UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2021-2022

Examen Final Extraordinario de Matemáticas F

Nombre y Apellidos:

Firma y DNI:

Nota: En esta prueba no se permiten libros o apuntes ni **calculadora**. La nota total de este examen son 10 puntos.

P1 [2pt] Sea la función

$$f(x) = \arctan(\log|x|),\ x \neq 0, \qquad f(0) = -\pi/2.$$

1) Probar que f es continua en toda la recta real.

2) Probar que f es derivable para todo $x \neq 0$ y calcular $f'(x)$.

3) Esbozar la gráfica de $f(x)$.

P2 [0.4pt+0.6pt] a) Calcular el resto de las siguientes razones trigonométricas: 1) $\tan x = -3/2$, $\pi/2 < x < \pi$, 2) $\csc x = -4/3$, $3\pi/2 < x < 2\pi$

b) Encontrar la asíntota oblicua de $f(x) = \dfrac{x^3 - 6x + 1}{x + 2}$

P3 [1pt+1pt+1pt] a) Sabiendo que $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$, calcular $\sum_{n=1}^{\infty} \dfrac{5n+6}{n^2(n+1)}$.

Nota: No se trata de decir si la serie es convergente o no, ni de estimar su suma. Se trata de calcular **exactamente** su suma.

b) Determinar la convergencia o divergencia de las series: 1) $\sum_{n=2}^{\infty}(n^{1/n} - 1)^n$, 2) $\sum_{n=1}^{\infty} \dfrac{1}{n^{1+1/n}}$. Especifique claramente los tests empleados para responder.

c) Encontrar todos los valores de x que satisfacen $\sum_{n=1}^{\infty} n x^n = 1/2$.

P4 [0.6pt+1.4pt] Calcular los **cuatro primeros términos no nulos** del desarrollo en serie de potencias en torno a $x = 0$ de

a) $f(x) = \sqrt{1+x}$,

b) $f(f(x))$.

P5 [2pt] Calcular el valor de la integral

$$I = \int_9^{13} \frac{dx}{x - 6\sqrt{x-9}}.$$

Solucions del Examen Extraordinari de MATEMÀTICAS 2022

(En algún orden...)

(3) a) Knowing that $\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find $\sum_{n=1}^{\infty} \frac{5n+6}{n^2(n+1)}$

Partial fractions

$$\frac{5n+6}{n^2(n+1)} = \frac{A}{n^2} + \frac{B}{n} + \frac{C}{n+1}, \quad A, B, C \text{ constants}$$

Then

$$5n+6 = A(n+1) + Bn(n+1) + Cn^2$$

an identity

$n=0 \quad 6 = A,$

$n=-1 \quad 1 = C,$

coef $n^2 \quad 0 = B + C \therefore B = -C = -1.$

Thus,

$$\frac{5n+6}{n^2(n+1)} = \frac{6}{n^2} + \left(-\frac{1}{n} + \frac{1}{n+1}\right)$$

$$\sum_{n=1}^{\infty} \frac{5n+6}{n^2(n+1)} = 6 \underbrace{\sum_{1}^{\infty} \frac{1}{n^2}}_{=\pi^2} + \sum_{n=1}^{\infty} \left(-\frac{1}{n} + \frac{1}{n+1}\right)$$

It is a telescoping series (of course!)

$\sum_{n=1}^{\infty} \left(-\frac{1}{n} + \frac{1}{n+1}\right) = -1$, why?

Partial sums:

$S_1 = -1 + 1/2$

$S_2 = -1 + \cancel{1/2} - \cancel{1/2} + 1/3 = -1 + 1/3$

$S_3 = -1 + \cancel{1/2} - \cancel{1/2} + \cancel{1/3} - \cancel{1/3} + 1/4 = -1 + 1/4$

