Solucions | 2021-2022

UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

Curso 2021-2022

Examen Final Ordinario de Matemáticas F

Nombre y Apellidos:

Firma y DNI:

Nota: En esta prueba no se permiten libros o apuntes ni **calculadora**. La nota total de este examen son 10 puntos.

P1 [1.75pt] Sean las funciones f y g definidas por

$$f(x) = \log(x-1), \quad x > 1,$$
$$g(x) = (x-2)^2 + 3, \quad x > 2.$$

a) Determinar el recorrido de $f(x)$

b) Justificar si existe f^{-1} y escribir $f^{-1}(x)$ en su caso.

c) Encontrar el valor de x que cumple $(f^{-1} \circ g)(x) = 1 + e^7$. El síimbolo $\circ$ denota *composición de funciones*.

P2 [2.25pt] Sea la función $f(x) = \dfrac{x}{1+\cos\frac{\pi}{2}x}$ definida en el intervalo $[0,1]$. ¿Cuántos valores de c en $(0,1)$ satisfacen las hipótesis del *Teorema del Valor Medio*? a) Ningún valor de c, b) sólo un valor, c) más de un valor.

Nota: Para responder a cualquiera de las preguntas hay que hacer cálculos, no hay otra alternativa.

P3 [0.75pt+1.25pt] a) Discutir, empleando al menos **dos tests diferentes/independientes**, si la serie numérica $\sum_{n=1}^{\infty}(-1)^n\left(\dfrac{n}{2n-1}\right)^n$ es convergente, absolutamente convergente o divergente.

b) ¿Para qué valores del parámetro p es la serie $\sum_{n=1}^{\infty}\left(\dfrac{1}{n} - \sin\left(\dfrac{1}{n}\right)\right)^p$ convergente?

P4 [2pt] Sea la función

$$F(x) = \int_x^{2x} dt\, e^{-t^2}$$

con x en el intervalo $[0,\infty)$. Estudiar los extremos absolutos y relativos de F, es decir, los máximos y mínimos absolutos y relativos.

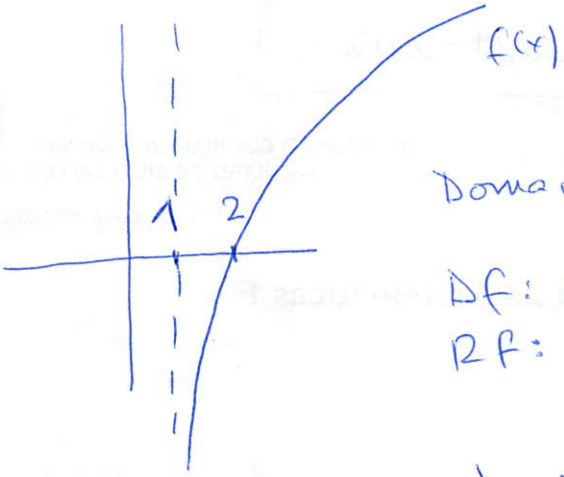
P5 [2pt] Calcular el valor de la integral

$$I = \int_{-1}^{1} dx\, \frac{1}{(x-2)(x^2+x+1)}.$$

2

(P1)

a)



Domain and range of f are:

$Df: (1, \infty)$
$Rf: (-\infty, \infty)$

f (see its graphic) is a strictly increasing function $\Rightarrow$ f is injective in its domain

[Another manner of proving injectivity: is to prove that $f(a) = f(b) \Rightarrow a = b \ \forall a, b$ in Df:

We see this point:

$f(a) = f(b)$ is $\log(a-1) = \log(b-1)$, or

$e^{\log(a-1)} = e^{\log(b-1)}$,

$\Rightarrow a - 1 = b - 1$ or

$\Rightarrow a = b$ " $a, b \in Df$

$\exists f^{-1}$ with $Df^{-1} = (-\infty, \infty)$ and $Rf^{-1} = (1, \infty)$

b) ~~$y = \log(x-1)$. Here $y = f(x)$~~ leave

do not consider this relation but

$x = \log(y-1)$. Here $y = f^{-1}(x)$

From $x = \log(y-1)$,

$e^x = y - 1$

$y = 1 + e^x$

$$\boxed{f^{-1}(x) = 1 + e^x}$$

se dice siempre (y puntúa por ello)

