.398 \simeq 4.39.$$
$$e^4-3 \simeq 51.74$$

d) on $[4,\infty)$, $g(x)$ is also $\log(x+3)$

and $$g'(60) = \left.\frac{1}{x+3}\right|_{x=60} = \frac{1}{63}. \qquad \boxed{g'(60) = \frac{1}{63}}$$

Also:

Proving injection on $[4,\infty)$: f is injective on $[4,\infty)$ if and only if

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

for all x_1, x_2 in $[4,\infty)$. Equivalently, f is injective if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Like this

$$f(x_1) = f(x_2)$$

$$e^{x_1} - \cancel{3} = e^{x_2} - \cancel{3} \quad \text{, take logs}$$

$$x_1 = x_2$$

$\therefore$ f is injective on $[4,\infty)$

Nótese que no hemos especificado allí si x_1, x_2 son el mismo número o no. No hemos dicho nada. Así se prueba inyección.

Aparte del test de la línea horizontal.

También puede usted calcular $g'(60)$ así: sin $g(x)$: sólo con $f(x)$ y cambiando $x \leftrightarrow y$

$$e^x - 3 = y$$

$$e^y - 3 = x \quad \text{, the inverse}$$

Derive w.r.t x:

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x+3}$$

✓ same result.

e) $f(x) = -e^x + 3$. At $x = -1$, $f(-1) = -e^{-1} + 3$

$f'(x) = -e^x$, $f'(-1) = -e^{-1}$

$$y - (3 - e^{-1}) = -e^{-1}(x+1)$$

$$\boxed{y = -e^{-1}(x+2) + 3}$$

Note: Also: a function that is strictly increasing (if $x_1 < x_2$ implies $f(x_1) < f(x_2)$) is injective

(2) a) $\lim_{n\to\infty} \sqrt[3]{n^4-n^2} - n e^{\arctan n} = \infty$ (as $n^{4/3}$)

b) $\lim_{n\to\infty} \frac{2n-n^{3/2}}{3n+\log n} = -\infty$ (as $-\frac{1}{3}n^{1/2}$)

c) $\lim_{x\to 0} \frac{6x-\sin 2x}{2x+3\sin 4x} = \frac{2}{7}$

d) $\lim_{x\to\infty} \tanh(\cosh x - \cos x) = 1$

a) $\sqrt[3]{n^4-n^2} - n e^{\arctan n}$

$= \sqrt[3]{n^4(1-\frac{1}{n^2})} - n e^{\arctan n}$

$\underset{n \text{ large}}{\simeq} n^{4/3} - n e^{\pi/2} \simeq n^{4/3}\left(1 - \frac{e^{\pi/2}}{n^{1/3}}\right) \simeq n^{4/3}$

tends to 0

$\simeq n^{4/3} \xrightarrow[n\to\infty]{} \infty.$

b) $\frac{2n-n^{3/2}}{3n+\log n} = \frac{n^{3/2}\left(\frac{2}{n^{1/2}}-1\right)}{3n\left(1+\frac{\log n}{n}\right)} \simeq \frac{-n^{3/2}}{3n} = -\frac{n^{1/2}}{3}$

tends to 0 when $n\to\infty$

$\xrightarrow[n\to\infty]{} -\infty$

c) $\lim_{x\to 0} \frac{6x-\sin 2x}{2x+3\sin 4x} = \lim_{x\to 0} \frac{6x-2x}{2x+12x} = \frac{2}{7}$

d) $\tanh\left(\cosh x\left(1-\frac{\cos x}{\cosh x}\right)\right) \underset{x \text{ large}}{\simeq} \tanh(\cosh x)$

$\frac{\text{acotada}}{\infty} \to 0$

$\simeq \tanh \infty \xrightarrow[x\to\infty]{} 1$

a) $\binom{n+2}{3} - \binom{n}{3} = n^2$. True or false ?
$n \geq 3$

checking for $n = 3, 4$

n	$\binom{n+2}{3} - \binom{n}{3}$	n^2	
3	$\binom{5}{3} - \binom{3}{3}$	9	ok
4	$\binom{6}{3} - \binom{4}{3}$	16	ok

$\binom{5}{3} - \binom{3}{3} = \frac{5!}{3!2!} - 1 = \frac{5 \cdot 4}{2} - 1 = 10 - 1 = 9$

$\binom{6}{3} - \binom{4}{3} = \frac{6!}{3!3!} - 4 = \frac{\cancel{6} \cdot 5 \cdot 4}{\cancel{3!}} - 4 = 20 - 4 = 16$

Assume it is true for n, $\binom{n+2}{3} - \binom{n}{3} = n^2$,
is it then true that
$$\binom{n+3}{3} - \binom{n+1}{3} - (n+1)^2$$
is zero?