$S_N = -1 + \frac{1}{N+1} \xrightarrow[N \to \infty]{} -1$

Conclusion: $\sum_{n=1}^{\infty} \frac{5n+6}{n^2(n+1)} = \pi^2 - 1$

b) b1) $\sum_{n=2}^{\infty} \left(n^{1/n} - 1\right)^n$

$a_n = \left(n^{1/n} - 1\right)^n$

$|a_n| = a_n$ because $n \geq 2$

The series is <u>convergent</u> by the <u>root test</u>

$$\rho = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \left(n^{1/n} - 1\right) = 1 - 1 = 0 < 1$$

because

$$\lim_{n \to \infty} n^{1/n} = 1$$

<u>Proof</u>: [to obtain full mark one needs to prove ... as one can ... this limit. It is not a limit like $\lim_{n \to \infty} \frac{3n^2+1}{8n^2+1}$, that is why we <u>want</u> to prove that $\left(n^{1/n}\right) \xrightarrow[n \to \infty]{} 1$.

$\lim_{n \to \infty} n^{1/n} = \infty^0$: It is an indeterminate form

<u>Acceptable answers</u>

1) $n^{1/n} = e^{\frac{\log n}{n}}$ "

$$\lim_{n \to \infty} \frac{\log n}{n} \underset{L'H\hat{o}}{=} \lim_{n \to \infty} \frac{1/n}{1} = 0 \text{ "}$$

$$\lim_{n \to \infty} n^{1/n} = \lim_{n \to \infty} e^{\frac{\log n}{n}} = e^0 = 1.$$

2) $\lim_{n\to\infty} n^{1/n} = l$

Take log ... it is basically as 1).

log lim = lim log (log is a continuous fction and preserves limits)

$$\lim_{n\to\infty} \log n^{1/n} = \lim_{n\to\infty} \frac{1}{n} \log n = 0 = \log l \Leftrightarrow \boxed{l=1}.$$

as before

3) [Some of you may have read in books that

$$1^{1/n} < n^{1/n} < \underbrace{1+\sqrt{\frac{2}{n}}}_{\text{No visto en clase, pero pueden haberlo leído.}}$$

No espero que ustedes lo supieran en el examen, sólo lo escribo...

$$\lim_{n\to\infty} 1^{1/n} = \lim_{n\to\infty} 1+\sqrt{\frac{2}{n}} = 1$$

$$\therefore \lim_{n\to\infty} n^{1/n} = 1$$ Squeeze theorem.

Non acceptable answers $\equiv$ not receiving full mark

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} n^0 = 1$$

(!!!!!!!!!!!!!!!!) [If $1/n \to 0$ also $n\to\infty$ so write ∞^0 instead of n^0 !!!]

]

b2) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

This series has the same character than $\sum_1^{\infty} \frac{1}{n}$

because

$$\lim_{n\to\infty} \frac{1/n^{1+1/n}}{\frac{1}{n^{1+1/n}}} = \lim_{n\to\infty} 1/n^{1/n} = \underline{\underline{1}},$$ finite, not ∞

Hence, $\sum_{1}^{\infty} \frac{1}{n^{1+1/n}}$ is divergent by comparison to the limit with $\sum_{1}^{\infty} \frac{1}{n}$. This is a surprising result! Because the exponent $1+\frac{1}{n}$ is bigger than 1 for positive n. It is positioned in between

$$\frac{1}{n^2} < \frac{1}{n^{1+1/n}} < \frac{1}{n},$$

it should not be convergent as the p-series are convergent?

$\sum_{1}^{\infty} \frac{1}{n^p}$ with $p>1$ that are convergent?

NO !!! In these series p is fixed and in $\sum \frac{1}{n^{1+1/n}}$ now "the p" is not fixed.

c) there are several methods... the thing is to recognize $\sum_{1}^{\infty} nx^n$ as a "known function". For example,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots = \frac{1}{1-x} \quad \text{" if } |x|<1,$$

otherwise the relation is false. Then

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x} - 1.$$

Deriv wrt x:

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \quad \text{"} \quad |x|<1 \text{ all the time.}$$

multiply by x

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} \quad \text{only if } |x|<1$$

because otherwise $\sum_{1}^{\infty} nx^n$ is divergent.