$Df^{-1}: (-\infty, \infty)$
$Rf^{-1}: (1, \infty)$

c) $f^{-1}(g(x)) = (f \circ g)(x) = 1 + e^{(x-2)^2+3}$,

$1 + e^{(x-2)^2+3} = 1 + e^7$,

$(x-2)^2 + 3 = 7$

$(x-2)^2 = 4$

$x - 2 \begin{cases} 2 \\ -2 \end{cases}$ " $x \begin{cases} 4 \\ 0 \end{cases}$ no vale?

But $x = 0$ is not in the domain of g,

and the value of x asked = the problem must be

$$\boxed{x=4}$$

[$f^{-1}(g(x))$ — x is in D_g that according to the problem is $(2,\infty)$.
]

(P2) $f(x) = \dfrac{x}{1+\cos\frac{\pi x}{2}}$.

The points where f is not a continuous function are the solutions x of

$$\cos\frac{\pi x}{2} = -1,$$

✓ none of them in $[0,1]$

that is,

$$x = \pm 2, \pm 6, \pm 10, \pm 14, \ldots$$

Consequently, f is a continuous function on $[0,1]$ and derivable on $(0,1)$. By the Mean Value Theorem, there exists a point c in $(0,1)$ s.t.

$$f'(c) = \frac{f(1)-f(0)}{1-0} = \frac{1-0}{1-0} = 1.$$

[(a) is not the correct answer:
]

How many points c in $(0,1)$ do satisfy $f'(c)=1$, or equivalently how many solutions x of

$$\frac{1+\cos\frac{\pi x}{2}+\frac{\pi x}{2}\sin\frac{\pi x}{2}}{\left(1+\cos\frac{\pi x}{2}\right)^2} = 1$$

do belong to $(0,1)$? [The MVT says that one for sure... but just one?]

↳ Si usted no me quiere decir dónde es f discontinua me tendrá que decir por qué f es continua en $[0,1]$, por ejemplo que $\cos \pi x/2 \geq 0$ si $x \in [0,1]$ y por lo tanto $1+\cos \pi x/2 \geq 1$

→ El denominador nunca se anula en $[0,1]$. Pero me tendrá que decir algo.

This condition is also

$$\cos\frac{\pi}{2}x + \cancel{1} + \frac{\pi}{2}x\sin\frac{\pi}{2}x = \cancel{1} + \cos^2\frac{\pi x}{2} + 2\cos\frac{\pi x}{2},$$

that reduces to

$$\frac{\pi}{2}x\sin\frac{\pi}{2}x = \cos^2\frac{\pi x}{2} + \cos\frac{\pi x}{2}$$

⌈ Let analyze this equation, or better

$$u\sin u = \cos^2 u + \cos u \quad , \; u \equiv \frac{\pi}{2}x$$

with

$$u \in (0, \pi/2).$$

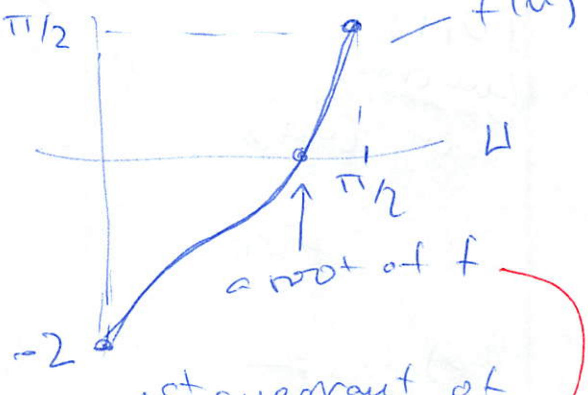
Define

$$g(u) \equiv u\sin u - \cos^2 u - \cos u \quad , \; u \in [0, \tfrac{\pi}{2}],$$

