Solución: 2º Parcial Enero 2021
Matemáticas 2020-2021

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2020-2021

Segundo Examen Parcial de Matemáticas-F

Nombre y Apellidos:

Firma y DNI:

P1 [0.5 punto] Hallar, si es posible, el valor del número c que satisface el *Teorema del Valor Medio* aplicado a $f(x) = 1 + \sqrt{x-1}$ en $[2, 9]$.

P2 [1 punto] Determinar el número de raíces que tiene la ecuación $3x - 2 + \cos\frac{\pi x}{4} = 0$ en $[0, 10]$.

P3 [1 punto] Sumar la serie $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\sqrt{2}}{2}\right)^n$

P4 [2 puntos] a) Determinar el conjunto de todos los valores de x para los que la serie $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(3x-1)^n}{(2^n+1)(n^2+1)}$ converge

P5 [1.5+0.5 puntos] Calcular: a) el coeficiente de x^5 en el desarrollo en serie de potencias de x de

$$\frac{x^2}{4x^3 - 3x - 1}.$$

b) el coeficiente de x^4 en $\tan 2x + \cos 2x$.

P6 [1.5] Estudiar el crecimiento y decrecimiento de $H(x) = \displaystyle\int_{e^{x^2}}^{4} dt\,\frac{\log t}{t+t^2}$ en $[-1, 1]$.

P7 [1.5] Calcular $\int_0^{\pi/2} dx\,\dfrac{\sin^3 x}{8\cos^2 x + 9\sin^2 x}$

P8 [0.5 puntos] ¿A qué valor convergen las series

a) $1 - \frac{4^2}{3!} + \frac{4^4}{5!} - \frac{4^6}{7|} + \cdots$

b) $\log 3 + \frac{(\log 3)^2}{2!} + \frac{(\log 3)^3}{3!} + \frac{(\log 3)^4}{4!} + \cdots$?

P1 $f(x) = 1+\sqrt{x-1}$ es continua en $[2,9]$, derivable en $(2,9)$, por lo tanto cumple las hipótesis del Teorema del Valor Medio (MVT).
Sin ni siquiera hacer una gráfica de $1+\sqrt{x-1}$
"existe c en $[2,9]$ tal que

$$f'(c) = \frac{f(9)-f(2)}{9-2}."$$

En nuestro caso,

$$f(x) = 1+\sqrt{x-1}, \qquad f'(x) = \frac{1}{2\sqrt{x-1}}$$

$$f(9) = 1+\sqrt{8}, \qquad f(2) = 2,$$

luego

$$\frac{1}{2\sqrt{c-1}} = \frac{\sqrt{8}-1}{7}$$

$$= \frac{\sqrt{8}-1}{7} \cdot \frac{\sqrt{8}+1}{\sqrt{8}+1} =$$

$$= \frac{\cancel{7}}{\cancel{7}(\sqrt{8}+1)}$$

ó

$$2\sqrt{c-1} = \sqrt{8}+1.$$

Elevando al cuadrado

$$4(c-1) = (\sqrt{8}+1)^2 = 9+2\sqrt{8}$$

$$c-1 = \frac{9}{4}+\frac{\sqrt{8}}{2} =$$

$$\boxed{c = \frac{13}{4}+\sqrt{2}}$$

Nota: Se calcule (o no) en aritmética decimal pueden estar seguros de que

$$c = 3+\frac{1}{4}+\sqrt{2}$$

está en $[2,9]$ por el MVT. ($c \simeq 4.66$ porque $\sqrt{2} \simeq 1.4142$).

P2 $f(x) = 3x - 2 + \cos\frac{\pi x}{4}$ tiene exactamente una raíz en $[0,10]$. Y es la única raíz de $f(x)$ en todo $\mathbb{R}$. No hay más.

Prueba: $f(x)$ es continua en todo $\mathbb{R}$ por ser suma de funciones continuas en todo $\mathbb{R}$. Además es derivable en todo $\mathbb{R}$ también.