Then, $\sum_{n=1}^{\infty} n x^n = 1/2$ is equivalent to

$$\frac{x}{(1-x)^2} = \frac{1}{2},$$

that is

$$x^2 - 4x + 1 = 0 \qquad x \begin{cases} \cancel{2+\sqrt{3}} & \text{discarded} \\ 2-\sqrt{3} \end{cases}$$

Solution: $\boxed{x = 2-\sqrt{3}}$ because satisfies $|x|<1$.

The solution is the unique of $\sum_{1}^{\infty} n x^n = 1/2$

[Another manner

$$S \equiv x + 2x^2 + 3x^3 + 4x^4 + \dots$$

útil

we are said that $S = 1/2$. But S satisfies

$$S = x + x^2 + x^3 + x^4 + \dots$$
$$\quad + x^2 + 2x^3 + 3x^4 + \dots$$

correct manipulations if the series converges

$$= x(1 + x + x^2 + \dots)$$
$$\quad + x(x + 2x^2 + 3x^3 + \dots)$$

$$= \frac{x}{1-x} + xS,$$

true if $|x|<1$ ... a fortiori (siempre se lo encuentra uno)

that is, if $|x|<1$

$$S = \frac{x}{1-x} + xS$$

or

$$S(1-x) = \frac{x}{1-x}$$

or

$$S(1-x)^2 = x = \frac{1}{2}(1-2x+x^2)$$

set by the problem: $S = 1/2$

Finally, x is a solution of $x^2-4x+1=0$, not bigger than 1 : $x = 2-\sqrt{3}$

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(5) $$\int_9^{13} \frac{dx}{x-6\sqrt{x-9}}$$

The good change of variables is
$$u=\sqrt{x-9}$$
$$u^2=x-9$$
$$2u\,du=dx.$$

The integral is equal to

$$\int_0^2 \frac{2u\,du}{u^2+9-6u} = 2\int_0^2 \frac{u\,du}{(u-3)^2} = 2\int_0^2 du\left[\frac{1}{u-3}+\frac{3}{(u-3)^2}\right],$$

because in simple fractions

$$\frac{u}{(u-3)^2} = \frac{1}{u-3}+\frac{3}{(u-3)^2}.$$

Truco en los exámenes: No vuelva a la variable x: ¿para qué va a volver? No deshaga el cambio, su trabajo. Ya ha puesto los límites correctos en la nueva variable, ¡pues ya ha cumplido! Y si hubiera hecho cinco cambios para llegar al resultado ¿los deshacemos todos? No, no, no y no!!! Es matemáticamente innecesario. Y lo que es aún peor: me deja ver a mí, que ha hecho pocas integrales en casa. Lo siento, pero es así.

Y poco profesional...

cont

$$= 2\left[\log|u-3| - \frac{3}{u-3}\right]_0^2$$

absolute value.

$$= 2(\log 1 - \log 3 + 3 - 1)$$

$$= 4 - 2\log 3.$$

Result:
$$\boxed{\int_9^{13}\frac{dx}{x-6\sqrt{x-9}} = 4-2\log 3}$$

SOLUTIONS MATEMÁTICAS

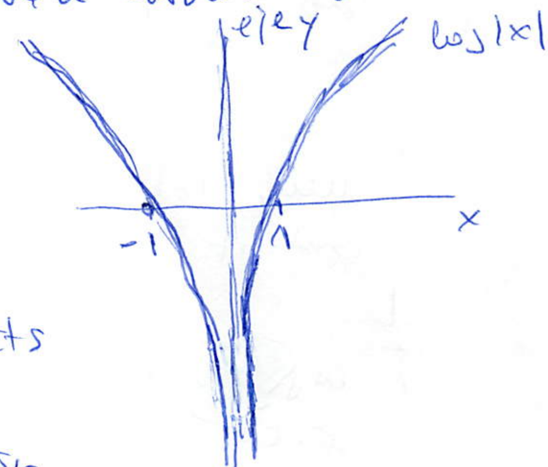
Examen Extraordinario 2022

(1) $f(x) = \arctan(\log|x|)$ if $x \neq 0$.
$f(0) = -\frac{\pi}{2}$

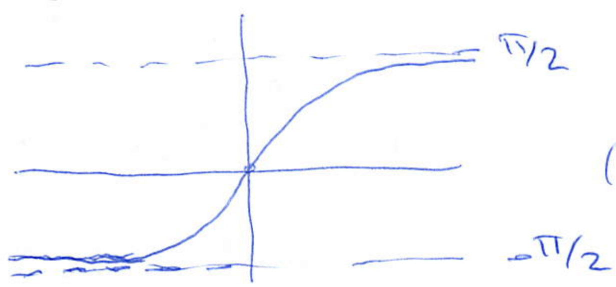
1) f is the composition of two functions,