since

$$g(0) = -1-1 = -2 < 0,$$
$$g(\tfrac{\pi}{2}) = \pi/2 > 0,$$

g continuous on $[0, \frac{\pi}{2}]$, according to Bolzano theorem there exists a point in $(0, \pi/2) \ni u$ where $g(u) = 0$ (what we already knew by MVT). In addition



$$g'(u) = \sin u + u\cos u + 2\cos u\sin u + \sin u$$

$' \equiv \frac{d}{du}$

is positive if $u \in (0, \pi/2)$ [cos u and sin u are positive on the 1st quadrant of the circle) indicating that g is strictly increasing from $u = 0$ to $u = \pi/2$.
⌊

esto es quien dica "una solución sólo"

"the" root of f in $(0, \pi/2)$

Therefore the point c in $(0,1)$ whose existence predicts the MVT is unique. The correct answer is ⓑ

⌈ Additional information:

$$g\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\frac{\sqrt{2}}{2} - \frac{1}{2} - \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{2}\underbrace{\left(\frac{\pi}{4} - 1\right)}_{\text{a negative number}} \underbrace{- \frac{1}{2}}_{\text{a negative number}} < 0$$

The point c that ensures the MVT is in $\left(\frac{1}{2}, 1\right)$. Unique and in $\left(\frac{1}{2}, 1\right)$

⌊

Ⓟ3 a) In the case of the series $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{2n-1}\right)^n$ it is <u>more reasonable</u> to study the absolute series of positive terms

$$\sum_{n=1}^{\infty} \left(\frac{n}{2n-1}\right)^n$$

than the alternating series of the problem.

the simplest one in this particular problem

i) By the root test [this test is for series of positive <u>terms</u>, therefore written as $\sqrt[n]{|a_n|}$ the sequence to consider]

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{n}{2n-1}\right)^n} = \boxed{\frac{n}{2n-1}} \xrightarrow[n\to\infty]{} \frac{1}{2}.$$

always positive $\forall n \geq 1$

Since $0<\frac{1}{2}<1$, the series $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{2n-1}\right)^n$ is absolutely convergent, therefore <u>convergent</u>.

[This means that we do <u>not</u> need to study $\sum_{1}^{\infty} (-1)^n \left(\frac{n}{2n-1}\right)^n$ via the AST, which is a pain, the pain being in how to show that

$a_n > a_{n+1}$ $\forall n$

[This is a delicate point given $a_n = \left(\frac{n}{2n-1}\right)^n$]

[If you insist (mal hecho!!) in the use of the AST theorem go to the end of this exercise.

one two three

I now write ii) iii) iv) three independent test to the root test. All of them applied to the series

$$\sum^{\infty} |a_n| \quad \text{and not to} \quad \sum^{\infty} (-1)^n a_n$$

ii) By comparison to the limit.

since

$$\frac{n}{2n-1} \xrightarrow[n\to\infty]{} 1/2 , \qquad \left(\frac{n}{2n-1}\right)^n \xrightarrow[n\to\infty]{} \frac{1}{2^n}$$

the series

$$\sum^{\infty} |a_n| = \sum^{\infty} \left(\frac{n}{2n-1}\right)^n$$

and $\sum^{\infty} \left(\frac{1}{2}\right)^n$ have the same character.