Como

$$\underset{0}{|}\longrightarrow\underset{10}{|}$$

$f(0) = -1$, $f(10) = 28$, la función
$\uparrow$ negativo $\quad$ $\uparrow$ positivo

$f(x)$ [lo dice el teorema de Bolzano] que es continua pasa necesariamente por 0, o sea, existe al menos un c en $[0,10]$ tal que $f(c)=0$, c es la raíz.

Bolzano

Pero $f(x)$ es creciente en todo $\mathbb{R}$,

$$f'(x) = 3 - \frac{\pi}{4}\sin\frac{\pi x}{4}$$

ya que

$$f'(x) > 0 \quad \forall x \text{ en } \mathbb{R}.$$

De ahí que tenga exactamente sólo una raíz y que ésta se halle en $[0,10]$ como hemos visto.

creciente

Veamos que $f'(x) > 0$:

$$-1 \le -\sin\frac{\pi x}{4} \le 1$$

$$\underbrace{3 - \frac{\pi}{4}}_{\text{a +ve number}} \le 3 - \frac{\pi}{4}\sin\frac{\pi x}{4} \le \underbrace{3 + \frac{\pi}{4}}_{\text{+ve number}}$$

multiply by $\pi/4$, add 3.

Hence, $f'(x) > 0$.

(P4) The series is geometric of ratio

$$0 < \frac{\sqrt{2}}{2} < 1,$$

thus convergent to

$$S = \frac{\sqrt{2}}{2}\left[1 + \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2}\right)^2 + \cdots\right]$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{1-\frac{\sqrt{2}}{2}} \qquad \text{“} \frac{a_1}{1-r} \text{”}$$

$$= \frac{\sqrt{2}}{2-\sqrt{2}} = \frac{1}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{\sqrt{2}+1}{1}.$$

<u>conclusion</u>: The sum is $\boxed{1+\sqrt{2}}$

Also

$$S = \frac{\sqrt{2}}{2}\Big[1 + \underbrace{\frac{\sqrt{2}}{2}\Big(1 + \frac{\sqrt{2}}{2} + \cdots\Big)}_{S}\Big],$$

$$S = \frac{\sqrt{2}}{2}\left[1 + S\right],$$

that gives

$$S = 1+\sqrt{2}.$$

(P5) With the change

$$y \equiv 3x-1,$$

the original series reduces to the power series

$$\sum_{0}^{\infty} (-1)^n \frac{y^n}{(2^n+1)(n^2+1)}. \qquad [*]$$

Consider y a fixed but arbitrary <u>number</u> and apply the ratio test to [*] with

$$b_n \equiv (-1)^n \frac{y^n}{(2^n+1)(n^2+1)}.$$

From

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{y^{n+1}}{y^n} \cdot \frac{(2^n+1)(n^2+1)}{(2^{n+1}+1)\left((n+1)^2+1\right)}\right|$$

$b_{n+1} \equiv$ bomb eneme's uno.

deduce that

$$\rho = \lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = |y| \underbrace{\lim_{n\to\infty} \frac{(2^n+1)(n^2+1)}{(2^{n+1}+1)\left[(n+1)^2+1\right]}}_{= \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \frac{1}{2}} .$$

n large:

$$2^n+1 \simeq 2^n$$
$$(n+1)^2 \simeq n^2$$

The series [*] with $\rho = \frac{|y|}{2}$ is

1) Absolutely convergent if $|y|<2$
2) Divergent if $|y|>2$
3) $y=2,-2$ needs further study.