$h(x) = \arctan x$,
and
$g(x) = \log|x|$. " $f = h \circ g$

Function $g(x) = \log|x|$ is even around $x=0$ and its graph is →
It is continuous everywhere except at $x=0$.



Function $h(x)$ is continuous everywhere (for all x real). Its graph is



(odd fction)

Since f is the composition of two continuous functions when $x \neq 0$ is continuous.
At $x=0$, g is discontinuous, so $x=0$ is the unique point that deserves further study.
We calculate, the

$$\lim_{x \to 0} \arctan(\underbrace{\log|x|}_{\text{tends to } -\infty})$$

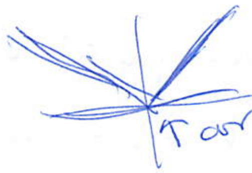
$= \arctan(-\infty)$

$= -\pi/2$

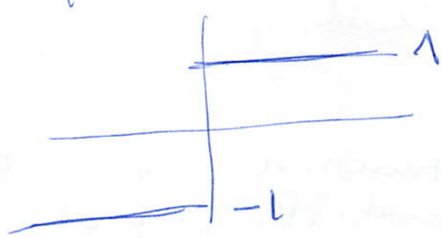
Since the limit $\lim_{x \to 0} f(x)$ coincides with the definition $f(0) = -\pi/2$, we conclude that f is continuous at $x=0$ too.

$\boxed{\text{Conclusion: } f \text{ is continuous } \forall x \text{ real}}$

2) Γ $|x| = \begin{cases} x & \text{if } x>0 \\ -x & \text{if } x<0 \end{cases}$



$\frac{d|x|}{dx} = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$



Then, $|x|$ is not derivable at $x=0$ (has a corner!) but it is derivable at $x \neq 0$.

L

Γ $\log|x|$ inherits the non-derivability at $x=0$ since

$$\frac{d \log|x|}{dx} = \frac{\frac{d|x|}{dx}}{|x|} = \begin{cases} \frac{1}{|x|} = \frac{1}{x} & \text{if } x>0 \\ -\frac{1}{|x|} = \frac{1}{x} & \text{if } x<0 \end{cases}$$

$= \frac{1}{x}$ singular at $x=0$

L

Γ $\arctan x$ is derivable everywhere since

$(\arctan x)' = \frac{1}{1+x^2}$ never zero for x real.

L

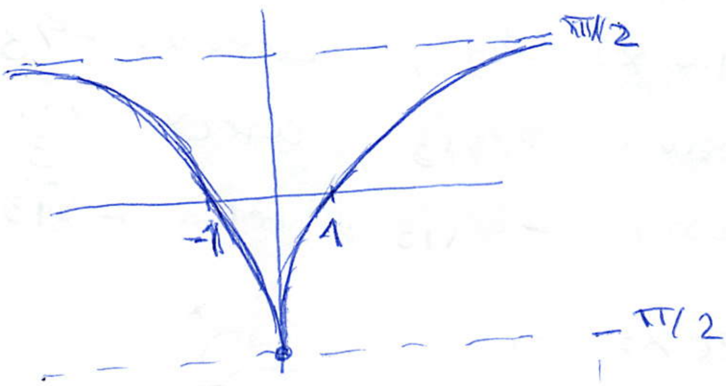
The function $f(x) = \arctan(\log|x|)$ is derivable everywhere except at $x=0$ where $\log|x|$ is not even continuous. If $x \neq 0$,

$$f'(x) = \frac{1/x}{1+(\log|x|)^2}, \quad \text{odd function.}$$

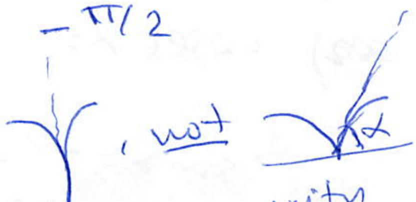
3) $f(x)$ is even because $f(x) = f(-x)$.