⌈ This is so because

$$\lim_{n\to\infty} \frac{\left(\frac{n}{2n-1}\right)^n}{\left(\frac{1}{2}\right)^n} = 1$$

⌊

The series $\sum^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric convergent series $\Rightarrow$

$$\sum^{\infty} \left(\frac{n}{2n-1}\right)^n \text{ converges}$$

$$\Rightarrow \sum^{\infty} (-1)^n \left(\frac{n}{2n-1}\right)^n \text{ converges.}$$

iii) By inequalities

$$2n-1 > \frac{3}{2}n \quad \text{if } n \geq 2$$

$$\frac{n}{2n-1} < \frac{n}{\frac{3}{2}n} = \frac{2}{3} < 1$$

and

$$\left(\frac{n}{2n-1}\right)^n < \left(\frac{2}{3}\right)^n.$$

Again

$$0 < \sum_{n=2}^{\infty} \left(\frac{n}{2n-1}\right)^n < \sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^n .$$

↑ geometric series of ratio $2/3 < 1$ convergent

The series $\sum^{\infty} \left(\frac{n}{2n-1}\right)^n$ = convergent,

therefore $\sum^{\infty} (-1)^n \left(\frac{n}{2n-1}\right)^n$ = convergent.

iv) By the ratio test (not advisable for this a_n but some students have tried it):

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left(\frac{n+1}{2n+1}\right)^{n+1}}{\left(\frac{n}{2n-1}\right)^n}$$

$$= \left(\frac{n+1}{2n+1}\right)\left(\frac{(n+1)(2n-1)}{n\;(2n+1)}\right)^n$$

todo son iguales → (=), o sea, álgebra pura y dura

$$= \left(\frac{n+1}{2n+1}\right)\left(\frac{\cancel{n}(1+1/n)}{\cancel{n}}\right)^n \left(\frac{\cancel{2n}(1-1/2n)}{\cancel{2n}(1+1/2n)}\right)^n$$

Now we take the limit when $n \to \infty$:

$$\lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2}$$

$$\lim_{n\to\infty} (1+1/n)^n = e$$ and not 1 as somebody writes in an exam

$$\lim_{n\to\infty} (1-1/2n)^n = e^{-1/2}$$ and not 1

$$\lim_{n\to\infty} (1+1/2n)^n = e^{1/2}$$. and not 1

Thus $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} \cdot \frac{e \cdot e^{-1/2}}{e^{1/2}} = 1/2 \cdot 1$$

↑ milagro!!

Tampoco

$$\lim_{n\to\infty} \frac{\left(1-\cancel{1/2n}\right)^n}{\left(1+\cancel{1/2n}\right)^n}$$

se cancelan, tampoco. Son $\frac{e^{-1/2}}{e^{1/2}}$. No se cancela nada...

Since $\rho = 1/2 < 1$, the serie $\sum^{\infty} |a_n|$ is convergent, therefore $\sum^{\infty} (-1)^n a_n$ is convergent.

If you insist in using the AST or Leibnitz theorem:

Define: $a_n \equiv \left(\frac{n}{2n-1}\right)^n$

1) obviously $a_n > 0$ since $n > 0$, $2n-1 > 0$ for all $n \geq 1$.

3) $\lim_{n\to\infty} \left(\frac{n}{2n-1}\right)^n = \lim_{n\to\infty} \frac{1}{2^n} = \underline{\underline{0}}$

2) We prove that $\underbrace{a_n > a_{n+1}}_{\text{sin límite, para cualquier } n \text{ finito}}$ $\underline{\underline{\forall n}}$

Consider

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{2n+1}\right)\left(\frac{(2n-1)(n+1)}{n(2n+1)}\right)^n$$

as before

$$= \left(\frac{n+1}{2n+1}\right)\left(\frac{2n^2+n-1}{2n^2+n}\right)^n$$

obviamente $\frac{n+1}{2n+1} = \frac{n+1}{n+1+n}$ es siempre < 1 $\forall n$

$\frac{2n^2+n-1}{2n^2+n} < 1$. $\left[\begin{array}{l} 0 < a < 1 \Rightarrow \\ 0 < a^n < 1 \quad n = 1, 2, 3, \ldots \end{array}\right.$

$\left(\frac{2n^2+n-1}{2n^2+n}\right)^n < 1$

Condition:

$$\frac{a_{n+1}}{a_n} < 1 \quad \underline{\forall n}$$

Therefore,

$$a_n > a_{n+1} \quad \forall n,$$

as we wanted to prove.