Notice that $|y|<2$ is

$-\frac{1}{3} < x < 1$ Absolutely convergent $\Rightarrow$ convergent

$x>1$ or $x<-\frac{1}{3}$ Divergent

When $x = -\frac{1}{3}, 1$ the series are

$$\sum_0^\infty \frac{(-1)^n(-2)^n}{(2^n+1)(n^2+1)}, \quad \sum_0^\infty \frac{(-1)^n 2^n}{(2^n+1)(n^2+1)}, \text{ respectively,}$$

that have the same character because

$$\sum_0^\infty \frac{2^n}{(2^n+1)(n^2+1)} = \text{convergent. [**]}$$

one only studies absolute convergence. The series [**] converges:

1) By comparison to the limit behaves as $\sum^\infty \frac{1}{n^2}$ = convergent = Basilea series.
2) By inequalities:

$(2^n+1)(n^2+1) > 2^n \cdot n^2$ for all n

$$0 < \frac{2^n}{(2^n+1)(n^2+1)} < \frac{2^n}{2^n \cdot n^2} = \frac{1}{n^2}.$$

$\boxed{\begin{array}{l}\text{Conclusion} \quad -\frac{1}{3} \le x \le 1 \quad \text{convergent (absolutely)} \\ x \text{ in } (-\infty, -\frac{1}{3}) \text{ or } (1, \infty) : \text{divergent}\end{array}}$

[Also is valid the calculation of

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = 2$$

with $a_n = \frac{(-1)^n}{(2^n+1)(n^2+1)}$

]

(P5) a) Just divide (long division = à la rule)

$$\begin{array}{l|l} x^2 & -1-3x+4x^3 \\ \hline \cancel{-x^2} - \cancel{3x^3} + 0 \quad 4x^5 & -x^2+3x^3-9x^4+23x^5 \\ \cancel{3x^3} \quad \cancel{9x^4} \quad 0 \quad -12x^6 & \\ \cancel{-9x^4} - 27x^5 \quad 0 \quad 28x^7 & \\ -23x^5 \quad 69x^6 & \\ \quad 57x^6 & \end{array}$$

The result is

$$\frac{x^2}{-1-3x+4x^3} = -x^2+3x^3-9x^4+23x^5+57x^6+\dots$$

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b) $\tan 2x + \cos 2x$

$\tan 2x$ is an odd function and has only odd powers: $x, x^3, x^5, \dots$

$\cos 2x$ is an even function and has <u>even</u>

powers, so the $\underline{\text{even}}$ (par) powers of

$\tan 2x + \cos 2x$

come from $\cos 2x$ only,

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots$$

the coefficient of x^4 in $\tan 2x + \cos 2x$ is (red)

$$\frac{2^4}{4!} = \frac{\cancel{2}\cdot\cancel{2}\cdot 2\cdot\cancel{2}}{\cancel{4}\cdot 3\cdot\cancel{2}} = \frac{2}{3}$$

P6 $f(x) = \dfrac{\log x}{x+x^2}$ is continuous for all x in $(0,\infty)$ and the Fundamental Theorem of Infinitesimal calculus (part I) states that

$$F(y) \equiv \int_4^y dt\, \frac{\log t}{t+t^2}$$

is derivable (therefore continuous) when $y \in (0,\infty)$. That is precisely our case since

$$y = e^{x^2} \text{ is on } (0,\infty).$$

Besides

$$F'(y) = \frac{\log y}{y+y^2} \text{ on } (0,\infty) \ni y$$

[y, x who cares? Da lo mismo. Pongo y para que ustedes vean que $y = e^{x^2}$ aquí.]