$$\lim_{x\to\infty} f(x) = \arctan(\infty) = \pi/2$$

The profile of $f(x)$ is then



Observe this "peak". It is



with $\tan\alpha$ = finite

with a vertical asymptote

I check this point, namely, that

$$\lim_{x\to 0^+} f'(x) = \infty$$

and not a finite value (as in this picture)

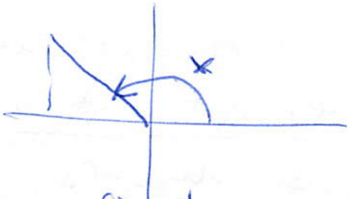
$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \frac{1/x}{1+(\log x)^2} = \frac{\infty}{\infty} \underset{\text{L'Hôpital}}{=} \lim_{x\to 0^+} \frac{-1/x^2}{2\log x \cdot \frac{1}{x}}$$

$$= \lim_{x\to 0^+} \frac{-1/x}{2\log x} = -\frac{\infty}{\infty} \underset{\text{L'Hôp again}}{=} \lim_{x\to 0^+} \frac{1/x^2}{2/x}$$

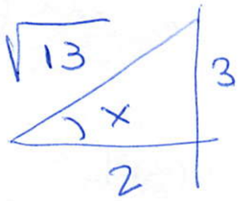
$$= \lim_{x\to 0^+} \frac{1}{2x} = \underline{\underline{\infty}}.$$

L

(2) a1) $\tan x = -3/2$:



Suppose x were in the first quadrant for a moment:



$\tan x = 3/2$ $\quad$ $\cot{a}n\, x = 2/3$

$\sin x = 3/\sqrt{13}$, $\operatorname{cosec} x = \sqrt{13}/3$

$\cos x = 2/\sqrt{13}$, $\sec x = \frac{\sqrt{13}}{2}$

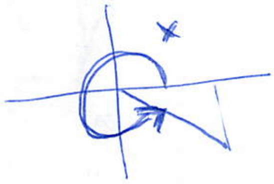
In the second quadrant:

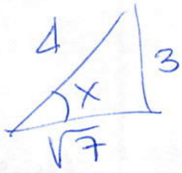
$\tan x = -3/2$, $\operatorname{cotan} x = -2/3$

$\sin x = 3/\sqrt{13}$, $\operatorname{cosec} x = \frac{\sqrt{13}}{3}$

$\cos x = -2/\sqrt{13}$, $\sec = -\sqrt{13}/2$

a2) $\operatorname{cosec} x = -4/3$,





x is the first quadrant

$\sin x = -3/4$

$\cos x = \sqrt{7}/4$

$\tan x = -3/\sqrt{7}$

$\operatorname{cotan} x = -\sqrt{7}/3$

$\sec x = \frac{1}{\cos x} = 4/\sqrt{7}$

$\operatorname{cosec} x = -4/3$

b) $\frac{x^3 - 6x + 1}{x+2}$:

	1	0	-6	1
-2		-2	4	4
	1	-2	-2	5

$x^2 - 2x - 2 + \frac{5}{x+2}$

$$\frac{x^3-6x+1}{x+2} \overset{\text{an identity}}{=} x^2-2x-2+\frac{5}{x+2}$$

$$\lim_{x\to+\infty}\left(x^2-2x-2+\frac{5}{x+2}\right) = x^2-2x-2$$

or $x\to-\infty$

The oblique asymptote in both $x\to+\infty$ or $x\to-\infty$ is the parabola x^2-2x-2. Plotee con Maple, Mathematica, MATLAB, lo que sea ambos gráficos:

$$\frac{x^3-6x+1}{x+2} \quad y \quad x^2-2x-2$$

y lo verá. Es prodigioso.