Thus, $\sum^{\infty} (-1)^n \left(\frac{n}{2n-1}\right)^n$ = convergent

by the AST theorem.

Obviamente que i, ii, iii, iv dan més informació. Porque prueban <u>convergencia absolute</u>

b) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right)^p$,

since $\frac{1}{n} - \sin\frac{1}{n} = \cancel{\frac{1}{n}} - \left[\cancel{\frac{1}{n}} - \frac{1}{n^3}\cdot\frac{1}{3!} + \frac{1}{n^5}\cdot\frac{1}{5!} - \frac{1}{n^7}\cdot\frac{1}{7!} + \dots\right]$

for <u>all</u> n

$$= \frac{1}{n^3}\left[\frac{1}{3!} - \frac{1}{n^2 5!} + \frac{1}{n^4 \cdot 7!} - \dots\dots\right]$$

$$\simeq \frac{1}{n^3}\cdot\frac{1}{3n} \quad \underline{\text{if}}\ n \to \infty$$

By comparison to the limit

$$\lim_{n\to\infty} \frac{\frac{1}{n} - \sin\frac{1}{n}}{1/n^3} = \frac{1}{6}$$

finite and non-zero as the theorem states!

indicating that

$\sum_1^\infty \left(\frac{1}{n} - \sin\frac{1}{n}\right)^p$ and $\sum_1^\infty \frac{1}{n^{3p}}$ have the same character. Conclusion

$$\sum_1^\infty \left(\frac{1}{n} - \sin\frac{1}{n}\right)^p = \begin{cases} \text{convergent if } 3p > 1, \\ \quad \text{ie if } \boxed{p > 1/3} \\ \text{divergent if } \boxed{p \leq 1/3} \end{cases}$$

If p is negative o zero is divergent, obviously.

(P4) $F(x) = \int_x^{2x} dt\, e^{-t^2}$ (the integrand)

The function e^{-x^2} is continuous everywhere then by the FTC1 (Fundamental theorem of Infinitesimal Calculus, part 1) F(x) given as above is continuous and derivable ⊛ ∀ x in ℝ, and in particular in [0,∞).

However [0,∞) is not bounded nor closed so the EVT, Extreme Value Theorem, does not apply, meaning that F(x) in [0,∞) may

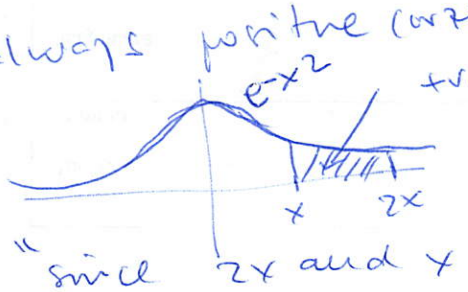
- have both absolute maximum and absolute minimum or
- have absolute maximum but not absolute minimum (or viceversa) or
- have neither absolute maximum nor absolute minimum.

O sea, que no sabemos lo que tiene. Hay en [0,1) que echar las cuentas.

What we know: a) $\boxed{F(0) = 0}$ because

$$F(0) = \int_0^0 e^{-t^2} = 0.$$

b) F(x) is always positive (or zero) because it is the area of



+ve or zero

⊛ omitido: "since 2x and x are also derivable"

c) $F(\infty) = \lim_{b\to\infty} \int_b^{2b} e^{-t^2} < \lim_{b\to\infty} \int_b^{2b} e^{-t}$