In our case

$$H(x) = -\int_4^{e^{x^2}} dt\, \frac{\log t}{t+t^2} = -F(e^{x^2}).$$

From

$$H(x) = -F(e^{x^2})$$

we have that

$$H'(x) = -F'(e^{x^2})\, e^{x^2} \cdot 2x$$

esta ' indica con respecto al argumento.

$$= -\frac{\log(e^{x^2})}{e^{x^2} + (e^{x^2})^2} \cdot e^{x^2} \cdot 2x$$

$$= \frac{-x^2 \cdot \cancel{e^{x^2}}}{\cancel{e^{x^2}}[1+e^{x^2}]} \cdot 2x$$

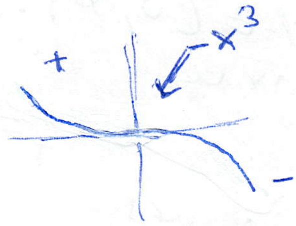
$$= \frac{-2x^3}{1+e^{x^2}}$$

Notice that

$$H'(x) = \frac{-2x^3}{1+e^{x^2}}$$

↑ always a positive number

behaves in sign as "$-x^3$"



and $H(x)$ increases on $[-1,0]$ and decreases on $[0,1]$.

P7

$$I = \int_0^{\pi/2} \frac{\sin^3 x\, dx}{8\cos^2 x + 9\sin^2 x}$$

$$= \int_0^{\pi/2} \frac{\sin^3 x\, dx}{8 + \sin^2 x}$$

← integrand is odd in "$\sin x$" and the change

$u = \cos x$

should work.

$$u = \cos x$$
$$du = -\sin x\,dx,$$

thus

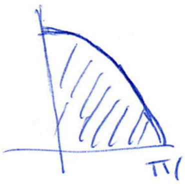
$$\frac{\sin^3 x\,dx}{8+\sin^2 x} = \frac{\sin^2 x \cdot \sin x\,dx}{8+(1-\cos^2 x)}$$
$$= \frac{(1-\cos^2 x)\sin x\,dx}{9-\cos^2 x}$$
$$= -\frac{(1-u^2)\,du}{9-u^2}$$

and

$$I = \int_0^{\pi/2} \frac{\sin^3 x\,dx}{8+\sin^2 x} = -\int_1^0 \frac{(1-u^2)}{9-u^2}\,du$$
$$= \int_0^1 du\,\frac{1-u^2}{9-u^2}.$$

$$\frac{1-u^2}{9-u^2} = \frac{9-u^2-8}{9-u^2} = 1-\frac{8}{9-u^2} = 1+\frac{8}{u^2-9}$$
$$= 1+\frac{4}{3}\left(\frac{1}{u-3}-\frac{1}{u+3}\right)$$

$$I = \int_0^1 \frac{1-u^2}{9-u^2} = \underbrace{\int_0^1 1\,dx}_{=1} + \frac{4}{3}\int_0^1 du\left(\frac{1}{u-3}-\frac{1}{u+3}\right)$$
$$= 1+\frac{4}{3}\Big[\log(u-3)-\log(u+3)\Big]_0^1$$
$$= 1+\frac{4}{3}\left[\log\left|\frac{u-3}{u+3}\right|\right]_0^1$$
$$= 1+\frac{4}{3}\left(\log\frac{2}{4}-0\right)$$
$$= 1-\frac{4}{3}\log 2$$



$\pi/2$

$$\log\frac{2}{4} = \log\frac{1}{2} = -\log 2$$

$$\boxed{I = 1-\frac{4}{3}\log 2} \simeq 0.08$$

$\log 2 \simeq 0.69$
$\log 2/3 \simeq 0.23$, $I = 1 - \underset{0.92}{0.23 \times 4} = 0.08.$

P8

a) $$1-\frac{4^2}{3!}+\frac{4^4}{5!}-\frac{4^6}{7!}+\dots = \frac{\sin 4}{4}$$

Denominators 1, −3!, 5!, −7! … like sin x

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots$$

converges for all x, in particular for $x=4$ and

$$\sin 4 = 4-\frac{4^3}{3!}+\frac{4^5}{5!}-\frac{4^7}{7!}+\dots$$

b) $$\log 3+\frac{(\log 3)^2}{2!}+\frac{(\log 3)^3}{3!}+\dots = e^{\log 3}-1 = 3-1 = 2$$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$ converges for all x and in particular for $x=\log 3$.