---

(4) $$\sqrt{1+x} = (1+x)^{1/2}$$

$$= \binom{1/2}{0} + \binom{1/2}{1}x + \binom{1/2}{2}x^2 + \binom{1/2}{3}x^3 + \dots$$

via binomial expansion, for instance. Not deriving!!!!!

$$\binom{1/2}{0} = 1$$

$\binom{1/2}{1} = 1/2$, because $\binom{m}{1} = m$ always

$\binom{1/2}{2} = -\frac{1}{8}$ because $\binom{m}{2} = \frac{m(m-1)}{2!} \underset{m=1/2}{=} -\frac{1}{8}$

$\binom{1/2}{3} = \frac{1}{16}$ " $\binom{m}{3} = \frac{m(m-1)(m-2)}{3!} \underset{m=1/2}{=} \frac{1}{16}$

Then

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} + \dots$$

← 4 non zero terms →

6

b) $f(f(x)) = \sqrt{1+\sqrt{1+x}}$

Do not use derivatives!!! It is a Titan work!!

We are asked to expand $\sqrt{1+\sqrt{1+x}}$ as

$$\sqrt{2} + ax + bx^2 + cx^3 + dx^4 + \dots \quad (*)$$

with a, b, c, d constants to be found

[$\sqrt{2}$ because if $x=0$,

$$\sqrt{1+\sqrt{1+0}} = \sqrt{2}.$$

]

There are two manners (at least) to find a, b, c, d, ... in (*) without too much pain:

(W1) Use a) and write $\sqrt{1+\sqrt{1+x}}$ as

$$\sqrt{2 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots} = \sqrt{2} + ax + bx^2 + cx^3 + \dots$$

square this relation and identify coefficients: that affords a, b, c ...

$$\underbrace{2 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots}_{LHS} = \underbrace{\left(\sqrt{2} + ax + bx^2 + cx^3 + \dots\right)^2}_{RHS}$$

$$\begin{aligned} RHS = {} & 2 \\ & + x\left[2a\sqrt{2}\right] \\ & + x^2\left[a^2 + 2\sqrt{2}\,b\right] \\ & + x^3\left[2\sqrt{2}\,c + 2ab\right] \\ & + x^4\left[\dots\right] + \dots \end{aligned}$$

$$2a\sqrt{2} = 1/2 \qquad \boxed{a = \frac{1}{4\sqrt{2}}}$$

$$a^2 + 2\sqrt{2}\,b = \frac{1}{32} + 2\sqrt{2}\,b = -\frac{1}{8} \;\text{"} \qquad \boxed{b = \frac{-5}{64\sqrt{2}}}$$

$$2\sqrt{2}c + 2ab = 2\sqrt{2}c + \frac{5}{64\cdot 4} = \frac{1}{16}, \quad \boxed{c = \frac{21}{512\sqrt{2}}}$$

a, b as before

Solution: $\sqrt{1+\sqrt{1+x}} = \sqrt{2} + \frac{1}{4\sqrt{2}}x - \frac{5x^2}{64\sqrt{2}} + \frac{21}{512\sqrt{2}}x^3 + \cdots$

$$= \frac{1}{\sqrt{2}}\Big(\underbrace{2 + \frac{x}{4} - \frac{5}{64}x^2 + \frac{21}{512}x^3}{} + \cdots\Big)$$

cuatro términos no malos! Y con un trabajo razonable.

(W2) $\sqrt{1+\sqrt{1+x}} = \sqrt{2 + \boxed{\frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots}}$ call this u

use a)

Notice that if $x=0$, $u=0$. It is important

$$= \sqrt{2+u}$$

$$= \sqrt{2}\sqrt{1+\frac{u}{2}}$$

as before!!

$$= \sqrt{2}\left(1 + \frac{1}{2}\left(\frac{u}{2}\right) - \frac{1}{8}\left(\frac{u}{2}\right)^2 + \frac{1}{16}\left(\frac{u}{2}\right)^3 + \cdots\right)$$

use again binomial series

$=$ collect powers of x using u as a function of x, check it, please

$$= \frac{2}{\sqrt{2}}\left[1 + \frac{x}{8} - \frac{5}{128}x^2 + \frac{21}{1024}x^3 + \cdots\right]$$

same result as in (W1).