Importante calcularlo
$[0,\infty) \equiv$ semirecta.
↑ el "borde" del intervalo

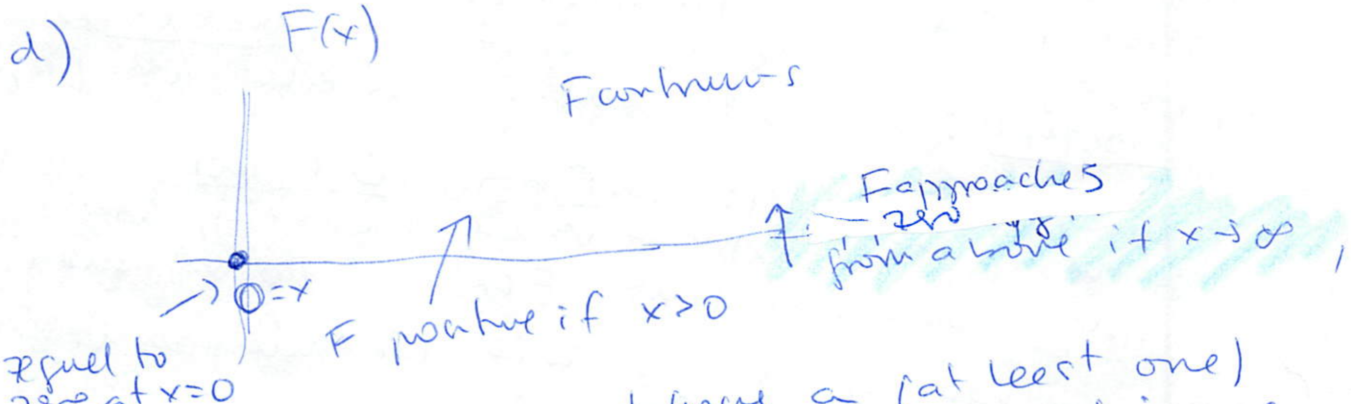
$e^{-t^2} < e^{-t}$ if $t \geq 2$ for instance

because $t^2 > t$
$e^{t^2} > e^{t}$

$= \lim_{b\to\infty} \left[-e^{-t}\right]_b^{2b} = \lim_{b\to\infty}\left(-e^{-2b} + e^{-b}\right)$

$= \lim_{b\to\infty} e^{-b}\left(1 - e^{-2b}\right) \simeq \lim_{b\to\infty} \underbrace{e^{-b}}_{\text{positive}} = \underline{\underline{0^+}}$

d)



F(x) must have a $\equiv$ necessarily maximum obtainable by (at least one) resolution of $F'(x)=0$ (there are no points where F' does not exist).

$F'(x) = f(2x)\frac{d(2x)}{dx} - f(x)\frac{dx}{dx}$

$= 2e^{-4x^2} - e^{-x^2}$

here $f(x) = e^{-x^2}$

Thus

$$\boxed{F'(x) = 2e^{-4x^2} - e^{-x^2}}$$

Resolution of $F'(x)=0$

$$F'(x) = \underbrace{e^{-x^2}}_{\text{never zero for finite } x,}\left[2e^{-3x^2} - 1\right]$$

then $F'(x)=0 \iff$

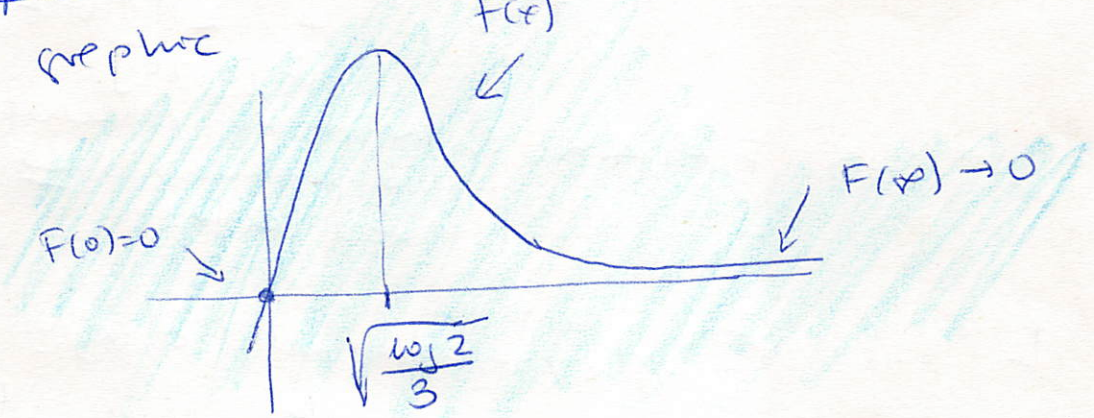
$$e^{-3x^2} = 1/2$$

or

$$3x^2 = \log 2$$

$$x = \pm\sqrt{\frac{\log 2}{3}}$$

only $x = \sqrt{\frac{\log 2}{3}}$ is in $(0,\infty)$ and it is necessarily, no second derivative to be calculated, what for?, a maximum. And it is the unique critical point of the problem, $F(x)$ with graphic