using the Pascal rule
$$\binom{m}{n} + \binom{m}{n+1} = \binom{m+1}{n+1}$$

1 5 10 10 ..... $m=5$
1 6 15 .... $m=6$

$\binom{5}{1} + \binom{5}{2} = \binom{6}{2}$

we have
$$\binom{n+3}{3} = \binom{n+2}{3} + \binom{n+2}{2}$$
$$\binom{n+1}{3} = \binom{n}{3} + \binom{n}{2},$$

then

$$\binom{n+3}{3} - \binom{n+1}{3} - (n+1)^2$$

$$= \underbrace{\binom{n+2}{3} + \binom{n+2}{2}}_{\text{Pascel}} - \underbrace{\left(\binom{n}{3} + \binom{n}{2}\right)}_{\text{Pascel}} - (n+1)^2$$

$$= \underbrace{\binom{n+2}{3} - \binom{n}{3}}_{\substack{\shortparallel \\ n^2 \\ \text{(induction)}}} + \binom{n+2}{2} - \binom{n}{2} - (n+1)^2$$

$$= n^2 + \frac{(n+2)!}{2!\,n!} - \frac{n!}{2!(n-2)!} - (n+1)^2$$

$$= n^2 + \frac{1}{2}(n+2)(n+1) - \frac{n(n-1)}{2} - (n+1)^2$$

$$= n^2 + \frac{1}{2}\left[\cancel{n^2} + 3n + 2 - \cancel{n^2} + n\right] - (n+1)^2$$

$$= n^2 + 2n + 1 - (n+1)^2$$

$$= 0$$

La relació dada es certa $\forall\, n \geq 3$.

Explanation.

```
            1
           1 1
          1 2 1
         1 3 3 1
        1 4 6 4 1
       1 5 10 10 5 1
      1 6 15 20 15 6 1
     1 7 21 35 35 21 7 1
    1 8 28 56 70 56 28 8 1
```

$10 - 1 = 9$
$20 - 4 = 16$
$35 - 10 = 25$
$56 - 20 = 36$
$\vdots$

b) All are odd functions since $f(x) = -f(-x)$. ($f(-x) = -f(x)$, vaja!)

1) $f(x) = \log \frac{1+x}{1-x}$

$f(-x) = \log\left(\frac{1-x}{1+x}\right) = \log(1-x) - \log(1+x)$

$= -\log\left(\frac{1+x}{1-x}\right)$

$= -f(x)$. odd

2) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$

$f(-x) = \sqrt{1-x+x^2} - \sqrt{1+x+x^2} = -f(x)$

odd

3) $f(x) = \log\left(x + \sqrt{1+x^2}\right)$, (*)

$f(-x) = \log\left(-x + \sqrt{1+x^2}\right)$,

Calculate $f(x) + f(-x)$

$= \log\left(x+\sqrt{1+x^2}\right) + \log\left(-x+\sqrt{1+x^2}\right)$

$= \log\left[\left(x+\sqrt{1+x^2}\right)\left(-x+\sqrt{1+x^2}\right)\right]$

$= \log\left[-\cancel{x^2} + 1 + \cancel{x^2}\right]$

$= \log 1$

$= 0$,

then $f(x) = -f(-x)$. Hence odd

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c) Prueba $\lim_{x\to 2} x^4 = 16$

Find δ : $0 < |x-2| < \delta$

$|x^4 - 16| = |(x-2)(x^3 + 2x^2 + 4x + 8)|$

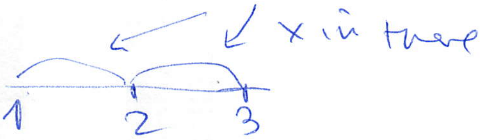
$x^4 - a^4 = (x-a)(x^3 + x^2 a + x a^2 + a^3)$

$x^4 + \cancel{x^3 a} + \cancel{x^2 a^2} + \cancel{x a^3}$ $- a^4$

$- \cancel{x^3 a} - \cancel{x^2 a^2} - \cancel{x a^3}$

(*) En realidad, esto aún no tienen que saberlo, $\log\left(x+\sqrt{1+x^2}\right) = \text{arc}~~sin~~\text{h}\,x$, and arcsinh x es impar, como lo es arcsin x, arctan x, arctanh x.

Assume $\delta = 1$



$$|x-2| < 1$$
$$1 < x < 3$$
$$|x| < 3$$

$$|x^4-16| = \underbrace{|x-2|}_{<1}\underbrace{|x^3+2x^2+4x+8|}_{<\cancel{27}+18+12+8=65} < 1 \cdot 65$$

Consider now $|x-2|<\delta$ we can take $|x|<3$ too because $|x|<3$ is valid if $\delta = 1$ or less then 1:

$$|x^4-16| < \delta \cdot 65 = \varepsilon,$$

then $$\delta = \min\left\{1, \frac{\varepsilon}{65}\right\}.$$

L

Limpro: For all ε choose

$$\delta = \min\left\{1, \frac{\varepsilon}{65}\right\}.$$

Assume $0 < |x-2| < \delta$, then,

$$|x^4-16| = |x-2||x^3+2x^2+4x+8| < \delta \cdot 65 = \varepsilon.$$

$\boxtimes$