[log 2 is a famous number. It is ≃ 0.69. Certainly a number in between 0 and 1

$$1 < 2 < e = 2.71828$$

$$\Rightarrow 0 = \log 1 < \log 2 < \log e = 1$$

$$\Rightarrow 0 < \log 2 < 1$$

call $a \equiv \sqrt{\frac{\log 2}{3}}$

$\frac{\log 2}{3} \simeq \quad 0.69 \underline{|3} \quad 0.23$

$a \simeq \sqrt{0.23} < \sqrt{0.25} < 0.5$

then $0 < a = \sqrt{\frac{\log 2}{3}} < 0.5$

└

┌ Maple says that

$$a = \sqrt{\frac{\log 2}{3}} = 0.4807$$

but Maple did not come to the exam.

└

┌ there is an absolute maximum obtained via $0 = F'(x)$ at $\underset{\text{maximum}}{\overset{\text{absolute}}{}} \equiv F(a)$ with value:

$$F(a) = F\left(\sqrt{\frac{\log 2}{3}}\right) = \int_a^{2a} e^{-t^2} dt < \int_a^{2a} dt = a$$

$$e^{-t^2} < 1 \quad \forall t$$

$\Rightarrow F(a) < a$: Mejor sin calculadora, no podemos o no nos hace falta.

└

In the interval $[0, \infty)$:

conclusion: F has an absolute minimum at $x_{min} = 0$ with value $F(0) = \underline{\underline{0}}$,

that $x_{min} = 0$ is extreme of the interval $[0, \infty)$.

F has an absolute maximum at $x_{max} \equiv a = \sqrt{\frac{\log 2}{3}}$ of value

$F\left(\sqrt{\frac{\ln 12}{3}}\right) < 0.5$. This absolute maximum obtained with $F'(x)=0$.

I do insist: No theorem forces F to have in $[0,\infty)$ an absolute maximum nor an absolute minimum. But it has both.

I = MJRPlaza

(P5) Calculate $I \equiv \int_{-1}^{1} dx \, \frac{1}{(x-2)\,(x^2+x+1)}$

Expression of the integrand

$$\frac{1}{(x-2)\,(x^2+x+1)} = \frac{1}{7}\left[\frac{1}{x-2} - \frac{x+3}{x^2+x+1}\right]$$

in partial fractions.

⌈It is mandatory to say that x^2+x+1 has no real roots therefore we write the identity

$x^2+x+1>0$ $\forall x$

$$\frac{1}{(x-2)\,(x^2+x+1)} = \frac{A}{x-2} + \frac{Bx+C_1}{x^2+x+1}$$

with A, B, C_1 constants to be determined: This identity holds for all x and

$$A(x^2+x+1) + (x-2)(Bx+C_1) = 1 \qquad [*]$$

Taking:

- $x=2$ we obtain that $7A=1$ or $\boxed{A = 1/7}$
- The coeff of x^2 in $[*]$ is $A+B=0$ then $\boxed{B = -1/7}$
- The independent term in $[*]$ is $A-2C_1=1$ or $\boxed{C = -3/7}$

⌊

We decompose our integral I in two integrals I_1, I_2:

$$I = \frac{1}{7}\Bigg[\underbrace{\int_{-1}^{1} \frac{dx}{x-2}}_{I_1} - \underbrace{\int_{-1}^{1} \frac{x+3}{x^2+x+1}}_{I_2}\Bigg]$$

$$I = \frac{1}{7}\big(\underset{\substack{\uparrow \\ \text{very} \\ \text{simple}}}{I_1} - \underset{\substack{\uparrow \\ \text{more} \\ \text{involved}}}{I_2}\big)$$

$$I_1 \equiv \int_{-1}^{1} dx \, \frac{1}{x-2} = \Big[\log \underset{\text{absolute value}}{|x-2|}\Big]_{-1}^{1} = \overset{0}{\cancel{\log 1}} - \log 3 = -\log 3$$

$$\boxed{I_1 = -\log 3}$$

$$I_2 \equiv \int_{-1}^{1} \frac{x+3}{x^2+1+x} = \int_{-1}^{1} \Bigg[\underset{\text{immediate}}{\frac{x+1/2}{x^2+x+1}} + \frac{5/2}{x^2+x+1}\Bigg]$$

$$= \frac{1}{2}\Big[\log(x^2+x+1)\Big]_{-1}^{1} + 5/2 \int_{-1}^{1} \frac{dx}{x^2+x+1}$$

$$= \frac{1}{2}\Big[\log 3 - \underset{0}{\cancel{\log 1}}\Big] + 5/2 \; I_3$$

$$I_3 \equiv \int_{-1}^{1} \frac{dx}{x^2+x+1},$$

and

$$\boxed{I_2 = \frac{1}{2}\log 3 + 5/2 \; I_3}$$

Calculation of I3:

$$x^2+x+1 \equiv \left(x+\frac{1}{2}\right)^2+3/4$$

(an identity)

[completing squares]

$$I3 = \int_{-1}^{1} dx \, \frac{1}{x^2+x+1} = \int_{-1}^{1} \frac{1}{(x+1/2)^2+3/4}$$

$$= 4/3 \int_{-1}^{1} \frac{dx}{\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1}$$

$$= \frac{4}{3} \int_{-1}^{1} \frac{dx}{1+\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2}$$

$$\left.\begin{aligned} t &\equiv \frac{2}{\sqrt{3}}(x+1/2) \\ dt &= \frac{2}{\sqrt{3}}\,dx \end{aligned}\right\} \rightarrow = \frac{4}{3}\cdot\frac{\sqrt{3}}{2} \int_{-1/\sqrt{3}}^{\sqrt{3}} \frac{1}{1+t^2}$$

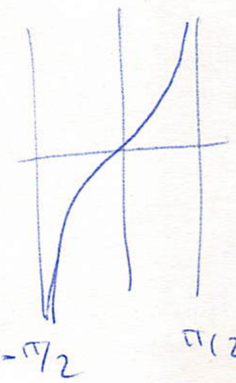
$$= \frac{2}{\sqrt{3}} \left[\arctan t\right]_{-1/\sqrt{3}}^{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} \left[\arctan\sqrt{3} + \arctan\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{2}{\sqrt{3}} \left[\frac{\pi}{3}+\frac{\pi}{6}\right]$$

$$= \frac{2}{\sqrt{3}} \left[\frac{\pi}{2}\right]$$

$$= \frac{\pi}{\sqrt{3}}$$



$\arctan\sqrt{3} = 60^\circ = \pi/3$

$\arctan(1/\sqrt{3}) = 30^\circ = \pi/6$

both values $(\frac{\pi}{3} \& \frac{\pi}{6})$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$: principal branch of tan x

Result: $\boxed{I_3 = \frac{\pi}{\sqrt{3}}}$

All together now,

$$I = \frac{1}{7}\left[-\log 3 - \frac{1}{2}\log 3 - \frac{5}{2} I_3 \right]$$

$$= \frac{1}{7}\left[-\frac{3}{2}\log 3 - \frac{5}{2}\frac{\pi}{\sqrt{3}} \right]$$

$$= -\frac{3}{14}\log 3 - \frac{5\sqrt{3}}{42}\pi$$

$$\boxed{I = -\frac{3}{14}\log 3 - \frac{5\sqrt{3}}{42}\pi}$$